American Journal of Computational Mathematics, 2021, 11, 53-63
https://www.scirp.org/journal/ajcm
ISSN Online: 2161-1211
ISSN Print: 2161-1203

Convergence and Superconvergence of Fully Discrete Finite Element for Time Fractional Optimal Control Problems

Yuelong Tang

College of Science, Hunan University of Science and Engineering, Yongzhou, China
Email: tangyuelonga@163.com

How to cite this paper: Tang, Y.L. (2021) Convergence and Superconvergence of Fully Discrete Finite Element for Time Fractional Optimal Control Problems. *American Journal of Computational Mathematics*, **11**, 53-63.

https://doi.org/10.4236/ajcm.2021.111005

Received: February 10, 2021
Accepted: March 22, 2021
Published: March 25, 2021

Abstract

In this paper, we consider a fully discrete finite element approximation for time fractional optimal control problems. The state and adjoint state are approximated by triangular linear finite elements in space and $L1$ scheme in time. The control is obtained by the variational discretization technique. The main purpose of this work is to derive the convergence and superconvergence. A numerical example is presented to validate our theoretical results.

Keywords

Time Fractional Optimal Control Problems, Finite Element, Convergence and Superconvergence

1. Introduction

There has been a great deal of research on finite element methods (FEMs) for partial differential equations (PDEs) or optimal control problems (OCPs) governed by PDEs, mostly focused on the case of integer order such as elliptic, parabolic and hyperbolic equations. Systematic introductions can be found in [1–3], and so on.

In the past two decades, numerical methods of fractional partial differential equations (FPDEs) have attracted much attention, since different FPDEs arise in various physical phenomena or processes with memory and hereditary. Many authors have investigated finite difference methods [4, 5], spectral methods [6–8], mixed FEMs [9, 10] and FEMs [11–13] to solve FPDEs. However, there are relatively few researches on numerical methods for OCPs governed by fractional PDEs.

There are some published papers on FEMs for fractional OCPs. A finite element approximation algorithm of optimal control problems governed by time fractional diffusion equation is presented in [14], but the authors do not provide error analysis results. In [15], a fast gradient projection method is investigated for a constrained fractional optimal control. Finite element approximation of space fractional optimal control problem is considered in [16]. In [17], a priori error estimates of time-stepping discontinuous Galerkin finite element approximation for time fractional optimal control problem is established. A fully discrete finite element approximation of optimal control problem governed by a time-fractional PDE is investigated in [18], where the time fractional derivative operates on the diffusion term. However, to our best of knowledge, superconvergence of finite element approximation for time fractional OCPs is rare. The main purpose of this work is to derive the convergence and superconvergence of fully discrete FEM for time fractional OCPs.

Let $J = [0, T]$ and $\Omega \subset \mathbf{R}^2$ be a bounded convex polygonal domain with smooth boundary $\partial\Omega$. We consider the following time fractional OCPs:

$$\begin{cases} \min\limits_{u \in U_{ad}} \dfrac{1}{2} \displaystyle\int_0^T \left(\|y(t,x) - y_d(t,x)\|^2_{L^2(\Omega)} + \lambda \|u(t,x)\|^2_{L^2(\Omega)} \right) dt, \\ {}_0\partial_t^\alpha y(t,x) - \Delta y(t,x) = f(t,x) + u(t,x), \quad t \in J,\, x \in \Omega, \\ y(t,x) = 0, \qquad t \in J,\, x \in \partial\Omega, \\ y(0,x) = 0, \qquad x \in \Omega. \end{cases} \tag{1}$$

Here ${}_0\partial_t^\alpha$ $(0 < \alpha < 1)$ denotes the α-order left Caputo derivative with respect to the time variable t and defined by

$${}_0\partial_t^\alpha y(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \frac{\partial y(s,x)}{\partial s} ds. \tag{2}$$

$y_d(t,x), f(t,x) \in C(J; L^2(\Omega))$ are given functions and the admissible set U_{ad} is defined by

$$U_{ad} = \{\ v(t,x) \in L^2(J; L^2(\Omega)) : v(t,x) \geq 0,\ \text{a.e. in } \Omega \times J\}. \tag{3}$$

Throughout the paper, $L^s(J; W^{m,q}(\Omega))$ denotes all L^s integrable functions from J into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,q}(\Omega))} = \left(\int_0^T \|v\|^s_{W^{m,q}(\Omega)} dt \right)^{\frac{1}{s}}$ for $s \in [1, \infty)$ and the standard modification for $s = \infty$, where $W^{m,q}(\Omega)$ is Sobolev spaces on Ω. Similarly, one can define $H^l(J; W^{m,q}(\Omega))$ and $C^k(J; W^{m,q}(\Omega))$ (see e.g. [19]). In addition, c or C is a generic positive constant.

The rest of this paper is organized as follows. In Section 2, we introduce a fully discrete approximation scheme of (1). In Section 3, we provide a convergence result of the control variable. In Section 4, we derive the superconvergence result of the state and adjoint state variables. Some numerical examples are provided in the last section to verify our theoretical results.

2. Finite Element Approximation of Time Fractional OPCs

For the state and co-state, we introduce triangular linear finite element for the spatial discretization and $L1$ scheme for the time discretization. The control is obtained by variational discretization (VD) technique [20].

For brevity, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and drop Ω or J whenever possible, *i.e.*,

$$\|\cdot\|_{m,2,\Omega} = \|\cdot\|_{m,2} = \|\cdot\|_m, \quad \|\cdot\|_0 = \|\cdot\|,$$
$$\|\cdot\|_{L^p(J;W^{m,2}(\Omega))} = \|\cdot\|_{L^p(H^m)}, \quad \|\cdot\|_{L^p(J;W^{m,q}(\Omega))} = \|\cdot\|_{L^p(W^{m,q})},$$
$$K = \left\{\ v(x) \in L^2(\Omega) : v(x) \geq 0,\ \text{a.e. in } \Omega\ \right\}.$$

Furthermore, we set $H_0^1(\Omega) \equiv \left\{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\right\}$, $W = H_0^1(\Omega)$, $U = L^2(\Omega)$. In addition,

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w, \qquad \forall\, v,\, w \in W,$$
$$(f_1, f_2) = \int_\Omega f_1 \cdot f_2, \qquad \forall\, f_1,\, f_2 \in U.$$

By using the variational principle [2], we recast (1) as the following weak formulation:

$$\begin{cases} \min\limits_{u \in U_{ad}} \dfrac{1}{2} \displaystyle\int_0^T \left(\|y - y_d\|^2 + \lambda \|u\|^2\right) dt, \\ ({}_0\partial_t^\alpha y, w) + a(y, w) = (f, w) + (u, w), \qquad t \in J,\, w \in W, \\ y(0, x) = 0, \qquad x \in \Omega. \end{cases} \tag{4}$$

It follows from (see e.g., [21]) that (4) has a unique solution (y, u), and that a pair (y, u) is a solution of (4), then there is a co-state p such that the triplet (y, p, u) fulfills the following optimality conditions:

$$({}_0\partial_t^\alpha y, w) + a(y, w) =(f, w) + (u, w), \qquad \forall\, w \in W, t \in J, \tag{5}$$
$$y(0, x) =0, \qquad x \in \Omega, \tag{6}$$
$$({}_t\partial_T^\alpha p, q) + a(q, p) =(y - y_d, q), \qquad \forall\, q \in W, t \in J, \tag{7}$$
$$p(x, T) =0, \qquad x \in \Omega, \tag{8}$$
$$(\lambda u + p, v - u) \geq 0, \qquad \forall\, v \in K, t \in J. \tag{9}$$

Here ${}_t\partial_T^\alpha$ denotes the α-order right Caputo derivative with respect to t and defined by

$${}_t\partial_T^\alpha p(t, x) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{1}{(t-s)^\alpha} \frac{\partial p(s, x)}{\partial s} ds.$$

Like in [22] , it is easy to prove that the inequality (9) is equivalent to

$$u = \max(0, -\frac{p}{\lambda}). \tag{10}$$

Let $\mathcal{T}^h$ be a family of quasi-uniform triangulations of Ω, such that $\bar{\Omega} = \bigcup\limits_{e \in \mathcal{T}^h} \bar{e}$ and $h = \max\limits_{\tau \in \mathcal{T}^h}\{h_e\}$, where h_e is the diameter of the element e. Furthermore, we set

$$W^h = \left\{\ v_h \in C(\bar{\Omega}) : v_h|_e \in \mathbb{P}_1,\ \forall\, e \in \mathcal{T}^h, v_h|_{\partial\Omega} = 0\ \right\},$$

where $\mathbb{P}_1$ is the space of polynomials whose degree at most 1.

Let P_h be the elliptic projection, defined as follows. For any $v \in W$,

$$a(v - P_h v, w_h) = 0, \qquad \forall w_h \in W^h.$$

It has the following error estimates (see, [12]):

$$\|v - P_h v\| + h\|\nabla(v - P_h v)\| \leq Ch^2 \|v\|_2. \tag{11}$$

In the next, we will consider $L1$ scheme for the time discretization.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a given uniform partition of $[0, T]$ with size $\tau = \frac{T}{N}$ and $t_n = n\tau, n = 0, 1, \cdots, N$. We set $\varphi^n = \varphi(t_n, x)$ and define the time-dependent maximum norm

$$\|v\|_{l^\infty(W^{m,q})} = \max_{0\le n\le N} \{\|v^n\|_{W^{m,q}}\}.$$

Then α-order left Caputo derivative can be approximated as follows

$$_0\partial_t^\alpha y^n = \frac{1}{\tau^\alpha\Gamma(2-\alpha)} \sum_{k=0}^{n} b_k^n y^k + r_{y,\tau}^n := F_t^\alpha y^n + r_{y,\tau}^n, \tag{12}$$

where $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$, $b_0^n = (n-1)^{1-\alpha} - n^{1-\alpha}$, $b_k^n = b_{n-k} - b_{n-k-1}$, $b_n^n = 1$ and $r_{y,\tau}^n$ is the truncation error. It follows from [5], if $y \in W^{2,\infty}(L^2)$, we have

$$\left|r_{y,\tau}^n\right| = |_0\partial_t^\alpha y^n - F_t^\alpha y^n| \le C\tau^{2-\alpha}. \tag{13}$$

Similarly, the α-order right Caputo derivative can be approximated as follows

$$_t\partial_T^\alpha p^n = \frac{1}{\tau^\alpha\Gamma(2-\alpha)} \sum_{k=n}^{N} \tilde{b}_k^n p^k + \tilde{r}_{p,\tau}^n := B_t^\alpha p^n + \tilde{r}_{p,\tau}^n, \tag{14}$$

where $\tilde{b}_n^n = -1$, $\tilde{b}_N^n = b_{N-n-1}$, $\tilde{b}_k^n = b_{k-n} - b_{k-n-1}$, $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$. If $p \in W^{2,\infty}(L^2)$, then

$$\left|\tilde{r}_{p,\tau}^n\right| = |_t\partial_T^\alpha p^n - B_t^\alpha p^n| \le C\tau^{2-\alpha}. \tag{15}$$

Then a fully discrete finite element approximation scheme of optimality conditions (5)-(9) writes as

$$(F_t^\alpha y_h^n, w_h) + a(y_h^n, w_h) = (f^n, w_h) + (u_h^n, w_h), \forall w_h \in W^h, \tag{16}$$

$$n = 1, 2, \cdots, N, y_h^0 = 0, \tag{17}$$

$$(B_t^\alpha p_h^n, q_h) + a\left(q_h, p_h^{n-1}\right) = (y_h^n - y_d^n, q_h), \qquad \forall q_h \in W^h, \tag{18}$$

$$n = N-1, \cdots, 1, 0, p_h^N = 0, \tag{19}$$

$$\left(\lambda u_h^n + p_h^{n-1}, v - u_h^n\right) \ge 0, \qquad \forall v \in K, n = 1, 2, \cdots, N. \tag{20}$$

As in reference [20], the inequality (20) is equivalent to

$$u_h^n = \max(0, -\frac{1}{\lambda} p_h^{n-1}), \qquad n = 1, 2, \cdots, N. \tag{21}$$

3. Convergence Analysis

In this section, we derive the convergence of the control for the approximation scheme (16)-(20). For ease of exposition, we set $\alpha_\tau = \tau^\alpha\Gamma(2-\alpha)$. We introduce some useful intermediate variables. Let $\left(y_h^n(u), p_h^{n-1}(u)\right) \in W^h \times W^h, n = 1, 2, \cdots, N$ fulfill the following equations:

$$(F_t^\alpha y_h^n(u), w_h) + a(y_h^n(u), w_h) = (f^n, w_h) + (u^n, w_h), \forall w_h \in W^h, \tag{22}$$

$$n = 1, 2, \cdots, N, y_h^0(u) = 0, \tag{23}$$

$$(B_t^\alpha p_h^n(u), q_h) + a\left(q_h, p_h^{n-1}(u)\right) = (y_h^n(u) - y_d^n, q_h), \quad \forall q_h \in W^h, \tag{24}$$

$$n = N-1, \cdots, 1, 0, p_h^N(u) = 0. \tag{25}$$

The following conclusions will be used in the following convergence analysis.

Lemma 1. [10] Let $\{\xi^n\}_{n=0}^N$ be a sequence of functions on Ω. Then

$$\left(\xi^n, \sum_{k=0}^{n} b_k^n \xi^k\right) = \frac{1}{2}\left(\|\xi^n\|^2 + \sum_{k=0}^{n-1} b_k^n \|\xi^k\|^2 - \sum_{k=0}^{n-1} b_k^n \|\xi^k - \xi^n\|^2\right).$$

Lemma 2. [23] Let $\varepsilon^k \geq 0$, satisfy $\varepsilon^n \leq \sum_{k=1}^{n-1}(b_{k-1} - b_k)\varepsilon^{n-k} + \gamma$ with $\gamma > 0$, $k = 1, 2, \cdots, L$. Then

$$\varepsilon^n \leq C\tau^{-\alpha}\gamma, \quad n = 1, 2, \cdots, L.$$

Lemma 3. Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (5)-(9) and (16)-(20), respectively. Suppose that $y, p \in W^{2,\infty}(L^2) \cap W^{1,\infty}(H^2)$. Then

$$\|y^n - y_h^n(u)\| + \|p^n - p_h^n(u)\| \leq C(h^2 + \tau^{2-\alpha}), \tag{26}$$

$$\|y^n - y_h^n(u)\|_1 + \|p^n - p_h^n(u)\|_1 \leq C(h + \tau^{2-\alpha}). \tag{27}$$

Proof. Set $y^n - y_h^n(u) = y^n - P_h y^n + P_h y^n - y_h^n(u) := \eta^n + \zeta^n$. Choosing $v = w_h$ in (5) and subtracting (22) from (5), we obtain error equation

$$\left(F_t^\alpha\left(y^n - y_h^n(u)\right), w_h\right) + a\left(y^n - y_h^n(u), w_h\right) + \left(r_{y,\tau}^n, w_h\right) = 0. \tag{28}$$

By the definition of P_h and (28), we have

$$\left(F_t^\alpha \zeta^n, w_h\right) + a\left(\zeta^n, w_h\right) = -\left(F_t^\alpha \eta^n, w_h\right) - \left(r_{y,\tau}^n, w_h\right). \tag{29}$$

Let $w_h = F_t^\alpha \zeta^n$ and use Lemma 1, we get

$$\begin{aligned} & a\left(\zeta^n, F_t^\alpha \zeta^n\right) \\ =& \frac{1}{2\alpha_\tau}\left(\|\nabla\zeta^n\|^2 + \sum_{k=0}^{n-1} b_k^n \|\nabla\zeta^k\|^2 - \sum_{k=0}^{n-1} b_k^n \|\nabla\left(\zeta^k - \zeta^n\right)\|^2\right). \end{aligned} \tag{30}$$

From $y \in W^{1,\infty}(H^2)$ and (11), there holds

$$\begin{aligned} \|F_t^\alpha \eta^n\| =& \left\|\frac{1}{\tau^{\alpha-1}\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \partial_t \eta^{n-k}\right\| \\ \leq& \frac{1}{\tau^{\alpha-1}\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \|\partial_t \eta^{n-k}\| \\ \leq& \frac{1}{\alpha_\tau} \sum_{k=0}^{n-1} b_k \int_{t_{n-k-1}}^{t_{n-k}} \|\eta_t\| dt \\ \leq& Ch^2 \|y_t\|_{L^\infty(H^2)}. \end{aligned} \tag{31}$$

Then, by applying Hölder's inequality $\|f \cdot g\|_{L^1(\Omega)} \leq \|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}$, Young's inequality and (31), we arrive at

$$\left(F_t^\alpha \eta^n, F_t^\alpha \zeta^n\right) \leq \|F_t^\alpha \eta^n\| \|F_t^\alpha \zeta^n\| \leq \frac{1}{2}\|F_t^\alpha \zeta^n\|^2 + \frac{C}{2}h^4. \tag{32}$$

Likewise

$$\left(r_{y,\tau}^n, F_t^\alpha \zeta^n\right) \leq \|r_{y,\tau}^n\| \|F_t^\alpha \zeta^n\| \leq \frac{1}{2}\|F_t^\alpha \zeta^n\|^2 + \frac{C}{2}\tau^{4-2\alpha}. \tag{33}$$

Combining (29)-(33) and notice that $b_k^n < 0\,(0 \leq k < n)$, we obtain

$$\|\nabla\zeta^n\|^2 \leq -\sum_{k=0}^{n-1} b_k^n \|\nabla\zeta^k\|^2 + C\tau^\alpha (h^2 + \tau^{2-\alpha})^2. \tag{34}$$

It follows from (34), Lemma 2 and Poincaré's inequality that

$$\|\zeta^n\|_1 \leq C\|\nabla\zeta^n\| \leq C(h^2+\tau^{2-\alpha}). \tag{35}$$

In the same way, we can derive

$$\|P_hp^n - p_h^n(u)\|_1 \leq C\|\nabla(P_hp^n - p_h^n(u))\| \leq C(h^2+\tau^{2-\alpha}). \tag{36}$$

Hence, (26) and (27) follow from embedding theorem, triangle inequality, (11) and (35)-(36). □

Theorem 4. Let (y,p,u) and (y_h,p_h,u_h) be the solutions of (5)-(9) and (16)-(20), respectively. Assume all the conditions in Lemma 3 are true. Then

$$\|u^n - u_h^n\| \leq C\left(h^2+\tau\right). \tag{37}$$

Proof. Set $v = u_h^n$ and $v = u^n$ in (9) and (20) respectively, we have

$$\begin{aligned}
&\lambda\|u^n - u_h^n\|^2 \\
&= \lambda\left(u^n - u_h^n, u^n - u_h^n\right) \\
&\leq \left(p_h^{n-1} - p_h^{n-1}(u), u^n - u_h^n\right) + \left(p_h^{n-1}(u) - p^{n-1}, u^n - u_h^n\right) \\
&\quad + \left(p^{n-1} - p^n, u^n - u_h^n\right) \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{38}$$

For the first term I_1, it follows from (16)-(19) and (22)-(25) that

$$\begin{aligned}
I_1 &= \left(p_h^{n-1} - p_h^{n-1}(u), u^n - u_h^n\right) \\
&= \left(F_t^\alpha(y_h^n - y_h^n(u)), p_h^{n-1}(u) - p_h^{n-1}\right) \\
&\quad - \left(B_t^\alpha(p_h^n(u) - p_h^n), y_h^n - y_h^n(u)\right) \\
&\quad - \left(y_h^n - y_h^n(u), y_h^n - y_h^n(u)\right) \\
&\leq - \|y_h^n - y_h^n(u)\|^2.
\end{aligned} \tag{39}$$

According to Hölder's inequality and Young's inequality, we get

$$\begin{aligned}
I_2 &= \left(p_h^{n-1}(u) - p^{n-1}, u^n - u_h^n\right) \\
&\leq \frac{1}{2}\|p_h^{n-1}(u) - p^{n-1}\|^2 + \frac{1}{2}\|u^n - u_h^n\|^2.
\end{aligned} \tag{40}$$

For the last term, we get

$$\begin{aligned}
I_3 &= \left(p^{n-1} - p^n, u^n - u_h^n\right) \\
&\leq \frac{1}{2}\|p^{n-1} - p^n\|^2 + \frac{1}{2}\|u^n - u_h^n\|^2 \\
&\leq \frac{1}{2}\tau^2\|p_t\|^2_{L^2(L^2)} + \frac{1}{2}\|u^n - u_h^n\|^2.
\end{aligned} \tag{41}$$

From (26) and (38)-(41), we obtain (37). □

4. Superconvergence Analysis

In this section, we derive the superconvergence of the state and adjoint state variables.

Theorem 5. Let (y,p,u) and (y_h,p_h,u_h) be the solutions of (5)-(9) and (16)-(20), respectively. Assume all the conditions in Theorem 4 are true. Then

$$\|P_hy^n - y_h^n\|_1 + \|P_hp^n - p_h^n\|_1 \leq C\left(h^2+\tau\right). \tag{42}$$

Proof. Let $P_h y^n - y_h^n = \theta^n$. From (5), (16) and the definition of P_h, for $n = 1, 2, \cdots, N$ and $\forall\, w_h \in W^h$, we get

$$\begin{aligned} &\left(F_t^\alpha \theta^n, w_h\right) + a\left(\theta^n, w_h\right) \\ &= -\left(F_t^\alpha \eta^n, w_h\right) + \left(u^n - u_h^n, w_h\right) - \left(r_{y,\tau}^n, w_h\right). \end{aligned} \tag{43}$$

According Lemma 1, we have

$$\left(F_t^\alpha \theta^n, \theta^n\right) = \frac{1}{2\alpha_\tau}\left(\|\theta^n\|^2 + \sum_{k=0}^{n-1} b_k^n \|\theta^k\|^2 - \sum_{k=0}^{n-1} b_k^n \|\theta^k - \theta^n\|^2\right). \tag{44}$$

Note that

$$a(\theta^n, \theta^n) \ge c\|\theta^n\|_1^2. \tag{45}$$

Set $w_h = \theta^n$ in (43), then from (44)-(45), Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned} &\|\theta^n\|^2 + 2\alpha_\tau c\|\theta^n\|_1^2 \\ &\le -\sum_{k=0}^{n-1} b_k^n \|\theta^k\|^2 - 2\alpha_\tau\left(F_t^\alpha \eta^n - (u^n - u_h^n) + r_{y,\tau}^n, \theta^n\right) \\ &\le -\sum_{k=0}^{n-1} b_k^n \|\theta^k\|^2 + 2\alpha_\tau\left(\|F_t^\alpha \eta^n\| + \|u^n - u_h^n\| + \|r_{y,\tau}^n\|\right)\|\theta^n\| \\ &\le -\sum_{k=0}^{n-1} b_k^n \|\theta^k\|^2 + \alpha_\tau\left(\|F_t^\alpha \eta^n\|^2 + \|u^n - u_h^n\|^2 + \|r_{y,\tau}^n\|^2 + \|\theta^n\|^2\right). \end{aligned} \tag{46}$$

From (13), (31), Lemma 2, Theorem 4 and embedding theorem, we derive

$$\|P_h y^n - y_h^n\|_1 \le C\left(h^2 + \tau\right). \tag{47}$$

Similarly, we can get

$$\|P_h p^n - p_h^n\|_1 \le C\left(h^2 + \tau\right). \tag{48}$$

Then (42) follows from (47)-(48). □

5. Numerical Experiments

The optimal control problem was dealt numerically with codes developed based on AFEPack. The package is freely available and the details can be found at [24]. We solve the following time fractional optimal control problem:

$$\begin{aligned} &\min_{u \in U_{ad}} \frac{1}{2}\int_0^T \left(\int_\Omega (y(t,x) - y_d(t,x))^2 + \lambda \int_\Omega (u(t,x))^2\right) dt, \\ &{}_0\partial_t^\alpha y(t,x) - \Delta y(t,x) = f(t,x) + u(t,x), \quad \text{in}\, \Omega \times (0,T], \\ &y(t,x) = 0, \quad \text{on}\, \partial\Omega \times (0,T], \\ &y(x,0) = y_0(x), \quad \text{in}\, \Omega. \end{aligned}$$

The discretization was described in Section 2. For ease of exposition, we denote $\|u - h_h\|_{l^\infty(L^2)}$, $\|P_h y - y_h\|_{l^\infty(H^1)}$ and $\|P_h p - p_h\|_{l^\infty(H^1)}$

by $|||e_u|||$, $|||e_y|||_1$ and $|||e_p|||_1$, respectively. The spatial and time convergence rates are computed by the following formula:

$$R_h = \frac{\ln(e_{i+1}) - \ln(e_i)}{\ln(h_{i+1}) - \ln(h_i)} \text{ and } R_\tau = \frac{\ln(e_{i+1}) - \ln(e_i)}{\ln(\tau_{i+1}) - \ln(\tau_i)},$$

where e_i (e_{i+1}) denotes the error when the spatial partition size is h_i (h_{i+1}) or the time step size is $\tau_i(\tau_{i+1})$.

Example 1 Let $\Omega = [0,1] \times [0,1]$, $T = 1$, $\lambda = 1$. The data are as follows:

$$\begin{aligned} &y(t,x) = \sin(2\pi x_1)\sin(2\pi x_2)t, \\ &p(t,x) = \sin(2\pi x_1)\sin(2\pi x_2)(1-t), \\ &u(t,x) = \max(0, -p(t,x)), \\ &f(t,x) = {}_0\partial_t^\alpha y(t,x) - \Delta y(t,x) - u(t,x), \\ &y_d(t,x) = y(t,x) - {}_t\partial_T^\alpha p(t,x) - \Delta p(t,x). \end{aligned}$$

For different α, the errors $|||e_u|||$, $|||e_y|||_1$ and $|||e_p|||_1$ are shown in Tables 1-3, where $h_{i+1} = h_i/2$ and $\tau_{i+1} = \tau_i/4$.

Table 1. The convergence rate for Example 1 with $\alpha = 0.05$.

h	τ	$\|\|\|e_u\|\|\|$	R_h	R_τ	$\|\|\|e_y\|\|\|_1$	R_h	R_τ	$\|\|\|e_p\|\|\|_1$	R_h	R_τ
$\frac{1}{10}$	$\frac{1}{10}$	4.6742e−2	-	-	8.2685e−3	-	-	7.4163e−3	-	-
$\frac{1}{20}$	$\frac{1}{40}$	1.3123e−2	1.83	0.92	2.3021e−3	1.84	0.92	2.0014e−3	1.90	0.94
$\frac{1}{40}$	$\frac{1}{160}$	3.3262e−3	1.98	0.99	5.9132e−4	1.96	0.98	5.1089e−4	1.97	0.99
$\frac{1}{80}$	$\frac{1}{640}$	8.4136e−4	1.98	0.99	1.4876e−4	1.99	1.00	1.2846e−4	1.99	1.00

Table 2. The convergence rate for Example 1 with $\alpha = 0.5$.

h	τ	$\|\|\|e_u\|\|\|$	R_h	R_τ	$\|\|\|e_y\|\|\|_1$	R_h	R_τ	$\|\|\|e_p\|\|\|_1$	R_h	R_τ
$\frac{1}{10}$	$\frac{1}{10}$	2.1405e−2	-	-	6.0628e−3	-	-	6.3384e−3	-	-
$\frac{1}{20}$	$\frac{1}{40}$	5.3908e−3	1.99	0.99	1.5228e−3	1.99	1.00	1.5903e−3	1.99	1.00
$\frac{1}{40}$	$\frac{1}{160}$	1.3485e−3	2.00	1.00	3.8158e−4	2.00	1.00	3.9828e−4	2.00	1.00
$\frac{1}{80}$	$\frac{1}{640}$	3.3385e−4	2.01	1.01	9.5292e−5	2.00	1.00	9.7726e−5	2.03	1.01

Table 3. The convergence rate for Example 1 with $\alpha = 0.95$.

h	τ	$\|\|\|e_u\|\|\|$	R_h	R_τ	$\|\|\|e_y\|\|\|_1$	R_h	R_τ	$\|\|\|e_p\|\|\|_1$	R_h	R_τ
$\frac{1}{10}$	$\frac{1}{10}$	4.5959e−2	-	-	7.2163e−3	-	-	9.5103e−3	-	-
$\frac{1}{20}$	$\frac{1}{40}$	1.2301e−2	1.90	0.95	1.7037e−3	2.08	1.04	2.2462e−3	2.08	1.04
$\frac{1}{40}$	$\frac{1}{160}$	3.1189e−3	1.97	0.99	4.2247e−4	2.01	1.01	5.5526e−4	2.01	1.01
$\frac{1}{80}$	$\frac{1}{640}$	7.8113e−4	1.99	1.00	1.0548e−4	2.00	1.00	1.3847e−4	2.00	1.00

From numerical results in Tables 1-3, it is clear that $|||u-u_h|||=\mathcal{O}(h^2+\tau)$, $|||P_h y-y_h|||_1=\mathcal{O}(h^2+\tau)$ and $|||P_h p-p_h|||_1=\mathcal{O}(h^2+\tau)$, which are consistent with our theoretical analysis.

6. Conclusion

In this paper, we establish the convergence and superconvergence results of a fully discrete finite element approximation for time fractional optimal control problems. Numerical experiment results verify the correctness of our theoretical results.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (11401201), the Natural Science Foundation of Hunan Province (2020JJ4323), the Scientific Research Project of Hunan Provincial Department of Education (20A211, 20C0854), the scientific research program in Hunan University of Science and Engineering (20XKY059), the construct program of applied characteristic discipline in Hunan University of Science and Engineering.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Ciarlet, P. (1978) The Finite Element Method for Elliptic Problems. Amstterdam, North-Holland.

[2] Liu, W. and Yan, N. (2008) Adaptive Finite Element Methods for Optimal Control Governed by PDEs. Science Press, Beijing.

[3] Hinze, M., Pinnau, R., Ulbrich, M. and Ulbrich S. (2009) Optimization with PDE Constraints. Springer, New York.

[4] Gao, G. and Sun, Z. (2011) A Compact Finite Difference Scheme for the Fractional Sub-Diffusion Equations. *Journal of Computational Physics*, **230**, 586-595.
https://doi.org/10.1016/j.jcp.2010.10.007

[5] Lin, Y. and Xu, C. (2007) Finite Difference/Spectral Approximation for the Time-Fractional Diffusion Equation. *Journal of Computational Physics*, **225**, 1533-1552.
https://doi.org/10.1016/j.jcp.2007.02.001

[6] Li, X. and Xu, C. (2009) A Space-Time Spectral Method for the Time Fractional Diffusion Equation. *SIAM Journal on Numerical Analysis*, **47**, 2108-2131. https://doi.org/10.1137/080718942

[7] Mao, Z. and Shen, J. (2017) Hermite Spectral Methods for Fractional PDEs in Unbounded Domains. *SIAM Journal on Scientific Computing*, **39**, A1928-A1950.
https://doi.org/10.1137/16M1097109

[8] Zheng, M., Liu, F., Anh, V. and Turner, I. (2016) A High-Order Spectral Method for the Multi-Term Time-Fractional Diffusion

Equations. *Applied Mathematical Modelling*, **40**, 4970-4985. https://doi.org/10.1016/j.apm.2015.12.011

[9] Shi, Z., Zhao, Y., Liu, F., Tang, Y., *et al.* (2017) High Accuracy Analysis of an H^1-Galerkin Mixed Finite Element Method for Two-Dimensional Time Fractional Diffusion Equations. *Computers and Mathematics with Applications*, **74**, 1903-1914. https://doi.org/10.1016/j.camwa.2017.06.057

[10] Zhao, Y., Chen, P., Bu, W., *et al.* (2017) Two Mixed Finite Element Methods for Time-Fractional Diffusion Equations. *Journal of Scientific Computing*, **70**, 407-428. https://doi.org/10.1007/s10915-015-0152-y

[11] Chen, L., Nochetto, R., Otárola, E. and Salgado, A. (2016) Multilevel Methods for Nonuniformly Elliptic Operators and Fractional Diffusion. *Mathematics of Computation*, **85**, 2583-2607. https://doi.org/10.1090/mcom/3089

[12] Jin, B., Lazarov, R., Pasciak, J. and Zhou, Z. (2015) Error Analysis of Semidiscrete Finite Element Methods for Inhomogeneous Time-Fractional Diffusion. *IMA Journal of Numerical Analysis*, **35**, 561-582. https://doi.org/10.1093/imanum/dru018

[13] Zeng, F., Li, C., Liu, F. and Turner, I. (2015) Numerical Alogrithms for Time Fractional Subdiffusion Equation with Second-Order Accuracy. *SIAM Journal on Scientific Computing*, **37**, A55-A78. https://doi.org/10.1137/14096390X

[14] Zhou, Z. and Gong, W. (2016) Finite Element Approximation of Optimal Control Problems Governed by Time Fractional Diffusion Equation. *Computers and Mathematics with Applications*, **71**, 301-318. https://doi.org/10.1016/j.camwa.2015.11.014

[15] Du, N., Wang, H. and Liu, W. (2016) A Fast Gradient Projection Method for a Constrained Fractional Optimal Control. *Journal of Scientific Computing*, **68**, 1-20. https://doi.org/10.1007/s10915-015-0125-1

[16] Zhou, Z. and Tan, Z. (2019) Finite Element Approximation of Optimal Control Problem Governed by Space Fractional Equation. *Journal of Scientific Computing*, **78**, 1840-1861. https://doi.org/10.1007/s10915-018-0829-0

[17] Zhang, C., Liu, H. and Zhou, Z. (2019) A Priori Error Analysis for Time-Stepping Discontinuous Galerkin Finite Element Approximation of Time Fractional Optimal Control Problem. *Journal of Scientific Computing*, **80**, 993-1018. https://doi.org/10.1007/s10915-019-00964-9

[18] Gunzburger, M. and Wang, J. (2019) Error Analysis of Fully Discrete Finite Element Approximations to an Optimal Control Problem Governed by a Time-Fractional PDE. *SIAM Journal on Control and Optimization*, **57**, 241-263. https://doi.org/10.1137/17M1155636

[19] Lions, J. and Magenes, E. (1972) Non Homogeneous Boundary Value Problems and Applications. Springer-Verlag, Berlin.

[20] Hinze, M., Yan, N. and Zhou, Z. (2009) Variational Discretization for Optimal Control Governed by Convection Dominated Diffusion Equations. *Journal of Computational Mathematics*, **27**, 237-253. http://global-sci.org/intro/article_detail/jcm/8570.html

[21] Tang, Y. and Hua, Y. (2014) Supercovergence Analysis for Parabolic Optimal Control Problems. *Calcolo*, **51**, 381-392. https://doi.org/10.1007/s10092-013-0091-7

[22] Chen, Y., Huang, Y. and Yi, N. (2008) A Posteriori Error Estimates of Spectral Method for Optimal Control Problems Governed by Parabolic Equations. *Science in China Series A: Mathematics*, **51**, 1376-1390. https://doi.org/10.1007/s11425-008-0097-9

[23] Jiang, Y. and Ma, J. (2011) High-Order Finite Element Methods for Time-Fractional Partial Differential Equations. *Journal of Computational and Applied Mathematics*, **235**, 3285-3290. https://doi.org/10.1016/j.cam.2011.01.011

[24] Li, R., Liu, W. and Yan, N. (2007) A Posteriori Error Estimates of Recovery Type for Distributed Convex Optimal Control Problems. *Journal of Scientific Computing*, **33**, 155-182. https://doi.org/10.1007/s10915-007-9147-7