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Discussion for the Solutions of Dyson-Schwinger Equations at $m \neq 0$ in QED$_3$

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ABSTRACT

In the case of nonzero fermion mass, within a range of *Ansätze* for the full fermion-boson vertex, we show that Dyson-Schwinger equation for the fermion propagator in QED$_3$ has two qualitatively distinct dynamical chiral symmetry breaking solutions. As the fermion mass increases and reaches to a critical value m_c, one solution disappears, and the dependence of m_c on the number of fermion flavors is also given.

Keywords: QED$_3$; Fermion Propagator; Solutions of DSEs

1. Introduction

Nowadays, it is widely accepted that Quantum Chromodynamics (QCD) in 3 + 1 dimensions is the fundamental theory for strong interaction. Dynamical chiral symmetry breaking (DCSB) is of fundamental importance for strong interaction physics. DCSB can be explored via the gap equation, viz., the Dyson-Schwinger equation (DSE) for the dressed-fermion self-energy. As is well known, the gap equation has two solutions in the chiral limit, *i.e.* the Nambu-Goldstone (NG) solution which is characterized by DCSB, and the Wigner (WN) solution in which chiral symmetry is not dynamically broken. However, when the current quark mass m is nonzero, the quark gap equation has only one solution which corresponds to the NG phase and the solution corresponding to the WN phase does not exist [1,2]. This conclusion is hard to understand and one will naturally ask why the Wigner solution of the quark gap equation only exists in the chiral limit and does not exist at finite current quark mass. The authors of Ref. [3] first discussed this problem and asked whether the quark gap equation has a Wigner solution in the case of nonzero current quark mass. Subsequently, the authors of Refs. [4-7] further investigated the problem of possible multi-solutions of the quark gap equation. As far as we know, partly due to the complexity of the non-Abelian character of QCD,

this problem has not been solved satisfactorily in the literature. In the present paper we try to propose a new approach to investigate this problem in the framework of a relatively simple Abelian toy model of QCD, namely, quantum electrodynamics in 2 + 1 dimensions (QED$_3$).

As a field-theoretical model, QED$_3$ has been extensively studied in recent years. It has many features similar to QCD in 3 + 1 dimensions. This is because QED$_3$ is known to have a phase where the chiral symmetry of the theory is spontaneously broken and the fermions are confined in this phase [8]. Moreover, QED$_3$ is supernormalizable, so it is not plagued with the ultraviolet divergences which are present in QED$_4$. These are the basic reasons why QED$_3$ is regarded as an interesting toy model: studying QED$_3$ it might be possible to investigate confinement [8-10] and dynamical chiral symmetry breaking (DCSB) [11-16] within a theory which is structurally much simpler than QCD while sharing the same basic nonperturbative phenomena. Herein we try to use the DSEs for the fermion and photon propagators in QED$_3$ to describe novel aspects of the interplay between explicit and dynamical chiral symmetry breaking.

2. Dyson-Schwinger Equation for Fermion Propagator

The Lagrangian of QED$_3$ with *N* flavors of fermions in a

general covariant gauge in Euclidean space, ignoring the issues discuss in Ref. [15], can be written as ifications that anticipate your paper as one part of the entire journals, and not as an independent document. Please do not revise any of the current designations.

$$L = \sum_{j=1}^{N} \bar{\psi}_j \left(\not{\partial} + ie\not{A} - m \right) \psi_j + \frac{1}{4} F_{\rho\nu}^2 + \frac{1}{2\xi} \left(\partial_\rho A_\rho \right)^2 \quad (1)$$

where the 4 × 1 spinor j is the fermion field with $j = 1, \cdots, N$ being the flavor indices.

Based on Lorentz structure analysis, the inverse fermion propagator in the chiral limit can be written as

$$S^{-1}(p) = i\gamma \cdot p A(p^2) + B(p^2) \quad (2)$$

One assumes that dressed fermion propagator at finite m $\left(S^{-1}(m, p) \right)$ is analytic in the neighborhood of $m = 0$, so the $S^{-1}(m, p)$ can be written as

$$S^{-1}(m, p) = S^{-1}(p) + \int_0^m \frac{\delta S^{-1}(m', p)}{\delta m'} dm'$$
$$= i\gamma \cdot p A(p^2) + B(p^2) + i\gamma \cdot p C(m, p^2) + D(m, p^2) \quad (3)$$
$$\equiv i\gamma \cdot p E(p^2) + F(p^2)$$

where

$$E(p^2) = A(p^2) + C(m, p^2),$$
$$F(p^2) = B(p^2) + D(m, p^2).$$

Setting $e^2 = 1$, the DSE for the fermion propagator can be written as

$$S^{-1}(m, p) = S_0^{-1}(m, p)$$
$$+ \int \frac{d^3 k}{(2\pi)^3} \times \left[\gamma_\rho S(m, k) \Gamma(m; p, k) D_{\rho\nu}(m, p - k) \right] \quad (4)$$

where $S_0^{-1}(m, p)$ is the bare inverse fermion propagator and $\Gamma(m; p, k)$ is the full fermion-photon vertex. Substituting Equation (3) into Equation (4), one can obtain

$$E(p^2) = 1 - \frac{1}{4p^2} \int \frac{d^3 k}{(2\pi)^3}$$
$$\times \text{Tr} \left[i(\gamma \cdot p) \gamma_\rho S(m, k) \Gamma(m; p, k) D_{\rho\nu}(m, q) \right] \quad (5)$$

and

$$F(p^2) = m + \frac{1}{4} \int \frac{d^3 k}{(2\pi)^3}$$
$$\times \text{Tr} \left[\gamma_\rho S(m, k) \Gamma(m; p, k) D_{\rho\nu}(m, q) \right] \quad (6)$$

where $q = p - k$. The full photon propagator can be written as

$$D(m, q) = \frac{\delta_{\rho\nu} - q_\rho q_\nu / q^2}{q^2 \left[1 + \Pi(m, q^2) \right]} + \xi \frac{q_\rho q_\nu}{q^4} \quad (7)$$

with the vacuum polarization $\Pi(m, q^2)$ defined by

$$\Pi_{\rho\nu}(m, q^2) = \left(\delta_{\rho\nu} q^2 - q_\rho q_\nu \right) \Pi(m, q^2) \quad (8)$$

The DSE satisfied by the photon vacuum polarization tensor reads

$$\Pi_{\rho\nu}(m, q^2) = -N \int \frac{d^3 k}{(2\pi)^3}$$
$$\times \text{Tr} \left[\gamma_\rho S(m, p) \Gamma(m; p, k) S(m, k) \right] \quad (9)$$

The boson polarization $\Pi(m, q^2)$ has an ultraviolet divergence which is present only in the longitudinal part. By applying the projection operator

$$P_{\rho\nu} = \delta_{\rho\nu} - 3 \frac{q_\rho q_\nu}{q^2} \quad (10)$$

one can remove this divergence and obtain a finite vacuum polarization $\Pi(m, q^2)$ [15].

The DSEs for the photon and fermion propagators form a set of coupled integral equations for the three scalar functions $\left(E(p^2), F(p^2) \text{ and } \Pi(m, q^2) \right)$ once the full fermion-photon-vertex $\Gamma(m; p, k)$ is known. Unfortunately, although several works attempts to resolve the problem, none of them are completely satisfactory [17-23]. Thus, in phenomenological applications, one often proceed by adopting reasonable approximation for $\Gamma(m; p, k)$ such that Equations (5), (6) and (9) are reduced to a closed system of equations which may be solved directly. In this letter, following Ref. [15], we choose the following AnsÄatzefor the full fermion-photon vertex

$$\Gamma_\nu(m; p, k) = f \left(E(p^2), E(k^2) \right) \gamma_\nu \quad (11)$$

and the form of function f is: 1) 1; 2) $\frac{1}{2} \left[E(p^2) + E(k^2) \right]$; 3) $E(p^2) E(k^2)$.

The first one is the bare vertex. This structure plays the most dominant role in the full fermion-photon vertex in high energy region and the full fermion-photon vertex reduces to it in large momentum limit. The second form is inspired by the BC-vertex [18]. Previous works [15,24] show that the numerical results of DSEs employing this Ansatz is as good as that employing BC and CP vertex [19]. Since the numerical results obtained using the last Ansatze coincide very well with earlier investigation [15,25], we choose this one as a reasonable Ansatze to be used in this work. Using those AnsÄatzefor the full fermion-photon vertex, the coupled DSEs for the fermion propagator and photon vacuum polarization reduce to the following form

$$E\left(p^2\right)=1+\frac{2}{p^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{E\left(k^2\right)(pq)(kq)f/q^2}{q^2\left[E^2\left(k^2\right)k^2+F^2\left(k^2\right)\right]\left[1+\Pi\left(q^2\right)\right]} \tag{12}$$

$$F\left(p^2\right)=m+2\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{F\left(k^2\right)f}{q^2\left[E^2\left(k^2\right)k^2+F^2\left(k^2\right)\right]\left[1+\Pi\left(q^2\right)\right]} \tag{13}$$

$$\Pi\left(q^2\right)=\frac{4N}{q^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{E\left(k^2\right)E\left(p^2\right)\left[k^2-2(kq)-3(kq)^2/q^2\right]f}{\left[E^2\left(k^2\right)k^2+F^2\left(k^2\right)\right]\left[E^2\left(p^2\right)p^2+F^2\left(p^2\right)\right]} \tag{14}$$

where the Landau gauge has been chosen. In the chira limit, $E\left(p^2\right)=A\left(p^2\right)$ and $F\left(p^2\right)=B\left(p^2\right)$. From Equations (12)-(14), it is not difficult to find that the above coupled equations have one Wigner solution $B\left(p^2\right)\equiv0$ and two dynamical symmetry breaking solutions: $B\left(p^2\right)\left(B\left(p^2\right)>0\right)$ and $-B\left(p^2\right)$. As was pointed out in Ref. [4], if $B\left(p^2\right)$ is a solution of the gap equation in the chiral limit, then so is $-B\left(p^2\right)$. While these two solutions are distinct, the chiral symmetry entails that each yields the same pressure. In the chiral limit, the two dynamical symmetry breaking solutions are symmetric about the Wigner solution $B\left(p^2\right)=0$. However, just as will be shown below, this might be changed when the fermion mass is not zero.

3. Numerical Results

Our next task is to solve for the two scalar functions $E\left(m,p^2\right)$ and $F\left(m,p^2\right)$. These two functions can be obtained by numerically solving the three coupled integral Equations (12)-(14). Starting from $E=1$, $F=1$ and $\Pi=1$, we iterate the three coupled integral equations until all the three functions converge to a stable solution which is plotted in **Figure 1** (solid line).

From **Figure 1**, it is easy to find that all the three scalar functions in the DCSB phase ($N=1$) are constant in the infrared region, while in the ultraviolet region the vector function behaves as $A\left(p^2\right)\to1$ and the photon vacuum polarization behaves as $\Pi\left(q^2\right)\propto1/q$. Nevertheless, in contrast to the case of massless QED$_3$ [15], in the large momentum region, the fermion self-energy reduces to the bare mass m in Equation (6). Since all the three functions are positive in the whole range of p^2, we define them as the "+" solution.

If we do interation starting from $F=-1$, $E=1$ and $\Pi=1$, we can obtain another stable solution. The typical behaviors of the three functions in the DCSB phase for a fixed mass and number of fermion flavors are also plotted in **Figure 1** (the dotted line). From **Figure 1**, we see that the DSEs for the fermion propagator has two distinct nonzero solutions. Especially, the infrared value of the fermion self-energy is negative, so we define it as the "−"
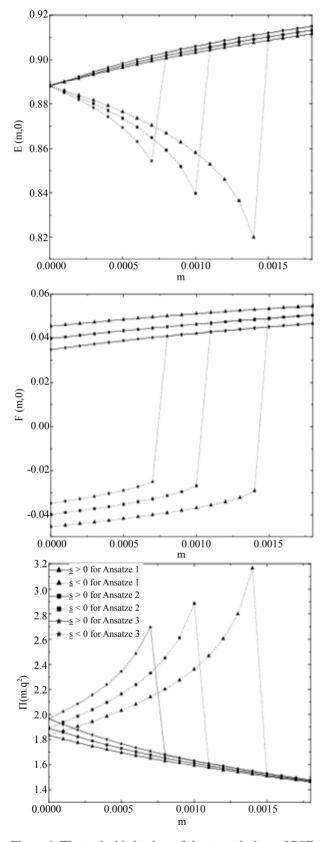
solution. In the low energy region, each of the three functions in the second solution is also almost constant, but it is different from the corresponding one in the "+" solution. As p^2 or q^2 increases, each function of the "−" solution approach to the corresponding one of the "+" solution.

To reveal the difference between these two solutions, we consider m as a continuous parameter in the DSEs. We plot the infrared value of E, F, Π in **Figure 2**. When $m=0$, from DSEs one obtains one E and Π, but two F which are symmetric about $F=0$ in **Figure 2** for each vertex ansatze. For the "+" solution of DSEs, as m increases, $E(0)$ and $F(0)$ increases while $\Pi(0)$ decreases. However, the three infrared values in the "−" solution show a different trend as m increases. When m reaches its critical value, we obtain only one solution for DSEs. In addition, from **Figure 2**, it can be seen that the critical mass exists for any truncated scheme of DSEs used in this work.

Furthermore, we investigate the influence of the number of fermion flavors on the critical mass. By employing ansatze 2, we can obtain the relation between the critical mass and the number of fermion flavors and it is plotted in **Figure 3**. We observe that the critical mass decreases as N increases and it vanishes at $N=N_c$, which is similar to the critical number of fermion flavors for DCSB in the chiral limit [16].

4. Conclusion

To summarize, in this paper, working in the framework of Dyson-Schwinger equations and employing a range of ansatze for the full fermion-photon vertex of QED$_3$, westudy the interplay between explicit and dynamical chiral symmetry breaking in QED$_3$. In the case of nonzero fermion mass, it is found that, besides the ordinary solution, the fermions gap equation has another solution which has not been reported in the previous work of QED$_3$. In the low energy region, one observes that these two solutions are apparently different, but in the high energy region they coincide with each other. In addition, it is found that this solution exists only when the mass is
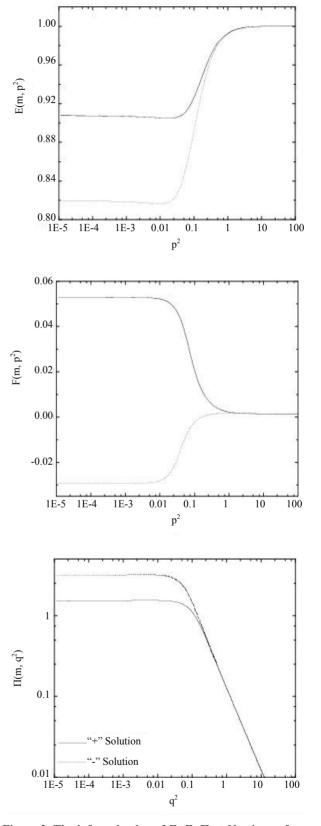
Figure 1. The typical behaviors of the two solutions of DSEs for the fermion propagator at $N = 1, m = 10^{-3}$ for Ansatze 2.

Figure 2. The infrared value of E; F; Π at $N = 1$ as a function of m ($S > 0$ represents the "+" solution and $S < 0$ represents the "−" solution).
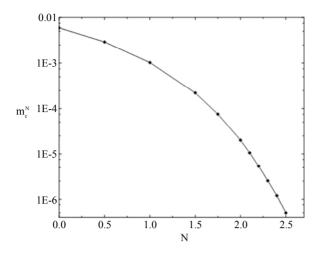
Figure 3. The relation between the critical mass and the number of fermion flavors.

smaller than a critical value. The critical mass decreases apparently with the rise of the number of fermion flavors and vanishes at a critical value N_c, which corresponds to the critical number of fermion flavors of QED$_3$ in the chiral limit. It is an interesting phenomena which deserve further investigations.

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