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Realization of the Linear Tree that Corresponds to a Fundamental Loop Matrix

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Abstract

Graph realization from a matrix is an important topic in network topology. This paper presents an algorithm for the realization of a linear tree based on the study of the properties of the number of the single-link loops that are incident to each tree branch in the fundamental loop matrix $\mathbf{B_f}$. The proposed method judges the pendent properties of the tree branches, determines their order one by one and then achieves the realization of the linear tree. The graph that corresponds to $\mathbf{B_f}$ is eventually constructed by adding links to the obtained linear tree. The proposed method can be extended for the realization of a general tree.

Keywords: Fundamental Loop Matrix, Linear Tree, Graph

1. Introduction

Graph realization from a matrix is an important topic in network topology. It has a broad application in electrical networks, switching networks, linear programming etc. The study of graph realization from a matrix is to judge whether a given matrix can be realized to be a graph. If yes, the method for realization needs to be found so that graph realization for a given matrix can be achieved. There are three aspects in graph realization from a matrix: a) the realizablity of the given matrix, b) the method for graph realization from a matrix, c) the unique correspondence between the realized graph and the given matrix. All the above issues have been under research by using algebra theory, graph theory, geometry structure etc [1–5]. While the necessary and sufficient conditions for the realizability of a graph from a matrix are proposed by many researchers from different viewpoints, the judgment of the realizability and the realization of the graph, in practice, are carried out simultaneously rather than in sequence.

Quite a few researchers studied the realization of a linear tree [6,7]. This paper presents an algorithm for the realization of a linear tree based on the study of the properties of the number of the single-link loops that are incident to each tree branch in the fundamental loop matrix $\mathbf{B_f}$. The proposed method judges the pendent properties of the tree branches, determines their order one by one and then achieves the realization of the linear tree. Since a general tree possesses a linear sub-tree, a general tree can then be realized by adding other tree branches after the linear sub-tree is realized. The graph that corresponds to an arbitrary fundamental loop matrix $\mathbf{B_f}$ is eventually constructed by adding links to the obtained linear tree. As is seen, the proposed method is simple, practical and efficient in realizing a general tree.

2. Pretreatment

For a graph that possesses n nodes and b branches, after a certain tree T is chosen, the fundamental loop matrix $\mathbf{B_f}$ has a standard form $[\mathbf{B_t}\ \mathbf{1}]$ where $\mathbf{B_t}$ is a $(b\text{-}n\text{+}1)\times(n\text{-}1)$ matrix. Assume that the columns $b_1, b_2, \ldots, b_{n-1}$ in $\mathbf{B_t}$ correspond to the branches $t_1, t_2, \ldots, t_{n-1}$ of T, while the columns $b_n, b_{n+1}, \ldots, b_b$ in $\mathbf{1}$ correspond to the links. As is known, $b_i^{\mathrm{T}}\cdot b_j$ indicates the number of the pairs of corresponding entries being all "1"s in the branch columns b_i and b_j. By realizing a graph from the matrix that is constructed by the rows of the aforementioned pairs as well as the same indexed rows in the corresponding link columns, we know that $b_i^{\mathrm{T}}\cdot b_j$ is the number of the single-link loops that pass t_i and t_j. Specially, $b_i^{\mathrm{T}}\cdot b_i$ indicates the number of entries "1" in b_i. By realizing a graph from the matrix that is constructed by the rows of the aforementioned "1"s as well as the same indexed rows in the corresponding link columns, we know that $b_i^{\mathrm{T}}\cdot b_i$ is the number of the single-link loops that pass t_i. To facilitate our discussions, it is assumed that the tree that corresponds to $\mathbf{B_t}$ is a linear tree. For example:

$$\mathbf{B_f} = [\mathbf{B_t}, \mathbf{1}] = \begin{array}{c} \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The number of the pairs of corresponding entries being all "1"s in b_1 and b_2 (i.e., row 1 and row 2) is 2. Thus, $b_1^T \cdot b_2 = 2$. By realizing a graph (Figure 1) from the matrix $\begin{array}{c} \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$ that is constructed by the rows of the aforementioned pairs as well as the same indexed rows in the corresponding link columns, we know that $b_1^T \cdot b_2 = 2$ is the number of the single-link loops that pass t_1 and t_2. Specially, $b_1^T \cdot b_1 = 2$ is the number of the single-link loops that pass t_1. The following theorems present the relationship among the branches of T (Due to limitation of space, the proofs are not presented in this article).

Theorem 1: Suppose $\mathbf{B_t}$ is a $(b\text{-}n+1)\times 3$ matrix and t_p is a pendent branch of the linear tree T that corresponds to $\mathbf{B_t}$. If and only if $b_p^T \cdot b_q \geq b_p^T \cdot b_r$ ($p \neq q \neq r$, p, q, r=1, 2, 3), the order of the branches of T is t_p, t_q, t_r.

As is seen in Figure 2, while there is only one single-link loop passing t_p and t_q, there are two passing t_p and t_r. Therefore, $b_p^T \cdot b_r \geq b_p^T \cdot b_q$ and the order of the branches of T is t_p, t_r, t_q according to Theorem 1.

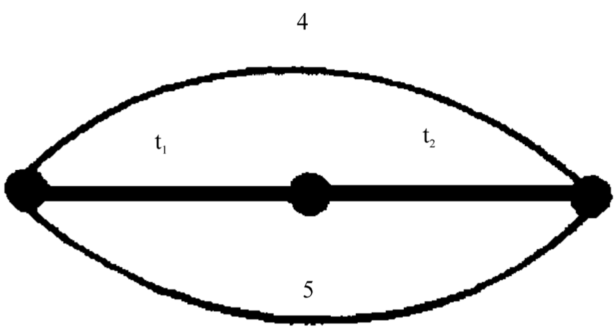

Figure 1. The explanation of $b \cdot b_j$.

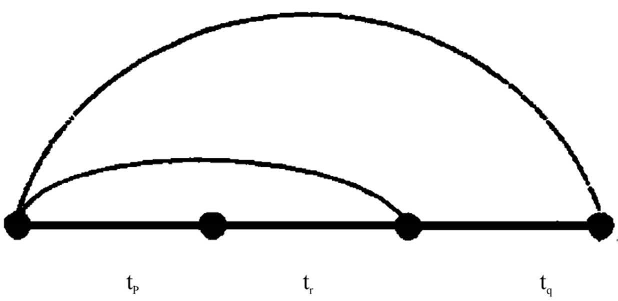

Figure 2. The explanation of Theorem 1.

Theorem 2: Suppose $\mathbf{B_t}$ is a $(b\text{-}n+1)\times(n\text{-}1)$ matrix and t_p is a pendent branch of the linear tree T that corresponds to $\mathbf{B_t}$. If and only if $b_p^T \cdot b_q \geq b_p^T \cdot b_r$ ($p \neq q \neq r$, $1 \leq p$, q, $r \leq n\text{-}1$), the order of the branches of T is t_p, …, t_q, …, t_r (i.e., t_q is closer to t_p than t_r is).

Theorem 3: Suppose $\mathbf{B_t}$ is a $(b\text{-}n+1)\times(n\text{-}1)$ matrix. For a certain column b_l arbitrarily chosen, if $b_l^T \cdot b_p = \min\limits_{i \neq l,\, i = 1,2,\ldots n-1} b_l^T \cdot b_i$ ($p \neq l$, $1 \leq p \leq n\text{-}1$), t_p is a pendent branch of the linear tree T that corresponds to $\mathbf{B_t}$.

Theorem 4: Suppose $\mathbf{B_t}$ is a $(b\text{-}n+1)\times(n\text{-}1)$ matrix. For a certain column b_l arbitrarily chosen, if $b_l^T \cdot b_j = \min\limits_{i \neq l,\, i = 1,2,\ldots n-1} b_l^T \cdot b_i$ where $j \in U = \{p, q, r, \ldots, w\}$ ($p \neq q \neq r \neq \ldots \neq w \neq l$, $1 \leq p, q, r, \ldots, w \leq n\text{-}1$), there must exist a certain $s \in U$ such that t_s is a pendent branch of the linear tree T that corresponds to $\mathbf{B_t}$.

As is seen in Figure 3, while $b_l^T \cdot b_m = b_l^T \cdot b_n = 2$, $b_l^T \cdot b_p = b_l^T \cdot b_q = 1$. Thus, there must exist a certain $s \in U = \{p, q\}$ such that t_p or t_q is a pendent branch of the linear tree T.

Theorem 5: Suppose $\mathbf{B_t}$ is a $(b\text{-}n+1)\times(n\text{-}1)$ matrix. For a certain column b_l arbitrarily chosen, if $b_l^T \cdot b_j = \min\limits_{i \neq l,\, i = 1,2,\ldots n-1} b_l^T \cdot b_i$ where $j \in U = \{p, q, r, \ldots, w\}$ ($p \neq q \neq r \neq \ldots \neq w \neq l$, $1 \leq p, q, r, \ldots, w \leq n\text{-}1$), then the t_h, t_m, t_n, …, t_l that correspond to $\bar{U} = \{h, m, n, \ldots, l\}$ construct a linear sub-tree of T, where $U \cap \bar{U} = \phi$ and $|U \cup \bar{U}| = n\text{-}1$.

As is seen in Figure 3, $U = \{p, q\}$ and $\bar{U} = \{m, n, l\}$. Since $b_l^T \cdot b_m = b_l^T \cdot b_n = 2$ but $b_l^T \cdot b_p = b_l^T \cdot b_q = 1$, t_m, t_n and t_l construct a linear sub-tree of T.

Theorem 6: Suppose $\mathbf{B_t}$ is a $(b\text{-}n+1)\times(n\text{-}1)$ matrix. For a certain column b_l arbitrarily chosen, $b_l^T \cdot b_j = \min\limits_{i \neq l,\, i = 1,2,\ldots n-1} b_l^T \cdot b_i$ where $j \in U = \{p, q, r, \ldots, w\}$ ($p \neq q \neq r \neq \ldots \neq w \neq l$, $1 \leq p, q, r, \ldots, w \leq n\text{-}1$). Thus, the t_h, t_m, t_n, …, t_l that correspond to $\bar{U} = \{h, m, n, \ldots, l\}$ construct

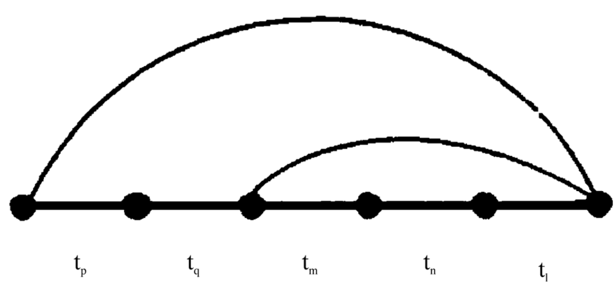

Figure 3. The explanation of Theorem 4.

a linear sub-tree of T, where $U \cap \bar{U} = \phi$ and $|U \cup \bar{U}| = n-1$. Then for a certain $k \in \bar{U}$, if there are certain s_1 and $s_2 \in U^* \subseteq U$ such that $b_k^T \cdot b_{s_1} = b_k^T \cdot b_{s_2}$, after the links L's that correspond to $\max_{k=h,m,n,\ldots,l} b_k^T \cdot b_{s_1}$ in the graph G that corresponds to $\mathbf{B_f}$ are deleted, the graph constructed by G_1 that corresponds to U^* and the graph G_2 that corresponds to $(U-U^*) \cup \bar{U}$ is a separable graph, where G_1 and $G_2 \subset G$, $G_1 \cup G_2 \cup G_L = G$, $G_1 \cap G_2 \cap G_L = \phi$ and G_L is the graph that corresponds to the links L's. If G_1 and G_2 are inseparable graphs, respectively, then G_1 and G_2 have a common cut-point.

As is seen in Figure 4(a), $b_l^T \cdot b_j = 1$ where $j \in U = \{p, q, r, w\}$. Thus, the t_h, t_m, t_n, …, t_l that correspond to $\bar{U} = \{h, m, n, l\}$ construct a linear sub-tree of T. Let $k = h \in \bar{U}$, and s_1 and $s_2 \in U^* = \{p, q\} \subseteq U$. Since $b_k^T \cdot b_{s_1} = b_k^T \cdot b_{s_2} = 3$, links 1, 2 and 3 are deleted. The remaining G_1 and G_2 are separable graphs as shown in Figure 4(b).

From the above facts, it is seen that when the conditions in Theorem 6 are satisfied, G_1 and G_2 are 2-isomorphic. Therefore, in the ordering of the branches of the linear tree, the branches of the linear sub-tree that corresponds to G_1 are first put into order separately, then those of the linear sub-tree that corresponds to G_2. The ordering of the branches of the linear tree is now reduced to the ordering of the branches for each linear sub-tree. The solution of this problem is depending on the following Theorem 7, where it is assumed that the sub-trees do not form 2-isomorphism.

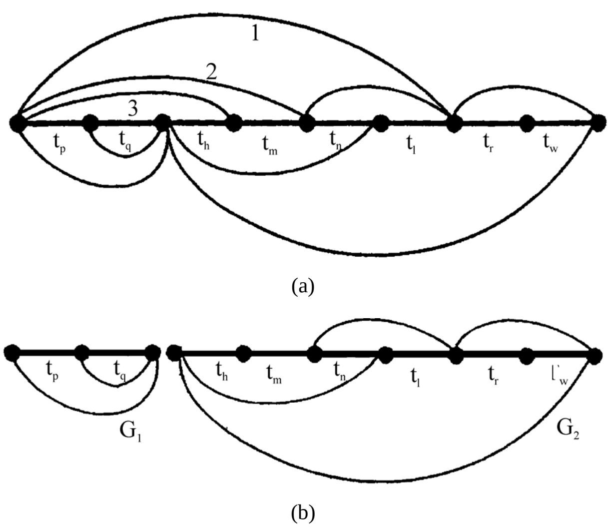

Figure 4. The explanation of Theorem 6.

Theorem 7: Suppose $\mathbf{B_t}$ is a $(b-n+1)\times(n-1)$ matrix. For a certain column b_l arbitrarily chosen, $b_l^T \cdot b_j = \min_{i \neq l,\, i=1,2,\ldots n-1} b_l^T \cdot b_i$ where $j \in U = \{p, q, r, \ldots, w\}$ $(p \neq q \neq r \neq \ldots \neq w \neq l,\ 1 \leq p, q, r, \ldots, w \leq n-1)$. Thus, the t_h, t_m, t_n, …, t_l that correspond to $\bar{U} = \{h, m, n, \ldots, l\}$ construct a linear sub-tree of T, where $U \cap \bar{U} = \phi$ and $|U \cup \bar{U}| = n-1$. Then for a certain $k \in \bar{U}$, if there is a certain $s \in U$ such that $b_k^T \cdot b_s = \min_{j \neq s,\, j=p,q,r,\ldots,w} b_k^T \cdot b_j$, t_s is a pendent branch of the linear tree T that corresponds to $\mathbf{B_t}$.

As is seen in Figure 5, $U = \{p, q, r\}$ and $\bar{U} = \{h, m, n, l\}$. Let $k = h \in \bar{U}$ and $s = p \in U$. Since $b_h^T \cdot b_p = 1 < b_h^T \cdot b_q = 2$ and $b_h^T \cdot b_p = 1 < b_h^T \cdot b_r = 3$, t_p is a pendent branch of the linear tree T.

3. The Algorithm for the Construction of the Linear Tree

We propose the following algorithm for the construction of the linear tree T. The thread of thinking is that one of the two pendent branches of T, e.g., t_p is found first. The other pendent tree branch t_q is found by using t_p as a base. Then t_q is taken off, the other pendent tree branch t_r is found by using t_p as a base again. Keep on with this procedure until the order of all the branches of T is decided.

1) For a given fundamental loop matrix $\mathbf{B_f} = [\mathbf{B_t}\ \mathbf{1}]$, let $\mathbf{M} = \mathbf{B_t}^T\mathbf{B_t}$ where the entry $m_{ij} = b_i^T \cdot b_j$. Establish a matrix $\mathbf{B_t'}$ which is of the same dimension as $\mathbf{B_t}$ so that the columns of $\mathbf{B_t}$ after ordering can be put into $\mathbf{B_t'}$. Let the column index for $\mathbf{B_t'}$ be f. Set $f=1$.

2) For row i $(1 \leq i \leq n-1)$ in $\mathbf{M}$ (Usually, let $i=1$ first to follow the row order in $\mathbf{M}$), if there is only one entry m_{ip} in row i that takes the minimum value, t_p that corresponds to column p is a pendent branch of the linear tree

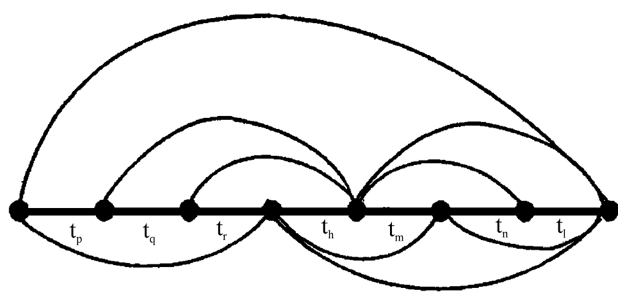

Figure 5. The explanation of Theorem 7.

T according to Theorem 3. Go to step (6). On the other hand, if there are multiple entries m_{ip_d}, m_{iq_d}, m_{ir_d}, …, m_{iw_d} in row i that take the minimum value, there must exist a certain $s \in U_d = \{p_d, q_d, r_d, \ldots, w_d\}$ (d is the iteration index. Let d=1 first.) such that t_s that corresponds to column s is a pendent branch of the linear tree T according to Theorem 4.

3) For row k in **M** where $k \in \bar{U}_d = \{h, m, n, \ldots, l\}$(Usually, let k follow the row order in $\bar{U}_d$), if there is only one entry m_{ks} in row k that takes the minimum value where $s \in U_d$, t_s that corresponds to column s is a pendent branch of the linear tree T according to Theorem 7. Go to step (4). On the other hand, if there are multiple entries m_{kp_d}, m_{kq_d}, m_{kr_d}, …, m_{kw_d} in row k that take the minimum value, there must exist a certain $s \in U_d = \{p_d, q_d, r_d, \ldots, w_d\}$ (d=d+1) such that t_s that corresponds to column s is a pendent branch of the linear tree T according to Theorem 4. Repeat step (3) until k takes all the elements in $\bar{U}_d$. At that time, if there are still multiple entries in row k that take the minimum value, go to step (8).

4) If the last column of $\mathbf{B_t}$**’** is not filled by a column from $\mathbf{B_t}$ yet, set p=s. Go to step (6). Otherwise, judge the adjacency of t_k, t_s and t_p according to Theorem 2.

5) If t_k and t_p are at the same side of t_s, i.e., the order is $t_s, \ldots, t_k, \ldots, t_p$, set p=s. Go to step (6). On the other hand, if t_k and t_p are at the different sides of t_s, i.e., the order is $t_k, \ldots, t_s, \ldots, t_p$, use t_k as a pendent tree branch to find the order of the tree branches corresponding to U_d and put them into the columns of $\mathbf{B_t}$**’**, i.e., column f to column f’ where f’=f+(number of elements in U_d)-1. Set all the entries in the columns of **M** corresponding to U_d to be ∞. If the entries in **M** are all ∞, stop. If not, set f = f+ (number of elements in U_d). Go back to step (2).

6) If the last column of $\mathbf{B_t}$**’** is already filled by a column from $\mathbf{B_t}$, i.e., a pendent tree branch at one end is already decided, go to step (b). Otherwise

a) Assume the pendent branch of T is t_p. Put t_p into the last column of $\mathbf{B_t}$**’**. Set i=p. Go to step (7).

b) Put column p of $\mathbf{B_t}$ into column f of $\mathbf{B_t}$**’**. Set f=f+1.

7) If all the columns of $\mathbf{B_t}$ have been put into $\mathbf{B_t}$**’**, the ordered columns of $\mathbf{B_t}$**’** have already constitute a linear tree. Stop. Otherwise, set the entries in column p of **M** to be ∞. Go back to step (2).

8) When there are only two entries in U_d, choose arbitrarily $s \in U_d$. t_s is a pendent branch of T. Go to step (4). Otherwise, according to Theorem 5 and Theorem 6, the linear sub-tree in graph G_1 that corresponds to U_d*=U_d can be put into order separately. Thus, take the columns $\mathbf{B_t}^{(1)}$ in $\mathbf{B_t}$ that correspond to the elements in U_d to construct $\mathbf{B_t}^{(1)\mathrm{T}}\,\mathbf{B_t}^{(1)}$=$\mathbf{M}^{(1)}$. Repeat steps (2)-(8) for $\mathbf{M}^{(1)}$. If the number of elements in U_d is not changed after one iteration, the order of the corresponding tree branches is arbitrary. Put the ordered columns of $\mathbf{B_t}^{(1)}$ into column f to column f’ where f’=f+(number of elements in U_d)-1. Set all the entries in the columns of **M** corresponding to U_d to be ∞. If the entries in **M** are all ∞, stop. If not, set f=f+(number of elements in U_d). Go back to step (2).

4. An Example

Given a fundamental loop matrix

$$
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\mathrm{B_r}=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},
\end{array}
$$

we have

$$
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\mathbf{B_t} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}
\end{array}
\text{ and } \mathbf{B_t}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.
$$

a) According to step (1),

$$
\mathbf{M}=\mathbf{B_t}^{\mathrm{T}}\mathbf{B_t}=
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{bmatrix} 3 & 2 & 2 & 1 & 3 \\ 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 3 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 \\ 3 & 2 & 3 & 2 & 4 \end{bmatrix}
\end{array}.
$$

Also, establish a matrix $\mathbf{B_t}$**’** which is of the same dimension as $\mathbf{B_t}$ so that the columns of $\mathbf{B_t}$ after ordering can be put into $\mathbf{B_t}$**’**. Let the column index for $\mathbf{B_t}$**’** be f. Set f=1.

b) According to step (2), consider row 1 of **M**. As m_{14} is the only entry in row 1 that takes the minimum value, t_4 is one pendent branch of T. Go to step (6).

c) According to step (6)(a), put column 4 of $\mathbf{B_t}$ into the last column of $\mathbf{B_t}$**’**, i.e.,

$$
\mathbf{B_t}\text{’}=
\begin{array}{c}
\begin{array}{ccccc} & & & & 4 \end{array} \\
\begin{bmatrix} & & & & 1 \\ & & & & 0 \\ & & & & 0 \\ & & & & 1 \end{bmatrix}
\end{array}. \text{ Set } i=4.
$$

d) According to step (7), set all the entries in column 4 of **M** to be ∞, i.e.,

$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 3 & 2 & 2 & \infty & 3 \\ 2 & 2 & 2 & \infty & 2 \\ 2 & 2 & 3 & \infty & 3 \\ 1 & 1 & 2 & \infty & 2 \\ 3 & 2 & 3 & \infty & 4 \end{bmatrix} \end{array}$$

e) According to step (2), consider row 4 of **M**. As m_{41} and m_{42} are the entries in row 4 that take the minimum value, there must exist the other pendent branch of T among t_1 and t_2 that correspond to $U_1=\{1, 2\}$. Here, $\bar{U}_1=\{3, 4, 5\}$.

f) According to step (3), consider row 3 of **M**. As m_{31} and m_{32} are the entries in row 3 that take the minimum value, there must exist the other pendent branch of T among t_1 and t_2 that correspond to $U_2=\{1, 2\}$. Here, $\bar{U}_2=\{3, 4, 5\}$.

g) Repeat step (3). Consider row 5 of **M**. m_{52} is the only entry in row 5 that takes the minimum value.

h) According to step (4), as the last column of $\mathbf{B_t}$**'** is already filled by a column from $\mathbf{B_t}$, judge the adjacency of t_5, t_2 and t_4. As $m_{42}<m_{45}$ in m_4, the order is $t_2, \ldots, t_5, \ldots, t_4$.

i) According to step (5), as t_5 and t_4 are at the same side of t_2, t_2 is the other pendent branch of T that is based on t_4. Set p=2.

j) According to step (6)(b), put column 2 of $\mathbf{B_t}$ into the first column of $\mathbf{B_t}$**'**, i.e., $\mathbf{B_t}' = \begin{array}{c} \begin{array}{cccc} 2 & & & 4 \end{array} \\ \begin{bmatrix} 1 & & & 1 \\ 0 & & & 0 \\ 1 & & & 0 \\ 0 & & & 1 \end{bmatrix} \end{array}$. Set f=1+1=2.

k) According to step (7), set all the entries in column 2 of **M** to be ∞, i.e.,

$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 3 & \infty & 2 & \infty & 3 \\ 2 & \infty & 2 & \infty & 2 \\ 2 & \infty & 3 & \infty & 3 \\ 1 & \infty & 2 & \infty & 2 \\ 3 & \infty & 3 & \infty & 4 \end{bmatrix} \end{array}.$$

l) According to step (2), consider row 4 of **M**. As m_{41} is the only entry in row 4 that takes the minimum value, t_1 is the other pendent branch of T that is based on t_4 with t_2 taken off.

m) According to step (6), put column 1 of $\mathbf{B_t}$ into the second column of $\mathbf{B_t}$**'**, i.e., $\mathbf{B_t}' = \begin{array}{c} \begin{array}{cccc} 2 & 1 & & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & & 1 \\ 0 & 1 & & 0 \\ 1 & 1 & & 0 \\ 0 & 0 & & 1 \end{bmatrix} \end{array}$. Set f=2+1=3.

n) According to step (7), set all the entries in column 1 of **M** to be ∞, i.e.,

$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \infty & \infty & 2 & \infty & 3 \\ \infty & \infty & 2 & \infty & 2 \\ \infty & \infty & 3 & \infty & 3 \\ \infty & \infty & 2 & \infty & 2 \\ \infty & \infty & 3 & \infty & 4 \end{bmatrix} \end{array}.$$

o) According to step (2), consider row 4 of **M**. As m_{43} and m_{45} are the entries in row 4 that take the minimum value, there must exist the other pendent branch of T that is based on t_4 with t_1 and t_2 taken off among t_3 and t_5 that correspond to $U_1=\{3, 5\}$. Here, $\bar{U}_1=\{1, 2, 4\}$.

p) According to step (3), consider row 1 of **M**. m_{13} is the only entry in row 1 that takes the minimum value.

q) According to step (4), as the last column of $\mathbf{B_t}$**'** is already filled by a column from $\mathbf{B_t}$, judge the adjacency of t_1, t_3 and t_4. As $m_{13}>m_{14}$ in m_1, the order is $t_1, \ldots, t_3, \ldots, t_4$.

r) According to step (5), as t_1 and t_4 are at the different sides of t_3, t_2 is used as the other pendent branch of T to find the order of the tree branches corresponding to U_d. Put column 3 of $\mathbf{B_t}$ into the fourth (f'=3+2-1=4) column of $\mathbf{B_t}$**'**, i.e.,

$$\mathbf{B_t}' = \begin{array}{c} \begin{array}{ccccc} 2 & 1 & & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & & 1 & 1 \\ 0 & 1 & & 0 & 0 \\ 1 & 1 & & 1 & 0 \\ 0 & 0 & & 1 & 1 \end{bmatrix} \end{array}.$$

Set f=3+1=4. Set all the entries in column 3 of **M** to be ∞, i.e.,

$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \infty & \infty & \infty & \infty & 3 \\ \infty & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & 3 \\ \infty & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & 4 \end{bmatrix} \end{array}.$$

s) According to step (2), consider row 4 of **M**. As m_{45} is the only entry in row 4 that takes the minimum value, t_5 is the other pendent branch of T that is based on t_1 with

t_3 taken off.

t) According to step (6), put column 5 of $\mathbf{B_t}$ into the third column of $\mathbf{B_t}$', i.e., $\mathbf{B_t}'= \begin{array}{c} \begin{matrix} 2 & 1 & 5 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$. Set f=4+1=5.

u) According to step (7), stop.

As a summary, we have the following table to achieve the order of the columns in $\mathbf{B_t}$'.

Colu. in $\mathbf{B_t}$'	5	1	2	4	3
Colu. in $\mathbf{B_t}$	4	2	1	3	5
Steps	(b),(c),(d)	(e),(f),(g),(h),(i),(j),(k)	(l),(m),(n)	(o),(p),(q),(r)	(s),(t),(u)
According to Algorithm Steps	(2),(6), (7)	(2),(3),(3),(4),(5),(6),(7)	(2),(6),(7)	(2),(3),(4),(5)	(2),(6),(7)

The column order [2, 1, 5, 3, 4] of $\mathbf{B_t}$' is thus the order of the branches of the linear tree as shown by the bold segments in Figure 6. The graph that corresponds to $\mathbf{B_f}$ can then be obtained by adding the links 6, 7, 8 and 9.

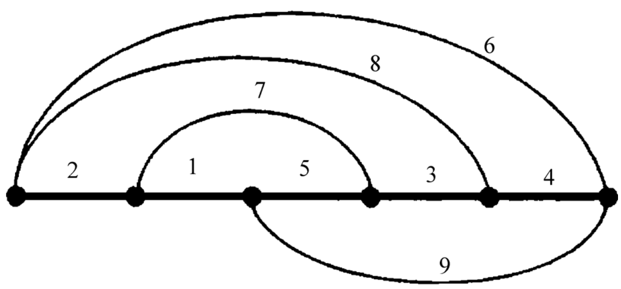

Figure 6. The graph that corresponds to $\mathbf{B_f}$.

5. Conclusions

This paper presents an algorithm for the realization of a linear tree based on the judgment of the pendent properties of the tree branches and the determination of their order one by one. The graph that corresponds to $\mathbf{B_f}$ is eventually constructed by adding links to the obtained linear tree. As an arbitrary tree contains a linear tree, the linear tree can then be realized first to realize a general tree. This will be discussed in another paper of ours.

The main contribution of this paper lies in the proposition of a new approach to the realization of a linear tree. Experiments validate the effectiveness of the proposed approach. This lays a foundation to the realization of a general tree and therefore a random graph from a given matrix.

6. References

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