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The Wiener Index of an Undirected Power Graph

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Abstract

The undirected power graph $P(Z_n)$ of a finite group Z_n is the graph with vertex set G and two distinct vertices u and v are adjacent if and only if $u \neq v$ and $\langle u\rangle \subseteq \langle v\rangle$ or $\langle v\rangle \subseteq \langle v\rangle$. The Wiener index $W(P(Z_n))$ of an undirected power graph $P(Z_n)$ is defined to be sum $\frac{1}{2}\sum_{\{u,v\}\subseteq V(P(Z_n))} d(u,v)$ of distances between all unordered pair of vertices in $P(Z_n)$. Similarly, the edge-Wiener index $W_e(P(Z_n))$ of $P(Z_n)$ is defined to be the sum $\frac{1}{2}\sum_{e,f} d(e,f)$ of distances between all unordered pairs of edges in $P(Z_n)$. In this paper, we concentrate on the wiener index of a power graph $P(Z_{p^k})$, $P(Z_{pq})$ and $P(Z_p)$. Firstly, we obtain new results on the wiener index and edge-wiener index of power graph $P(Z_n)$, using m,n and Euler φ function. Also, we obtain an equivalence between the edge-wiener index and wiener index of a power graph of Z_n.

Keywords

Wiener Index, Edge-Wiener Index, An Undirected Power Graph, Line Graph

1. Introduction

We define an undirected power graph $P(G)$ for a group G as follows. Let us denote the cylic subgroup genarated by $u \in G$ by $\langle u\rangle$, that is, $\langle u\rangle = \{u^m \mid m \in \mathbb{N}\}$, where $\mathbb{N}$ denotes the set of naturel numbers. The graph $P(G)$ is an undirected graph where vertex set is G and two vertices $u,v \in G$ are adjacent if and only if $u \neq v$ and $\langle u\rangle \subseteq \langle v\rangle$ or $\langle v\rangle \subseteq \langle v\rangle$ (which is equivalent to say $u \neq v$ and $u^m = v$ or $v^m = u$ for some positive integer m.) [1] [2] [3] [4].

For a graph G, let $\deg(u)$ and $d(u,v)$ denote the degree of a vertex $u \in V(G)$ and the distance between vertices $u,v \in V(G)$, respectively. Let $L(G)$ denote the line graph of G, that is, the graph with vertex set $E(G)$ and two distinct edges $e, f \in E(G)$ adjacent in $L(G)$ whenever they share an end-vertex in G. Furthermore, for, $f \in E(G)$, we let $d(e,f)$ denote the distance between e and f in the line graph $L(G)$.

We consider the power graph $P(Z_n)$ for the additive group Z_n of integers modulo n. The diameter of a graph G is the greatest distance between any pair of vertices, and denoted by $diam(G)$. In $P(Z_n)$, the distance is one if the vertices is adjacent and the distance is two if the vertices is non adjacent. Therefore, $diam(P(Z_n)) = 2$. The order an element $\overline{g}$ in Z_n is denoted by ($\overline{g}$) or $|g|$. For a positive integer n, $\phi(n)$ denotes the Euler's totient function of n.

In this paper, the wiener index and the edge-wiener index, denoted by $W(G)$ and $W_e(G)$, respectively and they are defined as follows:

$$W(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d(u,v)$$

$$W_e(G) = \frac{1}{2} \sum_{\{e,f\} \subseteq E(G)} d(e,f)$$

Now, we give some theorem and corollary in literature. Using our main theorems;

Theorem 1. ([5]) For each finite group, the number of edges of the undirected power graph $P(G)$ is given by the formula

$$E(P(G)) = \frac{1}{2} \sum_{g \in G} \{2o(g) - \phi(o(g)) - 1\}$$

Corollary 2. ([6]) The number of edges of the undirected power graph $P(Z_n)$ is given by $\frac{1}{2} \sum_{d|n} \{2d - \phi(d) - 1\} \phi(d)$.

Theorem 3. ([3]) Let G be connected graph with n vertices and m edges. If $diam(G) \le 2$, Then $W(G) = n(n-1) - m$.

Theorem 4. ([5]) A finite group has a complete undirected power graph if and only if it is cyclic and has order equal to p^k, where p is a prime and k is a nonnegative integer.

2. Main Results

In this section, our aim is to give our main results on the Wiener index and the edge-Wiener index of an undirected power graph $P(Z_n)$ for $n = p^k$, or $n = pq$, where p and q are distinct prime numbers and k is a nonnegative integer.

Theorem 5. Let $P(Z_n)$ be an undirected power graph of with n vertices and m edges. Then

$$W(P(Z_n)) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(P(Z_n))} \begin{cases} 1, & u \sim v \\ 2, & u \nsim v \end{cases}$$

Proof. Let

$R=\{\{u,v\}\subseteq V(P(Z_n)) \mid u\sim v \text{ if only if } u\neq v, \langle u\rangle\subseteq\langle v\rangle \text{ or } \langle v\rangle\subseteq\langle u\rangle\}$ be a set. In $P(Z_n)$, for $\{u,v\}\subseteq V(P(Z_n))$, there are two cases; If $u\nsim v$ then $d(u,v)=2$. Otherwise, *i.e.* $u\sim v$, then $d(u,v)=1$. Therefore

$$
\begin{aligned}
W(P(Z_n)) &= \frac{1}{2}\sum_{\{u,v\}\subseteq V(P(Z_n))} d(u,v)\\
&= \frac{1}{2}\left(\sum\nolimits_{\{u,v\}\subseteq R} d(u,v)+\sum\nolimits_{\{u,v\}\subsetneq R} d(u,v)\right)\\
&= \frac{1}{2}\sum\nolimits_{\{u,v\}\subseteq R}1+\frac{1}{2}\sum\nolimits_{\{u,v\}\subsetneq R}2\\
&= \frac{1}{2}\sum\nolimits_{\{u,v\}\subseteq V(P(Z_n))}\begin{cases}1, \{u,v\}\subseteq R\\ 2, \{u,v\}\subsetneq R\end{cases}
\end{aligned}
$$

For definition of *R*, we obtain. Thus

$$W(G)=\frac{1}{2}\sum_{\{u,v\}\subseteq V(P(Z_n))}\begin{cases}1, & u\sim v\\ 2, & u\nsim v\end{cases}$$

the proof is complete.

Corollary 6. Let *p* and *k* is prime number and nonnegative integer, respectively. For $P(Z_{p^k})$ power graph of order p^k and *m* edges,

$$W(P(Z_{p^k}))=\binom{p^k}{2}.$$

Proof. In [2], If $n=p^k$ then $P(Z_n)=K_n$. For any $u\in V(Z_{p^k})$, $d(u)=p^k-1$.

$$R^c=\{\{u,v\}\subseteq V(P(Z_n)) \mid u\nsim v\}=\varnothing$$

Thus

$$
\begin{aligned}
W(P(Z_{p^k})) &= \frac{1}{2}\sum_{\{u,v\}\subseteq V(P(Z_n))}\begin{cases}1, & u\sim v\\ 2, & u\nsim v\end{cases}\\
&= \frac{1}{2}\left(\sum_{\{u,v\}\subseteq R} d(u,v)+\sum_{\{u,v\}\subsetneq\varnothing} d(u,v)\right)\\
&= \frac{1}{2}\left(\sum_{\{u,v\}\subseteq R} 1+\sum_{\{u,v\}\subseteq\varnothing} d(u,v)\right)\\
&= \frac{1}{2}\sum_{\{u,v\}\subseteq R} 1=\frac{1}{2}p^k\left(p^k-1\right)=\binom{p^k}{2}
\end{aligned}
$$

Therefore the proof is proved.

Theorem 7. Let $P(Z_n)$ be a power graph of with *n* vertices and *m* edges. Then

$$W(P(Z_n))=\frac{1}{2}\left\{\binom{2n}{2}+\sum_{g=0}^{n-1}\left(\phi(|\bar{g}|)-2|\bar{g}|\right)\right\}$$

Proof. If we consider Theorem 3. for $=P(Z_n)$, we write

$$W(P(Z_n))=n(n-1)-m$$

$$m=\frac{1}{2}\sum_{g\in Z_n}\left\{2o(g)-\phi(o(g))-1\right\}.$$

If we put the value of m into the formula, we obtain

$$\begin{aligned}W(P(Z_n)) &= n(n-1)-m\\ &= n(n-1)-\frac{1}{2}\sum_{g\in Z_n}\left\{2o(g)-\phi(o(g))-1\right\}\\ &= n^2-n+\frac{1}{2}\sum_{g\in Z_n}\left\{\phi(o(g))-2o(g)\right\}-\frac{1}{2}\sum_{g\in Z_n}1\\ &= n^2-n+\frac{n}{2}+\frac{1}{2}\sum_{g\in Z_n}\left\{\phi(o(g))-2o(g)\right\}\\ &= \left\{n^2-\frac{n}{2}+\frac{1}{2}\sum_{g=0}^{n-1}\left(\phi(|\overline{g}|)-2|\overline{g}|\right)\right\}\end{aligned}$$

$$W(P(Z_n))=\frac{1}{2}\left\{\binom{2n}{2}+\sum_{g=0}^{n-1}\left(\phi(|\overline{g}|)-2|\overline{g}|\right)\right\}$$

Thus, the proof is complete.

Corollary 8. Let $P(Z_n)$ be a power graph of with $n=p$, where p is a prime number. Then

$$W(P(Z_n))=\binom{P}{2}.$$

Proof. Let $n=p$ be a prime number. Then

$$\begin{aligned}W(P(Z_p)) &= \frac{1}{2}\left\{\binom{2p}{2}+\sum_{g=0}^{p-1}\left(\phi(|\overline{g}|)-2|\overline{g}|\right)\right\}\\ &= \frac{1}{2}\left[\frac{2p(2p-1)}{2}+\phi(|\overline{0}|)+\phi(|\overline{1}|)+\cdots+\phi(|\overline{p-1}|)-2\left(|\overline{0}|+|\overline{1}|+\cdots+|\overline{p-1}|\right)\right]\\ &= \frac{1}{2}\left[2p^2-p-1+\left(\phi(|\overline{1}|)+\cdots+\phi(|\overline{p-1}|)\right)-2\left(|\overline{1}|+\cdots+|\overline{p-1}|\right)\right]\\ &= \frac{1}{2}\left[2p^2-p-1+(p-1)\phi(p)-2(p-1)p\right]\\ &= \frac{1}{2}\left[2p^2-p-1+(p-1)^2-2p^2+2p\right]=\binom{p}{2}\end{aligned}$$

Theorem 9. Let $P(Z_n)$ be a power graph of with n vertices and m edges. Then

$$W(P(Z_n))=\frac{1}{2}\left\{\binom{2n}{2}+\sum_{d|n}\phi(d)\left(\phi(d)-2d\right)\right\}.$$

Proof. Where $P(Z_n)$ is power graph $=P(Z_n)$, using theorem 3. And corollary 2, we obtain

$$W(P(Z_n))=n(n-1)-m$$

$$m=\frac{1}{2}\sum_{d|n}\left\{2d-\phi(d)-1\right\}\phi(d)$$

If we write this m in formula for $W\left(P\left(Z_n\right)\right)$

$$W\left(P\left(Z_n\right)\right)=n\left(n-1\right)-m$$
$$=n\left(n-1\right)-\frac{1}{2}\sum_{d|n}\left\{2d-\phi\left(d\right)-1\right\}\phi\left(d\right)$$
$$=n^2-n+\frac{1}{2}\sum_{d|n}\phi\left(d\right)^2+\frac{1}{2}\sum_{d|n}\phi\left(d\right)-\sum_{d|n}d\phi\left(d\right)$$
$$=n^2-\frac{n}{2}+\frac{1}{2}\sum_{d|n}\phi\left(d\right)\left(\phi\left(d\right)-2d\right)$$
$$W\left(P\left(Z_n\right)\right)=\frac{1}{2}\left\{\binom{2n}{2}+\sum_{d|n}\phi\left(d\right)\left(\phi\left(d\right)-2d\right)\right\}.$$

End of proof.

Corollary 10. Let $P\left(Z_n\right)$ be a power graph of with $n=pq$ vertices and m edges, where p and q are distinct prime numbers. Then

$$W\left(P\left(Z_{pq}\right)\right)=m+2\phi\left(pq\right)$$

or equiently

$$W\left(P\left(Z_{pq}\right)\right)=\binom{pq}{2}+\phi\left(pq\right).$$

Proof. If we write $n=pq$ in theorem 9., we obtain

$$W\left(P\left(Z_{pq}\right)\right)=\frac{1}{2}\left\{\binom{2pq}{2}+\sum_{d|pq}\phi\left(d\right)\left(\phi\left(d\right)-2d\right)\right\}$$
$$=\frac{1}{2}\left[pq\left(2\cdot pq-1\right)+\phi\left(1\right)\left(\phi\left(1\right)-2\cdot 1\right)+\phi\left(p\right)\left(\phi\left(p\right)-2\cdot p\right)\right.$$
$$\left.+\phi\left(q\right)\left(\phi\left(q\right)-2\cdot q\right)+\phi\left(pq\right)\left(\phi\left(pq\right)-2\cdot pq\right)\right]$$
$$=\frac{1}{2}\left[p^2q^2+pq-2\cdot p-2\cdot q+2\right] \qquad (*)$$
$$=\left[\frac{p^2q^2-pq}{2}+pq-p-q+1\right]$$
$$=\left[\binom{pq}{2}-\phi\left(pq\right)\right]+2\cdot\phi\left(pq\right)$$

On the other hand;

$$W\left(P\left(Z_{pq}\right)\right)=pq\left(pq-1\right)-m=\binom{pq}{2}+\phi\left(pq\right)$$

where

$$m=\binom{pq}{2}-\phi\left(pq\right) \qquad (**)$$

(**) equation put in (*) equation, we obtain,

$$W\left(P\left(Z_{pq}\right)\right)=m+2\phi\left(pq\right).$$

This completes the proof.

On the other hand using m in (**), we obtain

$$\begin{aligned} W\left(P\left(Z_{pq}\right)\right) &= m + 2\phi(pq) \\ &= \binom{pq}{2} - \phi(pq) + 2\phi(pq) \\ &= \binom{pq}{2} + \phi(pq) \end{aligned}$$

This completes the proof.

Theorem 11. If $P(Z_n)$ is a power graph of order $n = p^k$ or $n = pq$ and m edges, where p and q are distinct prime and k is a nonnegative integer. Then

$$maks\left\{W\left(P\left(Z_n\right)\right)\right\} = \binom{n+1}{2}$$

and

$$\min\left\{W\left(P\left(Z_n\right)\right)\right\} = \binom{n}{2}$$

Proof. If $n = p^k$ in Corollary 6.

$$W\left(P\left(Z_{p^k}\right)\right) = \binom{p^k}{2}.$$

And so

$$\min\left\{W\left(P\left(Z_n\right)\right)\right\} = \binom{n}{2}$$

And if $n = pq$ in Corollary 10.

$$W\left(P\left(Z_{pq}\right)\right) = \binom{pq}{2} + \phi(pq)$$

therefore

$$W\left(P\left(Z_n\right)\right) \le \binom{n}{2} + \phi(n).$$

Also

$$\phi(n) \le n .$$

We write

$$W\left(P\left(Z_n\right)\right) \le \binom{n}{2} + \phi(n) \le \binom{n}{2} + n .$$

And so,

$$maks\left\{W\left(P\left(Z_n\right)\right)\right\} = \binom{n+1}{2}.$$

Theorem 12. If $P(Z_n)$ is a power graph of order $n = p^k$ and m edges, where p is prime and k is a nonnegative integer. Then

$$W_e\left(P\left(Z_n\right)\right) = 3\left\{\binom{n}{3} + diam\left(L\left(P\left(Z_n\right)\right)\right)\binom{n}{4}\right\}.$$

Proof. For $P\left(Z_{p^k}\right)$ power graph, $E\left(P\left(Z_n\right)\right)=\binom{n}{2}$ and $\forall u\in V\left(P\left(Z_n\right)\right)$, $d\left(u\right)=n-1$.

Let's consider to this figure in $P\left(Z_{p^k}\right)$ power graph any $e_{\overline{n},\overline{n-1}}\in E\left(P\left(Z_{p^k}\right)\right)$. For $P\left(Z_{p^k}\right)$ power graph of Line graph as shown in **Figure 1**.

Choose the random $e_{\overline{n},\overline{n-1}}\in E\left(P\left(Z_{p^k}\right)\right)$ edge and this corner in neighborhood $L\left(P\left(Z_n\right)\right)$ line graph in **Figure 2**. In the same way, with $e_{\overline{n},\overline{n-1}}\in V\left(L\left(P\left(Z_{p^k}\right)\right)\right)$ point neighborhood amount of points $2\left(n-2\right)$. In the same way $e_{\overline{n},\overline{n-1}}$ neighborhood with corner amount of point $m-1-2\left(n-2\right)$ and therefore $V\left(L\left(P\left(Z_{p^k}\right)\right)\right)$ if each elements for calculated and if edge-Wiener index identified we have the following result.

In edge-Wiener index

$$W_e\left(P\left(Z_{p^k}\right)\right)=\frac{1}{2}\sum\nolimits_{\{e,f\}\subseteq E\left(P\left(Z_n\right)\right)}d\left(e,f\right)$$

$$=\frac{1}{2}\left\{\sum_{uv=e}\left[\left(d\left(u\right)+d\left(v\right)-2\right)\right]\right.$$

$$\left.+\sum_{uv=e}\left[diam\left(L\left(P\left(Z_n\right)\right)\right)\cdot\left(\left(m-1\right)-\left(d\left(u\right)+d\left(v\right)-2\right)\right)\right]\right\}$$

$$=\frac{1}{2}\left\{\binom{n}{2}\left[2\left(n-2\right)+diam\left(L\left(P\left(Z_n\right)\right)\right)\left(\binom{n}{2}-1-2\left(n-2\right)\right)\right]\right\}$$

$$=\left[\frac{n\left(n-1\right)\left(n-2\right)}{2}+\frac{n\left(n-1\right)}{4}diam\left(L\left(P\left(Z_n\right)\right)\right)\left(\frac{n^2-5n-6}{2}\right)\right]$$

$$=3\binom{n}{3}+\frac{n\left(n-1\right)\left(n-2\right)\left(n-3\right)}{8}diam\left(L\left(P\left(Z_n\right)\right)\right)$$

$$W_e\left(P\left(Z_{p^k}\right)\right)=3\left[\binom{n}{3}+diam\left(L\left(P\left(Z_n\right)\right)\right)\binom{n}{4}\right]$$

Concluded, namely the prove end.

Theorem 13. If $P\left(Z_n\right)$ is a power graph of order $n=p^k$ and m edges, where p is prime and k is a nonnegative integer. Then

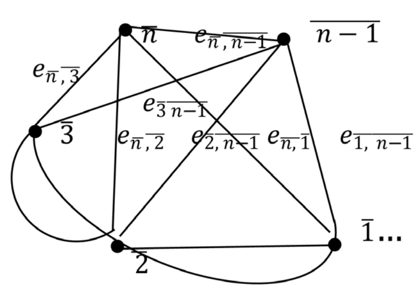

Figure 1. Power grap of Z_{p^k}.

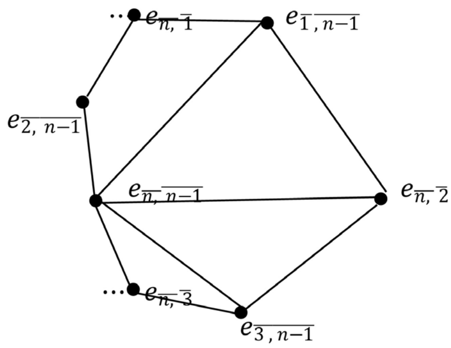

Figure 2. Line graph of $P(Z_n)$.

$$W_e(P(Z_n)) = \binom{n-1}{2} W(P(Z_n))$$

Proof. $n = p^k$ $(\in Z^+)$ is in $W(P(Z_n)) = \binom{n}{2}$. In the same way,

Case 1. for $n = 2,3$ and according to $diam(L(P(Z_n))) = 1$, $W_e(P(Z_2)) = 0$, therefore $W_e(P(Z_3)) = W(P(Z_3))$ ve $\binom{3-1}{2} = 1$, namely this equation the proof.

Case 2. For $n \neq 2,3$ is $diam(L(P(Z_n))) = 2$ in theorem 12.,

$$\begin{aligned} W_e(P(Z_n)) &= 3\left[\binom{n}{3} + diam(L(P(Z_n)))\binom{n}{4}\right] \\ &= 3\left[\binom{n}{3} + 2\binom{n}{4}\right] \\ &= \frac{1}{2}n(n-1)\left[(n-2) + \left(\frac{(n-2)(n-3)}{2}\right)\right] \\ &= \binom{n}{2}(n-2)\left[1 + \frac{n-3}{2}\right] \\ &= \binom{n-1}{2} W(P(Z_n)) \end{aligned}$$

Thus the proof is completed.

3. Conclusion

We will show the undirected power graph of a Group G with $P(G)$. Here, the undirected $P(Z_n)$ Power graph of the group $(Z_n, +)$ according to $N = p^k$ and $n = pq$, with p, q being different primes and k being positive integers, is considered and new theorems and results on the Wiener index calculations of these power graphs with the help of Euler function are have been obtained.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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