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Violation of Cluster Property in Heisenberg Antiferromagnet

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Abstract

There are concepts that are accepted in our daily life, but are not trivial in physics. One of them is the cluster property that means there exist no relations between two events which are sufficiently separated. In a paper recently published by the author, it has been pointed out that this cluster property violates in the correlation function of the spin operator for the spin 1/2 XXZ antiferromagnet on the square lattice. In this paper, we investigate the spin 1/2 Heisenberg antiferromagnet on the square lattice, which has SU(2) symmetry. In order to study the cluster property, we need to calculate the ground state accurately. For this purpose, we employ the effective model based on the magnetization of the sub-lattices. Then we can define the quasi-degenerate states to calculate the ground state. Including two kinds of interactions which break SU(2) symmetry into the Hamiltonian, we obtain the ground state quantitatively. We find that two kinds of spin correlation functions due to degenerate states are not zero when the lattice size is large but finite. The magnitude of one of them is the same as the one previously found in the XXZ antiferromagnet, while another one is much larger when the additional interaction is strong. We then conclude that in Heisenberg antiferromagnet correlation functions violate the cluster property and the magnitude of the violation qualitatively differs from the one in the XXZ antiferromagnet.

Keywords

Cluster Property, Entanglement, Spontaneously Symmetry Breaking, Heisenberg Antiferromagnet, Degenerate States

1. Introduction

Entanglement forces us to abandon the classical concept about locality [1] [2] [3]. Although there are many criticisms of abandoning the concept [4], experiments strongly support entanglement required by quantum mechanics [5]. Since

Deutsch [6] presented the concept of quantum computer [7], a huge number of researchers apply entanglement to quantum information [8] [9] [10].

Entanglement found in many-body systems is a quite interesting subject [11] [12] in terms of the quantum phase transition [13] [14] [15] [16], because the correlation must be found even at the long distance where the whole system changes entirely. The entangled correlation at infinity distance implies the violation of the cluster decomposition [17] or the cluster property [18], which is a fundamental concept of physics that there is no relation between two events that occur infinitely apart from each other. Therefore, we find active studies on the cluster property in the many-body systems [19] [20] and in quantum field theory [21] [22], including QCD [23] [24].

In the previous paper [25], we have investigated the cluster property of spin 1/2 XXZ antiferromagnet on the square lattice. For this antiferromagnet, the ground state realizes semi-classical Neel order [26], in other words, spontaneous symmetry breaking (SSB) [17] [27] of U(1) symmetry [28] [29] [30] [31] [32]. In the previous study, we supposed that the lattice size is very large but finite. On this condition, we clarified that the essential property of SSB is the existence of the quasi-degenerate states between which the expectation value of the local operator is not zero. Introducing an additional interaction that explicitly breaks the symmetry, we determined the ground state definitely using the quasi-degenerate states. Then we found that the correlation function in the ground state violated the cluster property and the magnitude of the violation depended on the strength of the symmetry breaking interaction.

In this paper, we investigate the violation of cluster property for Heisenberg quantum spin on the square lattice, whose continuous symmetry is SU(2). SSB in this system can be explained by the effective model which realizes the magnetization of the sub-lattices [26]. By this model, we can present a definite description of the quasi-degenerate states. Here we should note that the quasi-degenerate states are the essential ingredients of SSB, as stressed in the previous work [25].

On the square lattice of size N, we can divide the whole lattice into two sub-lattices, which are called A sub-lattice and B sub-lattice. The semi-classical order of this spin system implies that the magnitude of the total spin on each sub-lattice is the order of N. Therefore, we can denote the state of spin system on a sub-lattice by its magnitude and its z-component.

On the finite lattice, the researchers studying this system have shown that the energy depends only on the spin magnitude of the whole lattice. Therefore the ground state as well as the energy can be determined by the effective Hamiltonian [26] that contains the squared total spin. By this effective Hamiltonian $\hat{H}_{eff}$ and the set of the quasi-degenerate state, we can make a complete description for the ground state even if we include the additional interactions.

In order to obtain the ground state uniquely on the finite lattice we introduce two additional interactions $\hat{V}_1$ and $\hat{V}_2$. The former interaction eliminates the degeneracy on the z-component of the total spin, and the latter induces the non-zero expectation value of the spin operator in the ground state. Then we will consider the Hamiltonian $\hat{H}_{eff,V} \equiv \hat{H}_{eff} + \hat{V}_1 + \hat{V}_2$ and calculate the spin correlation functions due to the quasi-degenerate states. We conclude that these functions violate the cluster property in Heisenberg antiferromagnet and their magnitudes are much larger than the largest magnitude in XXZ model. This large correlation function encourages us to search for the violation of the cluster property in experiments.

Contents of this paper are as follows. In the next section we introduce the model studied in this paper, which is the spin 1/2 Heisenberg antiferromagnet on the square lattice, and define two sub-lattices. In Section 3 we give the effective Hamiltonian $\hat{H}_{eff}$ by which we calculate the ground state in terms of states that are eigen states of the sub-lattice magnetizations. In order to confirm the validity of $\hat{H}_{eff}$ we present numerical results which are calculated by the exact diagonalization on the quite small lattices. Using $\hat{H}_{eff}$ we show that the ground state is described by the associated Legendre polynomial when the lattice size is large [33]. In Section 4 we define the additional interactions $\hat{V}_1$ and $\hat{V}_2$ which explicitly break SU(2) symmetry. Then we calculate the matrix elements of these interactions by the eigen states of the sub-lattice magnetizations.

In Section 5, we calculate the ground state when the Hamiltonian contains the symmetry breaking interactions in addition to the effective Hamiltonian. We give the Hamiltonian in the continuous approximation which is reliable when the lattice size is large. Then obtaining the two-dimensional partial differential equation and solving it, we find the ground state. Also we discuss the conditions for our method to be reasonable. In Section 6, we calculate the correlation functions due to the degenerate states. Here one should notice that we have two kinds of the correlation functions because there are two kinds of Nambu-Goldstone mode in SU(2) symmetry. In Section 7, we employ linear spin wave theory [28] to calculate the correlation functions due to Nambu-Goldstone mode, which keep the cluster property. The violation of the cluster property due to the degenerate states, therefore, could be observable only when the former is smaller than the latter.

In Section 8, we discuss, using the correlation functions presented in Sections 6 and 7, the violation of the cluster property in the correlation functions due to the degenerate states. Then we numerically estimate the violation when the lattice size N is $10^{20} = (10^{10})^2$. This size is determined by considering that the molecular distance is the order of 10^{-10} m and the length of the macroscopic material is the order of 1 m. From our results we conclude that in Heisenberg antiferromagnet correlation functions due to the degenerate states violate the cluster property, and that the magnitude of the violation qualitatively differs from that in XXZ antiferromagnet. The final section is devoted to summary and discussion.

2. Heisenberg Antiferromagnet on Square Lattice

The model we study here is the spin 1/2 Heisenberg antiferromagnet on the square lattice. A site with a symbol i is defined by a pair of integers (i_x, i_y), by which positions are given on the $L_x \times L_y$ square lattice.

$$i \equiv i_x + i_y L_x, \quad i_x = 0,1,2,\cdots,L_x - 1, \quad i_y = 0,1,2,\cdots,L_y - 1. \tag{1}$$

The lattice size N is given by $N = L_x L_y$. The Hamiltonian $\hat{H}$ in our study is given by

$$\hat{H} = \sum_{(i,j)} \vec{\hat{S}}_i \cdot \vec{\hat{S}}_j \tag{2}$$

Here $\vec{\hat{S}}_i$ is the spin operator at the site i and (i,j) denotes the nearest neighbor pair on the square lattice. In order to make $\hat{S}_i^z$ a diagonal matrix, the basis state is defined by $|s_1, s_2, \cdots, s_N\rangle$ where $\hat{S}_i^z |s_i\rangle = |s_i\rangle s_i$ with $s_i = 1/2$ or $-1/2$. When we represent the state $|(+1/2)_i\rangle \beta_i + |(-1/2)_i\rangle \gamma_i$ at each site by the vector $[\beta_i, \gamma_i]^{\mathrm{T}}$, the spin operators with x-, y- and z-components are given by the Pauli matrix σ^l,

$$\hat{S}_i^x = -\sigma^y/2, \quad \hat{S}_i^y = \sigma^x/2, \quad \hat{S}_i^z = \sigma^z/2. \tag{3}$$

The reason for this assignment is that in the analysis by linear spin wave theory [28] we would like to use the real matrix element on $\hat{S}_i^y$ [25].

Since the Hamiltonian (2) has the SU(2) symmetry, the total spin operators commute with $\hat{H}$,

$$\left[\hat{H}, \hat{S}^\alpha\right] = 0, \quad \hat{S}^\alpha = \sum_i \hat{S}_i^\alpha, \quad \alpha = x, y, z. \tag{4}$$

Therefore the energy eigen state of $\hat{H}$ with the energy eigen value $E_{J,M}$ has also the quantum numbers J and M of SU(2) symmetry for the whole lattice.

$$\begin{aligned} &\hat{H}|\psi_{J,M}\rangle = |\psi_{J,M}\rangle E_{J,M}, \\ &\left(\vec{\hat{S}}\right)^2 |\psi_{J,M}\rangle = |\psi_{J,M}\rangle J(J+1), \\ &\hat{S}^z |\psi_{J,M}\rangle = |\psi_{J,M}\rangle M. \end{aligned} \tag{5}$$

For this antiferromagnet on the square lattice we divide the whole lattice into two kinds of sub-lattices, which are called A sub-lattice and B sub-lattice. In order to give a definition of A sub-lattice (B sub-lattice), we use integers i_x and i_y for the site i to introduce a symbol P_i

$$P_i \equiv \operatorname{mod}(i_x + i_y, 2) = \begin{cases} 0 & \text{for } i \in \text{A sub-lattice} \\ 1 & \text{for } i \in \text{B sub-lattice} \end{cases} \tag{6}$$

Using P_i we introduce the spin operators on each sub-lattice,

$$\begin{aligned} &\hat{S}_U^\alpha \equiv \sum_{i \in \text{U sub-lattice}} \hat{S}_i^\alpha = \sum_i \frac{1 + \varepsilon_U (-1)^{P_i}}{2} \hat{S}_i^\alpha, \\ &\varepsilon_U = \begin{cases} +1 & (\text{U} = \text{A}) \\ -1 & (\text{U} = \text{B}) \end{cases} \end{aligned} \tag{7}$$

The eigen state of $\left(\vec{\hat{S}}_U\right)^2$ and $\hat{S}_U^z$ will be used in the next section to construct $|\psi_{J,M}\rangle$ in (5).

3. Effective Hamiltonian

The model we study is based on the one where the spin system on each sub-lattice is ferromagnet [26] and is formulated by

$$\left(\hat{\vec{S}}_U\right)^2 \left|\Psi_{J,M}\right\rangle = \left|\Psi_{J,M}\right\rangle J_U\left(J_U+1\right). \tag{8}$$

We can insist that J_A is the order of the size N, because the semi-classical order is found in the Heisenberg antiferromagnet. Note that we have $J_B = J_A$ by the translational symmetry of $\hat{H}$.

The effective Hamiltonian [26] is

$$\hat{H}_{eff} \equiv E_0 + \frac{1}{2\chi_0 N}\left(\hat{\vec{S}}\right)^2. \tag{9}$$

Here E_0 is the lowest energy with $J=0$ and χ_0 is the uniform susceptibility. The energy eigen values of the state $\left|\Psi_{J,M}\right\rangle$ are given by

$$\hat{H}_{eff}\left|\Psi_{J,M}\right\rangle = \left|\Psi_{J,M}\right\rangle E_{J,M},$$

$$E_{J,M} = E_0 + \frac{J\left(J+1\right)}{2\chi_0 N} \equiv E_J\left(N\right), \tag{10}$$

$$J = 0,1,2,\cdots, \quad M = -J,-J+1,\cdots,0,1,\cdots,J-1,J.$$

Before proceeding to study $\left|\Psi_{J,M}\right\rangle$, let us make numerical calculation by the exact diagonalization on the quite small lattices, so that we can confirm the assumptions for $\hat{H}_{eff}$. First, we calculate the energy $E_J\left(N\right)$ on lattices of size N = 20, 26, 32 and 36 to examine the energy postulate in (10). The results are shown in **Figure 1**, where we find the remarkable agreement between our data and the postulate. Next, we calculate the squared magnitude of the ferromagnet on A sub-lattice with the lattice size N, which is given by

$$S_A^2\left(J,N\right) \equiv \left\langle\Psi_{J,M}\right|\left(\hat{\vec{S}}_A\right)^2\left|\Psi_{J,M}\right\rangle. \tag{11}$$

when the large ferromagnet is realized on A sub-lattice $S_A^2\left(J,N\right)$ should be proportional to N^2 with little dependence on J. **Figure 2** plots $S_A^2\left(J,N\right)$, which we obtain by the exact diagonalization on lattices of size N = 20, 26, 32 and 36 for J = 0 − 8, versus $J\left(J+1\right)$. In this figure we see that $S_A^2\left(J,N\right)$ scarcely depend on $J\left(J+1\right)$ as is expected. Also our data show that $S_A^2\left(J=0,N\right)/\left(N/2\right)^2$ are 0.258, 0.234, 0.219 and 0.210 for N = 20, 26, 32 and 36, respectively. Thus we see the little dependence of this quantity on N, which is consistent with (8). Sandvik [34] made the extensive study on this dependence to obtain the value on the infinitely large lattice accurately.

In order to construct $\left|\Psi_{J,M}\right\rangle$ we use a state $\left|J_U,M_U\right\rangle$, which is the eigen state of $\left(\hat{\vec{S}}_U\right)^2$ and $\hat{S}_U^z$, defined on the U sub-lattice,

$$\left(\hat{\vec{S}}_U\right)^2\left|J_U,M_U\right\rangle = \left|J_U,M_U\right\rangle J_U\left(J_U+1\right),$$

$$\hat{S}_U^z\left|J_U,M_U\right\rangle = \left|J_U,M_U\right\rangle M_U,$$

$$\hat{S}_U^{\pm}\left|J_U, M_U\right\rangle = \left|J_U, M_U \pm 1\right\rangle \varepsilon_U \sqrt{J_U\left(J_U+1\right) - M_U\left(M_U \pm 1\right)},$$
$$\hat{S}_U^{\pm} \equiv \hat{S}_U^{y} \mp i\hat{S}_U^{x}. \tag{12}$$

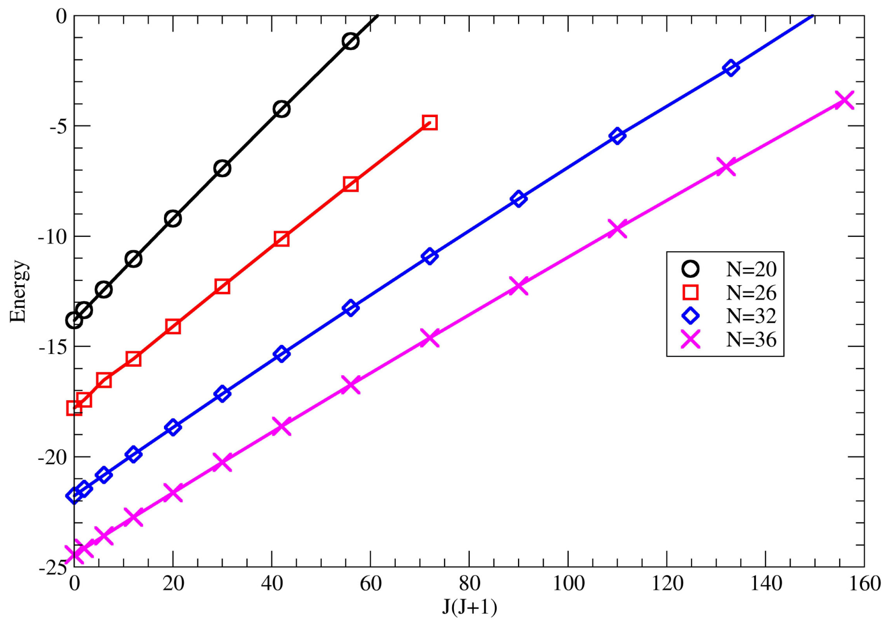

Figure 1. The energy $E_J(N)$ defined in (10) for the spin 1/2 Heisenberg antiferromagnet on the square lattice. In the horizontal axis we show $J(J+1)$. These results are calculated by the exact diagonalization on N = 20, 26, 32 and 36 lattices. The solid lines are results of the least square fit.

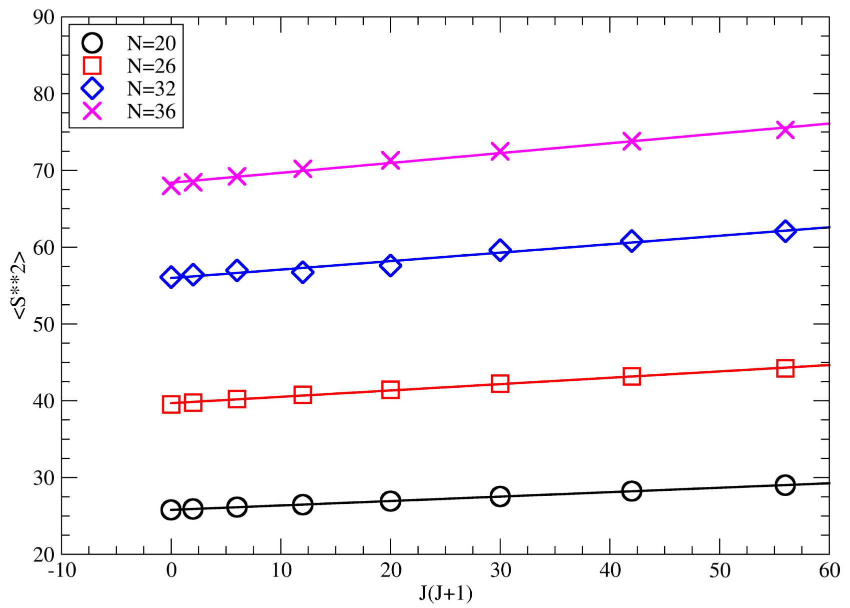

Figure 2. The expectation value $S_A^2(J,N)$ defined by (11) for the spin 1/2 Heisenberg antiferromagnet on the square lattice. In the horizontal axis we show $J(J+1)$. These results are calculated by the exact diagonalization on N = 20, 26, 32 and 36 lattices. The solid lines are results of the least square fit.

Here ε_U is defined in (7). Note that

$$\left(\hat{\vec{S}}\right)^2 = \left(\hat{\vec{S}}_A\right)^2 + \left(\hat{\vec{S}}_B\right)^2 + \hat{S}_A^+\hat{S}_B^- + \hat{S}_A^-\hat{S}_B^+ + 2\hat{S}_A^z\hat{S}_B^z. \tag{13}$$

Then the eigen state of $\hat{H}_{eff}$ should be the solution of the following equation

$$\left\{\left(\hat{\vec{S}}_A\right)^2 + \left(\hat{\vec{S}}_B\right)^2 + \hat{S}_A^+\hat{S}_B^- + \hat{S}_A^-\hat{S}_B^+ + 2\hat{S}_A^z\hat{S}_B^z\right\}\left|\Psi_{J,M}\right\rangle = \left|\Psi_{J,M}\right\rangle J(J+1). \tag{14}$$

In general the eigen state $\left|\Psi_{J,M}\right\rangle$ can be represented by a linear combination of products of $\left|J_A, M_A\right\rangle$ and $\left|J_B, M_B\right\rangle$,

$$\left|\Psi_{J,M}\right\rangle = \sum_{M_A+M_B=M}\left|J_A, M_A\right\rangle\left|J_B, M_B\right\rangle C(J, M : J_A, M_A, J_B, M_B). \tag{15}$$

As described in textbooks on quantum mechanics, $C(J, M : J_A, M_A, J_B, M_B)$ is the Clebsch-Gordan coefficient and it is obtained by algebraic calculations.

In our study $J_A(=J_B)$ is quite large, which is $O(N)$. In addition we assume that $J \ll J_A$ because there exists the suppression factor $J(J+1)/(2\chi_0)$ in the Hamiltonian. We do not impose, however, any restriction on the relation of J^2 and M^2. In this case M_A^2 and M_B^2 are $O(J_A^2)$, which will be understood in the following discussion. Since M^2/J_A^2 is quite small in our assumption we approximate the equation of the Clebsh-Gordan coefficients by neglecting the terms of M^2/J_A^2. The calculation of these coefficients in this approximation has been extensively studied by Sprung, van Dijk, Martorell and Criger [33], which we follow here. For a fixed M we introduce a variable Z defined by

$$Z \equiv M_A - M/2, \quad M_A = Z + M/2, \quad M_B = -Z + M/2. \tag{16}$$

Here Z is an integer when M is an even integer, while Z is a half-integer when M is an odd integer. For a fixed $J_A(=J_B)$ we can use $\left|Z\right\rangle$ to represent $\left|\Psi_{J,M}\right\rangle$. Let us start with, instead of (15),

$$\left|\Psi_{J,M}\right\rangle = \sum_Z \left|Z\right\rangle \psi_Z. \tag{17}$$

Here ψ_Z denotes the coefficient for the state $\left|Z\right\rangle$. The non-zero off-diagonal matrix element in the calculation is

$$\begin{aligned} a_Z &\equiv \left\langle Z+1\right|\hat{S}_A^+\hat{S}_B^-\left|Z\right\rangle \\ &= \left\langle J_A, M_A+1\right|\left\langle J_B, M_B-1\right|\hat{S}_A^+\hat{S}_B^-\left|J_A, M_A\right\rangle\left|J_B, M_B\right\rangle \\ &= \varepsilon_A\varepsilon_B\sqrt{J_A(J_A+1) - M_A(M_A+1)}\sqrt{J_B(J_B+1) - M_B(M_B-1)}. \end{aligned} \tag{18}$$

Since the matrix element is real we also have

$$a_{Z-1} = \left\langle Z\right|\hat{S}_A^+\hat{S}_B^-\left|Z-1\right\rangle^* = \left\langle Z-1\right|\hat{S}_A^-\hat{S}_B^+\left|Z\right\rangle. \tag{19}$$

For $\hat{S}_A^z\hat{S}_B^z$, we obtain the diagonal matrix element

$$\begin{aligned} b_Z &\equiv \left\langle Z\right|\hat{S}_A^z\hat{S}_B^z\left|Z\right\rangle \\ &= \left\langle J_A, M_A\right|\left\langle J_B, M_B\right|\hat{S}_A^z\hat{S}_B^z\left|J_A, M_A\right\rangle\left|J_B, M_B\right\rangle \\ &= M_AM_B = -Z^2 + \frac{M^2}{4}. \end{aligned} \tag{20}$$

Then the eigen equation for $\left|\Psi_{J,M}\right\rangle$ led from (14) is

$$\sum_Z \left\{ |Z\rangle 2J_A(J_A+1) + |Z+1\rangle a_Z + |Z-1\rangle a_{Z-1} + |Z\rangle 2b_Z \right\} \psi_Z = \sum_Z |Z\rangle J(J+1)\psi_Z. \tag{21}$$

This equation yields an equation of ψ_Z,

$$a_{Z-1}\psi_{Z-1} + a_Z\psi_{Z+1} = \left\{ J(J+1) - \left[2J_A(J_A+1) + 2b_Z \right] \right\} \psi_Z. \tag{22}$$

From (18) we obtain, based on the approximation to neglect the term of M/J_A,

$$a_Z \sim (-1)\left\{ \left[J_A(J_A+1) - \frac{M^2}{4} \right] - Z^2 - Z - \frac{M^2x^2}{2(1-x^2)} \right\}. \tag{23}$$

Here we introduce a variable x,

$$x \equiv Z\Delta x, \quad \Delta x = \frac{1}{\sqrt{J_A(J_A+1) - \frac{M^2}{4}}}. \tag{24}$$

Applying (20) and (23) to (22) the equation for ψ_Z becomes

$$\begin{aligned} &-\left\{ J_A(J_A+1) - \frac{M^2}{4} - Z^2 - \frac{M^2x^2}{2(1-x^2)} \right\} (\psi_{Z+1} + \psi_{Z-1}) + Z(\psi_{Z+1} - \psi_{Z-1}) \\ &= \left[J(J+1) - 2J_A(J_A+1) + 2Z^2 - \frac{M^2}{2} \right] \psi_Z. \end{aligned} \tag{25}$$

Let us replace the variable Z by x in (24). In addition we introduce $\phi(x) \equiv \psi_Z / \sqrt{\Delta x}$ in order to obtain the smooth function of x. Then we have

$$\begin{aligned} &-\left\{ \frac{1}{(\Delta x)^2}(1-x^2) - \frac{M^2x^2}{2(1-x^2)} \right\} \left[\phi(x+\Delta x) + \phi(x-\Delta x) \right] \\ &+ \frac{x}{\Delta x}\left[\phi(x+\Delta x) - \phi(x-\Delta x) \right] \\ &= \left\{ J(J+1) - \frac{2}{(\Delta x)^2}(1-x^2) - M^2 \right\} \phi(x). \end{aligned} \tag{26}$$

Expanding $\phi(x \pm \Delta x)$ we obtain

$$\begin{aligned} &-\left\{ (1-x^2) - (\Delta x)^2 \frac{M^2x^2}{2(1-x^2)} \right\} \frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}\phi(x)}{\mathrm{d}x} \\ &= \left\{ J(J+1) - \frac{M^2}{1-x^2} \right\} \phi(x). \end{aligned} \tag{27}$$

when $(\Delta x)^2 M^2 \ll 1$, which means $M^2 \ll J_A^2$, we can neglect the term with $(\Delta x)^2$. Finally we obtain the equation

$$-(1-x^2)\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}\phi(x)}{\mathrm{d}x} = \left\{ J(J+1) - \frac{M^2}{1-x^2} \right\} \phi(x). \tag{28}$$

The solution of this equation is the associated Legendre function $P_J^M(x)$ ($M > 0$),

$$P_J^M(x) = \frac{\left(1-x^2\right)^{M/2}}{J!2^J} \frac{\mathrm{d}^{J+M}\left(x^2-1\right)^J}{\mathrm{d}x^{J+M}}. \tag{29}$$

Taking the normalization into account we thus obtain

$$\phi_{J,M}(x) = \sqrt{\frac{2J+1}{2}} \sqrt{\frac{(J-M)!}{(J+M)!}} P_J^M(x). \tag{30}$$

Then the state $|\Psi_{J,M}\rangle$ is given by

$$|\Psi_{J,M}\rangle = \sum_Z |Z\rangle \phi_{J,M}(x = Z\Delta x)\sqrt{\Delta x}. \tag{31}$$

4. Additional Interactions

In this section we introduce two symmetry breaking interactions which are necessary to obtain the unique ground state. In SU(2) symmetry we have two independent generators among three generators, $\hat{Q}_\alpha = \hat{S}^\alpha (\alpha = x, y, z)$, $\hat{S}^\alpha$ being defined in (4). We adopt $\hat{Q}_y$ and $\hat{Q}_z$ as the independent generators. The additional interactions $\hat{V}_1$ and $\hat{V}_2$ have to break the symmetry related to these operators. Since we impose that $\hat{V}_1$ breaks the symmetry on $\hat{Q}_y$, we assume that $\hat{V}_1$ is given by

$$\hat{V}_1 \equiv g_1\left(\hat{Q}_z\right)^2 = g_1\left(\sum_i \hat{S}_i^z\right)^2 = g_1\left(\hat{S}^z\right)^2 \quad (g_1 > 0). \tag{32}$$

The parameter g_1 should be positive because states with large M should be suppressed. For its magnitude we do not impose any severe constraint because $\hat{V}_1$ commutes with the Hamiltonian $\hat{H}$. The reason to choose the squared operator is that this interaction gives a non-trivial effect to the correlation function, which will be described in Section 6. Another interaction $\hat{V}_2$, which has been already introduced in the study of XXZ antiferromagnet [25], is given by

$$\hat{V}_2 \equiv -g_2 \sum_i (-1)^{P_i} \hat{S}_i^y = -g_2\left(\hat{S}_A^y - \hat{S}_B^y\right) \quad (1 \gg g_2 > 0). \tag{33}$$

This interaction breaks the symmetry on $\hat{Q}_z$. The positivity of g_2 is conventional because it changes when the phase of states changes. The reason why $|g_2|$ should be small is that, since $\left[\hat{H}, \hat{V}_2\right] \neq 0$, this interaction modifies the quasi-degenerate states by inducing the magnon states.

The Hamiltonian $\hat{H}_V$ is the sum of the Hamiltonian (2) and these interactions, which is given by

$$\hat{H}_V = \hat{H} + \hat{V}_1 + \hat{V}_2. \tag{34}$$

Note that, because $\left[\hat{Q}_\alpha, \hat{H}_V\right] = \left[\hat{Q}_\alpha, \hat{V}_1 + \hat{V}_2\right] \neq 0$ for $\alpha = x, y$ and z, we completely break SU(2) symmetry by these interactions so that we can obtain the unique ground state with parameters g_1 and g_2. Replacing $\hat{H}$ by the effective Hamiltonian $\hat{H}_{eff}$ defined in (9), we can introduce $\hat{H}_{eff,V}$,

$$\hat{H}_{eff,V} = \hat{H}_{eff} + \hat{V}_1 + \hat{V}_2. \tag{35}$$

Here we can express the whole Hamiltonian only by the spin operators $\hat{S}_U^\alpha$ on U sub-lattice (U=A,B). Therefore the eigen states of $\hat{H}_{eff,V}$ are constructed by the set of states $\{|J_A, M_A\rangle |J_B, M_B\rangle\}$.

The operation of $\hat{V}_1$ to the state is easily calculated since, from (5), this interaction does not change the state $|\Psi_{J,M}\rangle$,

$$\hat{V}_1 |\Psi_{J,M}\rangle = |\Psi_{J,M}\rangle g_1 M^2. \tag{36}$$

The interaction $\hat{V}_2$, on the other hand, changes the state $|\Psi_{J,M}\rangle$ since it is represented as

$$\hat{V}_2 = -\frac{g_2}{2}\left[\left(\hat{S}_A^+ - \hat{S}_B^+\right) + \left(\hat{S}_A^- - \hat{S}_B^-\right)\right]. \tag{37}$$

By the algebraic argument on the spin-one operator $\hat{S}_A^+$, we find that

$$\hat{S}_A^+ |\Psi_{J,M}\rangle = |\Psi_{J+1,M+1}\rangle c_+^A + |\Psi_{J,M+1}\rangle c_0^A + |\Psi_{J-1,M+1}\rangle c_-^A. \tag{38}$$

Let us calculate these coefficients $c_\pm^A$ and c_0^A. Since $|\Psi_{J,M}\rangle$ is expressed by the function $\phi_{J,M}(x)$ as shown in (31), c_+^A is given by

$$\begin{aligned} c_+^A &= \langle \Psi_{J+1,M+1} | \hat{S}_A^+ | \Psi_{J,M} \rangle \\ &= \sum_{Z,Z'} \phi_{J+1,M+1}(x') \langle Z' | \hat{S}_A^+ | Z \rangle \phi_{J,M}(x) \left(\sqrt{\Delta x}\right)^2 \\ &= \sum_{Z,Z'} \phi_{J+1,M+1}(x') \langle Z' | Z+1 \rangle \sqrt{J_A(J_A+1) - M_A(M_A+1)} \phi_{J,M}(x) \Delta x \\ &\sim \sum_{Z} \phi_{J+1,M+1}(x+\Delta x) J_A \sqrt{1-x^2} \phi_{J,M}(x) \Delta x. \end{aligned} \tag{39}$$

For the associated Legendre function we have the following relation,

$$(2J+1)\sqrt{1-x^2} P_J^M(x) = P_{J+1}^{M+1}(x) - P_{J-1}^{M+1}(x). \tag{40}$$

Then from (30) we have

$$\begin{aligned} &\sqrt{1-x^2}\phi_{J,M}(x) \\ &= \sqrt{\frac{2J+1}{2J+3}} \frac{\sqrt{(J+M+2)(J+M+1)}}{2J+1} \phi_{J+1,M+1}(x) \\ &\quad - \sqrt{\frac{2J+1}{2J-1}} \frac{\sqrt{(J-M)(J-M-1)}}{2J+1} \phi_{J-1,M+1}(x). \end{aligned} \tag{41}$$

Assuming $J_A \gg J \gg |M| \gg 1$, which we can realize through the parameters g_1 and g_2, the above relation is replaced by

$$\sqrt{1-x^2}\phi_{J,M}(x) \sim \frac{1}{2}\phi_{J+1,M+1}(x) - \frac{1}{2}\phi_{J-1,M+1}(x). \tag{42}$$

Thus we obtain $c_+^A \sim J_A/2$. Applying the same discussion to c_0^A and c_-^A, we finally obtain that

$$c_+^A \sim +\frac{J_A}{2},\ c_0^A \sim 0,\ c_-^A \sim -\frac{J_A}{2}. \tag{43}$$

Similarly we obtain the coefficients for $\hat{S}_B^+$,

$$c_+^B \sim -\frac{J_A}{2},\ c_0^B \sim 0,\ c_-^B \sim +\frac{J_A}{2}. \tag{44}$$

Note that we adopt the convention in (12) for the phase on the states $|J_A, M_A\rangle$ and $|J_B, M_B\rangle$. From (43) and (44) we have

$$\left(\hat{S}_A^+ - \hat{S}_B^+\right)|\Psi_{J,M}\rangle \sim |\Psi_{J+1,M+1}\rangle J_A + |\Psi_{J-1,M+1}\rangle(-J_A). \tag{45}$$

As for $\hat{S}_A^-$, we use the relation

$$\langle\Psi_{J\pm1,M+1}|\left(\hat{S}_A^+ - \hat{S}_B^+\right)|\Psi_{J,M}\rangle^* = \langle\Psi_{J,M}|\left(\hat{S}_A^- - \hat{S}_B^-\right)|\Psi_{J\pm1,M+1}\rangle. \tag{46}$$

Then we obtain

$$\begin{aligned}\hat{V}_2|\Psi_{J,M}\rangle &= -g_2\left(\hat{S}_A^y - \hat{S}_B^y\right)|\Psi_{J,M}\rangle \\ &\sim -\frac{g_2}{2}J_A\left\{|\Psi_{J+1,M+1}\rangle - |\Psi_{J-1,M+1}\rangle + |\Psi_{J-1,M-1}\rangle - |\Psi_{J+1,M-1}\rangle\right\}.\end{aligned} \tag{47}$$

From this expression we see that the interaction $\hat{V}_2$ does not change the even-odd property of $J-M$.

5. Ground State

In the previous section we presented the matrix elements of the effective Hamiltonian with additional interactions. Using these matrix elements we now solve the eigen equation to find the ground state of $\hat{H}_{eff,V}$,

$$\hat{H}_{eff,V}|\Psi_V\rangle = |\Psi_V\rangle E_{\Psi_V},\ |\Psi_V\rangle = \sum_{J>0,M}|\Psi_{J,M}\rangle\Phi_{J,M}. \tag{48}$$

Here E_{Ψ_V} is the energy eigen value and $\Phi_{J,M}$ denotes the coefficient. Since $\hat{H}_{eff}$ gives the diagonal element $J(J+1)/(2\chi_0 N)$ for $|\Psi_{J,M}\rangle$ we have

$$\begin{aligned}\hat{H}_{eff,V}|\Psi_{J,M}\rangle &\sim \left[\frac{J(J+1)}{2\chi_0 N} + g_1 M^2\right]|\Psi_{J,M}\rangle \\ &-\frac{g_2}{2}J_A\left[|\Psi_{J+1,M+1}\rangle - |\Psi_{J-1,M+1}\rangle + |\Psi_{J-1,M-1}\rangle - |\Psi_{J+1,M-1}\rangle\right].\end{aligned} \tag{49}$$

From (48) and (49) we obtain the equation for $\Phi_{J,M}$, which is

$$\begin{aligned}&\left[\frac{J(J+1)}{2\chi_0 N} + g_1 M^2\right]\Phi_{J,M} - \frac{g_2}{2}J_A\left\{\Phi_{J-1,M-1} - \Phi_{J+1,M-1} + \Phi_{J+1,M+1} - \Phi_{J-1,M+1}\right\} \\ &= E_{\Psi_V}\Phi_{J,M}.\end{aligned} \tag{50}$$

Since the even-odd property of $J-M$ is kept in this equation, we examine the equation only for even values of $J-M$. We introduce $\tilde{\Phi}_{J,M}$ by

$$\Phi_{J,M} = (-1)^{(J-M)/2}\tilde{\Phi}_{J,M}. \tag{51}$$

Then the eigen equation becomes

$$\begin{aligned}&\left[\frac{J(J+1)}{2\chi_0 N} + g_1 M^2\right]\tilde{\Phi}_{J,M} - \frac{g_2}{2}J_A\left\{\tilde{\Phi}_{J-1,M-1} + \tilde{\Phi}_{J+1,M-1} + \tilde{\Phi}_{J-1,M+1} + \tilde{\Phi}_{J+1,M+1}\right\} \\ &= E_{\Psi_V}\tilde{\Phi}_{J,M}.\end{aligned} \tag{52}$$

Let us introduce new variables by

$$s \equiv J\Delta s,\ \ t \equiv M\Delta t,\ \ \phi(s,t) \equiv \tilde{\Phi}_{J,M} \big/ \sqrt{\Delta s \Delta t}. \tag{53}$$

The difference values Δs and Δt will be fixed later. Also using the staggered or the order parameter m^+ [26], J_A is estimated by

$$J_A = m^+ \frac{N}{2}. \tag{54}$$

Then we obtain the equation

$$\begin{aligned}&\left[\frac{1}{2\chi_0 N(\Delta s)^2}s^2 + \frac{g_1}{(\Delta t)^2}t^2\right]\phi(s,t) - \frac{g_2 m^+ N}{4}\{\phi(s-\Delta s,t-\Delta t)\\&+\phi(s+\Delta s,t-\Delta t)+\phi(s-\Delta s,t+\Delta t)+\phi(s+\Delta s,t+\Delta t)\} = E_{\Psi_V}\phi(s,t).\end{aligned} \tag{55}$$

Making the continuous approximation to expand $\phi(s\pm\Delta s, t\pm\Delta t)$ by the partial differential, we have the equation

$$\begin{aligned}&\left[\frac{1}{2\chi_0 N(\Delta s)^2}s^2 + \frac{g_1}{(\Delta t)^2}t^2\right]\phi(s,t) - \frac{g_2 m^+ N}{2}\left\{(\Delta s)^2\frac{\partial^2\phi}{\partial s^2} + (\Delta t)^2\frac{\partial^2\phi}{\partial t^2}\right\}\\&=\left(E_{\Psi_V} + g_2 m^+ N\right)\phi(s,t).\end{aligned} \tag{56}$$

In this equation we set

$$(\Delta s)^2 = \frac{1}{\sqrt{\chi_0 m^+ g_2 N^2}},\ \ (\Delta t)^2 = \sqrt{\frac{2g_1}{m^+ g_2 N}}. \tag{57}$$

As a result the eigen equation becomes

$$\begin{aligned}&\sqrt{\frac{g_2 m^+}{4\chi_0}}\left\{s^2 - \frac{\partial^2}{\partial s^2}\right\}\phi(s,t) + \frac{1}{\sqrt{2}}\sqrt{m^+ g_1 g_2 N}\left\{t^2 - \frac{\partial^2}{\partial t^2}\right\}\phi(s,t)\\&=\left(E_{\Psi_V} + g_2 m^+ N\right)\phi(s,t).\end{aligned} \tag{58}$$

The eigen solution of (58) is

$$\phi_{l_s,l_t}(s,t) = N_{l_s}H_{l_s}(s)N_{l_t}H_{l_t}(t)\mathrm{e}^{-s^2/2-t^2/2}. \tag{59}$$

Here $H_l(u)$ denotes the Hermite polynomial and N_l is the normalization factor. We should notice that there are some constraints on the wave function and the parameters. The constraint on the wave function comes from the positivity of $J = s/\Delta s$. By this constraint we impose that $\phi(s,t)=0$ for $s=0$ by the continuity of $\phi(s,t)$, because $\phi(s,t)=0$ for $s<0$. Therefore the quantum number l_s should be an odd integer. The energy eigen value in (58) is then given by, with $l_s = 1,3,5,\cdots$, and $l_t = 0,1,2,3,\cdots$,

$$E_{\Psi_V}(l_s,l_t) = -g_2 m^+ N + \sqrt{\frac{g_2 m^+}{4\chi_0}}(2l_s+1) + \frac{1}{\sqrt{2}}\sqrt{m^+ g_1 g_2 N}(2l_t+1). \tag{60}$$

Note that the values of the variables s and t are $O(1)$ because the solution (59) does not contain the parameters N, g_1 and g_2. The requirement $J \gg |M|$ is then satisfied by the condition

$$\left(\frac{\Delta s}{\Delta t}\right)^2 \ll \left(\frac{s}{t}\right)^2 \sim 1. \tag{61}$$

Thus the conditions we impose are

$$(\Delta s)^2 \ll (\Delta t)^2 \ll 1. \tag{62}$$

Here the second condition is necessary because of the continuous condition. The ground state $|G_V\rangle$ and the ground energy E_{G_V} are then given by

$$\begin{aligned} |G_V\rangle &= \sum_{J,M} |\Psi_{J,M}\rangle (-1)^{(J-M)/2} \phi_{l_s=1,l_t=0}(s = J\Delta s, t = M\Delta t)\sqrt{\Delta s \Delta t}, \\ E_{G_V} &= E_{\Psi_V}(l_s = 1, l_t = 0). \end{aligned} \tag{63}$$

6. Correlation Functions Due to Quasi-Degenerate States

In this and the next sections we calculate the spin correlation function $\Delta F^{\alpha\beta}(i,j)$, which is defined by

$$\begin{aligned} \Delta F^{\alpha\beta}(i,j) &\equiv F^{\alpha\beta}(i,j) - F^{\alpha}(i)F^{\beta}(j), \\ F^{\alpha\beta}(i,j) &\equiv \langle G_V|(-1)^{P_i}\hat{S}_i^{\alpha}(-1)^{P_j}\hat{S}_j^{\beta}|G_V\rangle, \\ F^{\alpha}(i) &\equiv \langle G_V|(-1)^{P_i}\hat{S}_i^{\alpha}|G_V\rangle. \end{aligned} \tag{64}$$

Here $|G_V\rangle$ is given by (63). As described in the previous work [25], we can divide the correlation function into the contributions $\Delta F_{DS}^{\alpha\beta}(i,j)$ and $\Delta F_{NG}^{\alpha\beta}(i,j)$, which originate from the quasi-degenerate states and Nambu-Goldstone mode, respectively.

$$\Delta F^{\alpha\beta}(i,j) = \Delta F_{DS}^{\alpha\beta}(i,j) + \Delta F_{NG}^{\alpha\beta}(i,j). \tag{65}$$

Since Nambu-Goldstone mode is realized in the operators of $\hat{S}_i^x$ and $\hat{S}_i^z$, we set $\alpha = \beta = x$ or z.

In this section we calculate the correlation functions due to the quasi-degenerate states, which are $\Delta F_{DS}^{xx}(i,j)$ and $\Delta F_{DS}^{zz}(i,j)$. Since we examine the contribution due to the quasi-degenerate states, we assume that we can replace the contribution of the local operator $\hat{S}_i^{\alpha}$ by the uniform contribution of the global operator $\hat{S}_U^{\alpha}/(N/2)$. Therefore we have the following replacement.

$$\hat{S}_i^{\alpha} \to \frac{1}{2}\left[1+(-1)^{P_i}\right]\hat{S}_A^{\alpha}/(N/2) + \frac{1}{2}\left[1-(-1)^{P_i}\right]\hat{S}_B^{\alpha}/(N/2). \tag{66}$$

By this replacement we obtain

$$\begin{aligned} F_{DS}^{\alpha}(i) &= \langle G_V|(-1)^{P_i}\hat{S}_i^{\alpha}|G_V\rangle_{DS} \\ &\to \langle G_V|(-1)^{P_i}\left\{\frac{1+(-1)^{P_i}}{2(N/2)}\hat{S}_A^{\alpha} + \frac{1-(-1)^{P_i}}{2(N/2)}\hat{S}_B^{\alpha}\right\}|G_V\rangle_{DS} \\ &= \frac{1}{N}\left\{\langle G_V|\left(\hat{S}_A^{\alpha} - \hat{S}_B^{\alpha}\right)|G_V\rangle_{DS} + (-1)^{P_i}\langle G_V|\left(\hat{S}_A^{\alpha} + \hat{S}_B^{\alpha}\right)|G_V\rangle_{DS}\right\} \\ &= \frac{1}{N}\left\{\langle G_V|\left(\hat{S}_A^{\alpha} - \hat{S}_B^{\alpha}\right)|G_V\rangle_{DS} + (-1)^{P_i}\langle G_V|\hat{S}^{\alpha}|G_V\rangle_{DS}\right\}. \end{aligned} \tag{67}$$

$$
\begin{aligned}
F_{DS}^{\alpha\alpha}(i,j) &= \langle G_V | (-1)^{P_i} \hat{S}_i^{\alpha} (-1)^{P_j} \hat{S}_j^{\alpha} | G_V \rangle_{DS} \\
&\to \frac{1}{N^2} \Big\{ \langle G_V | \big(\hat{S}_A^{\alpha} - \hat{S}_B^{\alpha}\big)^2 | G_V \rangle_{DS} + \langle G_V | \big(\hat{S}^{\alpha}\big)^2 | G_V \rangle_{DS} (-1)^{P_i} (-1)^{P_j} \\
&\quad + \langle G_V | \big(\hat{S}_A^{\alpha} - \hat{S}_B^{\alpha}\big) \hat{S}^{\alpha} | G_V \rangle_{DS} \big[(-1)^{P_i} + (-1)^{P_j} \big] \Big\}.
\end{aligned}
\tag{68}
$$

6.1. $\Delta F_{DS}^{xx}(i,j)$

Let us consider the first term of $\Delta F_{DS}^{x}(i)$ in (67) with $\alpha = x$. Repeating the discussion in Section 4, we have

$$
\big(\hat{S}_A^x - \hat{S}_B^x\big) |\Psi_{J,M}\rangle \sim \frac{i}{2} J_A \big(|\Psi_{J+1,M+1}\rangle - |\Psi_{J-1,M+1}\rangle - |\Psi_{J-1,M-1}\rangle + |\Psi_{J+1,M-1}\rangle \big). \tag{69}
$$

Notifying the convention (12) for the phase on the state $|J_A, M_A\rangle$ and the state $|J_B, M_B\rangle$, we have

$$
\begin{aligned}
\big(\hat{S}_A^x - \hat{S}_B^x\big) |G_V\rangle_{DS} &\sim \frac{i}{2} J_A \sum_{J,M} |\Psi_{J,M}\rangle (-1)^{(J-M)/2} \\
&\quad \times \big\{ \tilde{\Phi}_{J-1,M-1} + \tilde{\Phi}_{J+1,M-1} - \tilde{\Phi}_{J+1,M+1} - \tilde{\Phi}_{J-1,M+1} \big\} \\
&\to \frac{i}{2} J_A \sum_{J,M} |\Psi_{J,M}\rangle (-1)^{(J-M)/2} (-4)(\Delta t) \frac{\partial \phi_{1,0}}{\partial t} \sqrt{\Delta s \Delta t}.
\end{aligned}
\tag{70}
$$

Here the function $\phi_{1,0}(s,t)$ is defined in (59). Then we obtain

$$
\langle G_V | \big(\hat{S}_A^x - \hat{S}_B^x\big) | G_V \rangle_{DS} \to -2iJ_A(\Delta t) \int_0^{\infty} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}t \phi_{1,0} \frac{\partial \phi_{1,0}}{\partial t} = 0. \tag{71}
$$

For the second term in (67) with $\alpha = x$, we notice that

$$
\hat{S}^{\pm} |\Psi_{J,M}\rangle = |\Psi_{J,M\pm1}\rangle \sqrt{J(J+1) - M(M\pm1)}. \tag{72}
$$

We have

$$
\hat{S}^x |\Psi_{J,M}\rangle = \frac{-1}{2i} \big(\hat{S}^+ - \hat{S}^-\big) |\Psi_{J,M}\rangle \sim \frac{i}{2} J \big\{ |\Psi_{J,M+1}\rangle - |\Psi_{J,M-1}\rangle \big\}. \tag{73}
$$

By this expression we obtain

$$
\hat{S}^x |G_V\rangle_{DS} \sim \sum_{J,M} \frac{i}{2} J \big\{ |\Psi_{J,M+1}\rangle - |\Psi_{J,M-1}\rangle \big\} (-1)^{(J-M)/2} \tilde{\Phi}_{J,M}. \tag{74}
$$

Since $J - M$ is limited to be even, $\tilde{\Phi}_{J,M} = 0$ for odd $J - M$. Therefore when $J' - M'$ is even, the following product is zero.

$$
\tilde{\Phi}_{J',M'} \big\{ \tilde{\Phi}_{J,M-1} - \tilde{\Phi}_{J,M+1} \big\} = 0. \tag{75}
$$

This brings us

$$
\langle G_V | \hat{S}^x | G_V \rangle_{DS} = 0. \tag{76}
$$

From (71) and (76) we conclude that

$$
F_{DS}^{x}(i) = 0. \tag{77}
$$

Next, we calculate the first term of $F_{DS}^{xx}(i,j)$ in (68) with $\alpha = x$. Following the calculation for (70) we obtain

$$\left\langle G_V \left| \left(\hat{S}_A^x - \hat{S}_B^x\right)^2 \right| G_V \right\rangle_{DS} \to \left(\frac{J_A}{2}\right)^2 (4\Delta t)^2 \int_0^\infty \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}t \left[\frac{\partial \phi_{1,0}}{\partial t}\right]^2 = 2J_A^2 (\Delta t)^2. \quad (78)$$

For the second term in (68) we have

$$\begin{aligned} \left(\hat{S}^x\right)^2 \left| G_V \right\rangle_{DS} &\sim \sum_{J,M} \frac{i}{2} J \hat{S}^x \left\{ \left|\Psi_{J,M+1}\right\rangle - \left|\Psi_{J,M-1}\right\rangle \right\} (-1)^{(J-M)/2} \tilde{\Phi}_{J,M} \\ &\sim \sum_{J,M} \left(\frac{i}{2} J\right)^2 \left\{ \left(\left|\Psi_{J,M+2}\right\rangle - \left|\Psi_{J,M}\right\rangle \right) - \left(\left|\Psi_{J,M}\right\rangle - \left|\Psi_{J,M-2}\right\rangle \right) \right\} (-1)^{(J-M)/2} \tilde{\Phi}_{J,M} \\ &= \sum_{J,M} \left(\frac{i}{2} J\right)^2 \left|\Psi_{J,M}\right\rangle (-1)^{(J-M)/2} \left\{ -\tilde{\Phi}_{J,M-2} - 2\tilde{\Phi}_{J,M} - \tilde{\Phi}_{J,M+2} \right\}. \end{aligned} \quad (79)$$

Therefore we obtain

$$\left\langle G_V \left| \left(\hat{S}^x\right)^2 \right| G_V \right\rangle_{DS} \to \frac{1}{(\Delta s)^2} \int_0^\infty \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}t\, s^2 \left(\phi_{1,0}\right)^2 = \frac{1}{(\Delta s)^2} \frac{3}{2}. \quad (80)$$

Because $\left|\Psi_{J,M}\right\rangle$ in the ground state is limited to the state with even $J-M$, and $\hat{S}^x$ changes the even state to the odd state while $\left(\hat{S}_A^x - \hat{S}_B^x\right)$ does not change this even-odd property, the third term in the above Equation (68) vanishes. Thus we have, using (54),

$$\Delta F_{DS}^{xx}(i,j) = F_{DS}^{xx}(i,j) \to \frac{1}{2}\left(m^+\right)^2 (\Delta t)^2 + (-1)^{P_i+P_j} \frac{3}{2N^2} \frac{1}{(\Delta s)^2}. \quad (81)$$

6.2. $\Delta F_{DS}^{zz}(i,j)$

In calculations (67) and (68) with $\alpha = z$, we used the state $\left|Z\right\rangle$ introduced in Section 3. Then we have

$$\begin{aligned} \left\langle Z \left| \left(\hat{S}_A^z - \hat{S}_B^z\right) \right| Z \right\rangle &= \left\langle J_A, M_A \right| \left\langle J_B, M_B \right| \left(\hat{S}_A^z - \hat{S}_B^z\right) \left| J_A, M_A \right\rangle \left| J_B, M_B \right\rangle \\ &= M_A - M_B = 2Z. \end{aligned} \quad (82)$$

Since $\left|\Psi_{J,M}\right\rangle$ is expressed by $\left|Z\right\rangle$'s as shown in (31), we see by the algebraic argument on SU(2),

$$\left(\hat{S}_A^z - \hat{S}_B^z\right)\left|\Psi_{J,M}\right\rangle = c_+^z \left|\Psi_{J+1,M}\right\rangle + c_0^z \left|\Psi_{J,M}\right\rangle + c_-^z \left|\Psi_{J-1,M}\right\rangle. \quad (83)$$

We can calculate the coefficients by

$$c_\pm^z = \left\langle \Psi_{J\pm1,M} \left| \left(\hat{S}_A^z - \hat{S}_B^z\right) \right| \Psi_{J,M} \right\rangle, \quad c_0^z = \left\langle \Psi_{J,M} \left| \left(\hat{S}_A^z - \hat{S}_B^z\right) \right| \Psi_{J,M} \right\rangle. \quad (84)$$

The expectation value for c_+^z is, when $J \gg M \gg 1$,

$$\begin{aligned} c_+^z &= \sum_{Z,Z'} \phi_{J+1,M}(x') \left\langle Z' \middle| Z \right\rangle (2Z) \phi_{J,M}(x) \Delta x \\ &\sim 2J_A \sum_Z \phi_{J+1,M}(x)\, x \phi_{J,M}(x) \Delta x \\ &\sim 2J_A \sum_Z \phi_{J+1,M}(x) \left\{ \frac{1}{2}\phi_{J+1,M}(x) + \frac{1}{2}\phi_{J-1,M}(x) \right\} \Delta x. \end{aligned} \quad (85)$$

Here we used the relation on the associated Legendre polynomial $P_J^M(x)$,

$$(J-M+1) P_{J+1}^M(x) - (2J+1) x P_J^M(x) + (J+M) P_{J-1}^M(x) = 0. \quad (86)$$

Together with similar calculations, we finally obtain

$$c_{\pm}^{z} \sim J_A, \quad c_0^z \sim 0. \tag{87}$$

We then have, for the first term of $F_{DS}^{z}(i)$ in (67) with $\alpha = z$,

$$\begin{aligned}
&\langle G_V | \left(\hat{S}_A^z - \hat{S}_B^z\right) | G_V \rangle_{DS} \\
&= \sum_{J,M,J',M'} (-1)^{(J-M)/2} \tilde{\Phi}_{J,M} \left\langle \Psi_{J,M} \right| \left(\hat{S}_A^z - \hat{S}_B^z\right) \left| \Psi_{J',M'} \right\rangle (-1)^{(J'-M')/2} \tilde{\Phi}_{J',M'} \\
&\sim \sum_{J,M,J',M'} (-1)^{(J-M)/2} \tilde{\Phi}_{J,M} J_A \left\langle \Psi_{J,M} \right| \left\{ \left| \Psi_{J'+1,M'} \right\rangle + \left| \Psi_{J'-1,M'} \right\rangle \right\} (-1)^{(J'-M')/2} \tilde{\Phi}_{J',M'} \\
&= J_A \sum_{J,M,J',M'} (-1)^{(J-M)/2} \tilde{\Phi}_{J,M} \left(\delta_{J,J'+1} + \delta_{J,J'-1} \right) \delta_{M,M'} (-1)^{(J'-M')/2} \tilde{\Phi}_{J',M'} = 0.
\end{aligned} \tag{88}$$

The reason why the above quantity vanishes is that the operator $\left(\hat{S}_A^z - \hat{S}_B^z\right)$ changes the state with the even $J-M$ to the state with the odd $J-M$.

For the second term of $F_{DS}^{z}(i)$ in (67) with $\alpha = z$, we have

$$\hat{S}^z \left| \Psi_{J,M} \right\rangle = \left| \Psi_{J,M} \right\rangle M = \left| \Psi_{J,M} \right\rangle \frac{t}{\Delta t}. \tag{89}$$

Therefore

$$\langle G_V | \hat{S}^z | G_V \rangle_{DS} \to \frac{1}{\Delta t} \int_0^\infty \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}t \left(\phi_{1,0}\right)^2 t = 0. \tag{90}$$

Thus we obtain

$$F_{DS}^{z}(i) = 0. \tag{91}$$

Next let us consider the first term of $F_{DS}^{zz}(i,j)$ in (68) with $\alpha = z$. We repeat the discussion for (83) to yield

$$\left(\hat{S}_A^z - \hat{S}_B^z\right)^2 \left| \Psi_{J,M} \right\rangle \sim J_A^2 \left\{ \left| \Psi_{J+2,M} \right\rangle + \left| \Psi_{J-2,M} \right\rangle + 2 \left| \Psi_{J,M} \right\rangle \right\}. \tag{92}$$

Then we have, for even $J-M$,

$$\begin{aligned}
&\langle G_V | \left(\hat{S}_A^z - \hat{S}_B^z\right)^2 | G_V \rangle_{DS} \\
&= \sum_{J,M,J',M'} \tilde{\Phi}_{J,M} (-1)^{(J-M)/2} \left\langle \Psi_{J,M} \right| \left(\hat{S}_A^z - \hat{S}_B^z\right)^2 \left| \Psi_{J',M'} \right\rangle \tilde{\Phi}_{J',M'} (-1)^{(J'-M')/2} \\
&\sim \sum_{J,M,J',M'} \tilde{\Phi}_{J,M} (-1)^{(J-M)/2} J_A^2 \\
&\quad \times \left\langle \Psi_{J,M} \right| \left\{ \left| \Psi_{J'+2,M'} \right\rangle + \left| \Psi_{J'-2,M'} \right\rangle + 2 \left| \Psi_{J',M'} \right\rangle \right\} \tilde{\Phi}_{J',M'} (-1)^{(J'-M')/2} \\
&= J_A^2 \sum_{J,M,J',M'} \tilde{\Phi}_{J,M} (-1)^{(J-M)/2} \left(\delta_{J,J'+2} + \delta_{J,J'-2} + 2\delta_{J,J'} \right) \delta_{M,M'} \tilde{\Phi}_{J',M'} (-1)^{(J'-M')/2} \\
&= J_A^2 \sum_{J,M} \tilde{\Phi}_{J,M} \left\{ -\tilde{\Phi}_{J+2,M} - \tilde{\Phi}_{J-2,M} + 2\tilde{\Phi}_{J,M} \right\} \\
&\to J_A^2 \int_0^\infty \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}t \phi_{1,0}(s,t) \left\{ -\phi_{1,0}(s+2\Delta s,t) - \phi_{1,0}(s-2\Delta s,t) + 2\phi_{1,0}(s,t) \right\} \\
&= -J_A^2 (2\Delta s)^2 \int_0^\infty \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}t \phi_{1,0} \frac{\partial^2 \phi_{1,0}}{\partial s^2}.
\end{aligned} \tag{93}$$

Finally we obtain

$$\langle G_V | \left(\hat{S}_A^z - \hat{S}_B^z\right)^2 | G_V \rangle_{DS} \to 4J_A^2 (\Delta s)^2 \frac{3}{2}. \tag{94}$$

For the second term of $F_{DS}^{zz}(i,j)$ in (68) with $\alpha = z$ we obtain, using (89),

$$\langle G_V | \left(\hat{S}^z\right)^2 | G_V \rangle_{DS} \to \frac{1}{(\Delta t)^2} \int_0^\infty \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}t \left(\phi_{1,0}\right)^2 t^2 = \frac{1}{(\Delta t)^2} \frac{1}{2}. \tag{95}$$

The operator $\left(\hat{S}_A^z + \hat{S}_B^z\right)\left(\hat{S}_A^z - \hat{S}_B^z\right)$ changes the state with the even $J - M$ to that with the odd $J - M$. Then the third term vanishes.

$$\langle G_V | \left(\hat{S}_A^z + \hat{S}_B^z\right)\left(\hat{S}_A^z - \hat{S}_B^z\right) | G_V \rangle_{DS} = 0. \tag{96}$$

The final result is, using (54),

$$\Delta F_{DS}^{zz}(i,j) = F_{DS}^{zz}(i,j) \to \frac{3}{2}\left(m^+\right)^2 (\Delta s)^2 + (-1)^{P_i + P_j} \frac{1}{2N^2} \frac{1}{(\Delta t)^2}. \tag{97}$$

7. Correlation Function due to Nambu-Goldstone Mode

In this section we calculate the correlation function due to Nambu-Goldstone mode. When we add the symmetry breaking interaction $\hat{V}_2$ to the Hamiltonian, this interaction changes the gapless mode to the gapped state. We employ linear spin wave theory (LSWT) [28] to obtain the correlation function for the gapped state. Since there is little difference between LSWT for SU(2) symmetry and one for U(1) symmetry carried out in the previous paper [25], we present only a brief discussion. The Hamiltonian of the model with spin S is given by

$$\hat{H}_V^S = \sum_{(i,j)} \hat{\vec{S}}_i \hat{\vec{S}}_j + g_1 \left(\hat{S}^z\right)^2 - g_2 \sum_i (-1)^{P_i} \hat{S}_i^y, \quad \left(\hat{\vec{S}}_i\right)^2 = S(S+1). \tag{98}$$

We replace the spin operators by the annihilation operators $\hat{b}_i$ and the creation operators $\hat{b}_i^\dagger$ of the magnon,

$$\hat{S}_i^z + i(-1)^{P_i} \hat{S}_i^x = \sqrt{2S}\hat{b}_i, \quad \hat{S}_i^y = (-1)^{P_i}\left(S - \hat{b}_i^\dagger \hat{b}_i\right). \tag{99}$$

Instead of $\hat{b}_i$ we use $\hat{b}_{\vec{k}}$, which is Fourier transform of $\hat{b}_i$,

$$\hat{b}_{\vec{k}} = \frac{1}{\sqrt{N}} \sum_i \hat{b}_i \mathrm{e}^{-i\vec{k}\vec{x}_i}, \quad \vec{x}_i = \left(i_x, i_y\right). \tag{100}$$

Here the wave vector is defined by $\vec{k} = \left(k_x, k_y\right) = \left(2\pi n_x / \sqrt{N}, 2\pi n_y / \sqrt{N}\right)$ using integers n_x and n_y. One should note that in LSWT the additional interaction $\hat{V}_1 = g_1\left(\hat{S}^z\right)^2$ induces only the constant contribution to the ground state energy, and no effect to the excitation energy. In order to show it, we express $\hat{V}_1$ in terms of $\hat{b}_{\vec{k}}$,

$$\begin{aligned} \hat{V}_1 &= g_1 \left[\sqrt{\frac{S}{2}} \sum_i \left(\hat{b}_i + \hat{b}_i^\dagger\right)\right]^2 = g_1 \frac{S}{2}\left[\sqrt{N}\left(\hat{b}_{\vec{k}=0} + \hat{b}_{\vec{k}=0}^\dagger\right)\right]^2 \\ &= g_1 \frac{SN}{2}\left[\left(\hat{b}_{\vec{k}=0}\right)^2 + \left(\hat{b}_{\vec{k}=0}^\dagger\right)^2 + 2\hat{b}_{\vec{k}=0}^\dagger \hat{b}_{\vec{k}=0} + 1\right]. \end{aligned} \tag{101}$$

Since $\hat{V}_1$ is expressed by the magnon operator with the zero-wave vector, this interaction gives only the constant to the ground state energy. Therefore it is safe to neglect the contributions from $\hat{V}_1$ in the following study.

For the magnon excitation with the non-zero wave vector we diagonalize the

Hamiltonian by

$$\hat{b}_{\vec{k}} = \hat{\alpha}_{\vec{k}} \cosh\theta_{\vec{k}} + \hat{\alpha}^{\dagger}_{-\vec{k}} \sinh\theta_{\vec{k}}. \tag{102}$$

We obtain

$$\hat{H}^S_V = \sum_{\vec{k}} \omega_{\vec{k}} \hat{\alpha}^{\dagger}_{\vec{k}} \hat{\alpha}_{\vec{k}} + const. \tag{103}$$

Here the parameter $\theta_{\vec{k}}$ is determined by

$$\tanh\left(2\theta_{\vec{k}}\right) = -\frac{\gamma_{\vec{k}}}{1+\bar{g}_2}, \quad \bar{g}_2 \equiv \frac{g_2}{4S}, \quad \gamma_{\vec{k}} \equiv \frac{\cos k_x + \cos k_y}{2}. \tag{104}$$

Then the energy $\omega_{\vec{k}}$ of the magnon is given by

$$\omega_{\vec{k}} = 4S\sqrt{\left(1+\bar{g}_2\right)^2 - \gamma^2_{\vec{k}}}. \tag{105}$$

We would like to obtain the spin correlation function at the large distance on the quite large lattice. Therefore we are interested in the magnon with the small energy. In the case $|k_x| \ll 1$ and $|k_y| \ll 1$, the energy is given by

$$\omega_{\vec{k}} \sim 4S\sqrt{\bar{g}_2\left(2+\bar{g}_2\right) + \frac{1}{2}\left(k_x^2 + k_y^2\right)}. \tag{106}$$

In the spin system with SU(2) symmetry on the square lattice, we have another kind of the magnon with the small energy, whose wave vector $\vec{k}$ is near (π,π). In the case of $|\pi - k_x| \ll 1$ and $|\pi - k_y| \ll 1$, the energy is given by

$$\omega_{\vec{k}} \sim 4S\sqrt{\bar{g}_2\left(2+\bar{g}_2\right) + \frac{1}{2}\left[\left(k_x-\pi\right)^2 + \left(k_y-\pi\right)^2\right]}. \tag{107}$$

Note that $\gamma_{\vec{k}}$ is negative in this case. We see that the gap energy $e_g = 4S\sqrt{\bar{g}_2\left(2+\bar{g}_2\right)}$ is proportional to $\sqrt{g_2}$ when g_2 is small. By this gap energy we expect that the spin correlation function decreases exponentially at the large distance.

Let us explicitly calculate the spin correlation function of the spin operator $(-1)^{P_i} \hat{S}^x_i$. We express the spin operator using the operators $\hat{\alpha}_{\vec{k}}$ and $\hat{\alpha}^{\dagger}_{-\vec{k}}$,

$$(-1)^{P_i} \hat{S}^x_i = \frac{\sqrt{2S}}{2i\sqrt{N}} \sum_{\vec{k}} \left\{ \sqrt{\frac{1}{2}\left(1+\frac{1}{\bar{\omega}_{\vec{k}}}\right)} + \varepsilon\left(\gamma_{\vec{k}}\right)\sqrt{\frac{1}{2}\left(-1+\frac{1}{\bar{\omega}_{\vec{k}}}\right)} \right\} \left(\hat{\alpha}_{\vec{k}} - \hat{\alpha}^{\dagger}_{-\vec{k}}\right) e^{i\vec{k}\vec{x}_i}. \tag{108}$$

Here we introduce notation $\bar{\omega}_{\vec{k}}$ defined by

$$\frac{1}{\bar{\omega}_{\vec{k}}} \equiv \cosh\left(2\theta_{\vec{k}}\right) = 4S\frac{1+\bar{g}_2}{\omega_{\vec{k}}}. \tag{109}$$

Also we define that

$$\varepsilon(x) = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x \le 0 \end{cases} \tag{110}$$

From the expression (108) we can calculate the spin correlation function due to Nambu-Goldston mode. Since $\hat{\alpha}_{\vec{k}} |G_V\rangle = 0$ it is clear that

$$F^{S,x}_{NG}(i) \equiv \langle G_V | (-1)^{P_i} \hat{S}^x_i | G_V \rangle_{NG} = 0. \tag{111}$$

Similarly we obtain

$$
\begin{aligned}
F_{NG}^{S,xx}(i,j) &\equiv \langle G_V|(-1)^{P_i}\hat{S}_i^x(-1)^{P_j}\hat{S}_j^x|G_V\rangle_{NG} \\
&= \frac{S}{2N}\sum_{\vec{k}}\left\{\sqrt{\frac{1}{2}\left(1+\frac{1}{\bar{\omega}_{\vec{k}}}\right)}+\varepsilon(\gamma_{\vec{k}})\sqrt{\frac{1}{2}\left(-1+\frac{1}{\bar{\omega}_{\vec{k}}}\right)}\right\}^2 \mathrm{e}^{i\vec{k}(\vec{x}_i-\vec{x}_j)} \\
&\sim \frac{S}{N}\sum_{\vec{k}\sim\vec{0}}\frac{1}{\bar{\omega}_{\vec{k}}}\mathrm{e}^{i\vec{k}(\vec{x}_i-\vec{x}_j)}.
\end{aligned}
\tag{112}
$$

Here note that we can neglect the contribution by the magnon with $(k_x,k_y)\sim(\pi,\pi)$, because in the second expression the first term cancels with the second term due to $\varepsilon(\gamma_{\vec{k}})=-1$.

For the correlation function of the spin operator $\hat{S}_i^z$, Nambu-Goldstone mode with the wave vector near (π,π) gives the large contribution. This operator is given by

$$
\hat{S}_i^z=\frac{\sqrt{2S}}{2\sqrt{N}}\sum_{\vec{k}}\left\{\sqrt{\frac{1}{2}\left(1+\frac{1}{\bar{\omega}_{\vec{k}}}\right)}-\varepsilon(\gamma_{\vec{k}})\sqrt{\frac{1}{2}\left(-1+\frac{1}{\bar{\omega}_{\vec{k}}}\right)}\right\}\left(\hat{\alpha}_{\vec{k}}+\hat{\alpha}_{-\vec{k}}^\dagger\right)\mathrm{e}^{i\vec{k}\vec{x}_i}. \tag{113}
$$

Then we have

$$
F_{NG}^{S,z}(i)\equiv\langle G_V|(-1)^{P_i}\hat{S}_i^z|G_V\rangle_{NG}=0. \tag{114}
$$

In addition we obtain

$$
\begin{aligned}
F_{NG}^{S,zz}(i,j) &\equiv \langle G_V|(-1)^{P_i}\hat{S}_i^z(-1)^{P_j}\hat{S}_j^z|G_V\rangle_{NG} \\
&= \frac{S}{2N}\sum_{\vec{k}}\left\{\sqrt{\frac{1}{2}\left(1+\frac{1}{\bar{\omega}_{\vec{k}}}\right)}-\varepsilon(\gamma_{\vec{k}})\sqrt{\frac{1}{2}\left(-1+\frac{1}{\bar{\omega}_{\vec{k}}}\right)}\right\}^2 \mathrm{e}^{i\vec{k}(\vec{x}_i-\vec{x}_j)}(-1)^{P_i+P_j} \\
&\sim \frac{S}{N}\sum_{\vec{k}\sim(\pi,\pi)}\frac{1}{\bar{\omega}_{\vec{k}}}\mathrm{e}^{i\vec{k}(\vec{x}_i-\vec{x}_j)}(-1)^{P_i+P_j} \\
&= \frac{S}{N}\sum_{\vec{k}'\sim\vec{0}}\frac{1}{\bar{\omega}_{\vec{k}'}}\mathrm{e}^{-i\vec{k}'(\vec{x}_i-\vec{x}_j)}\mathrm{e}^{i(\pi,\pi)(\vec{x}_i-\vec{x}_j)}(-1)^{P_i+P_j}.
\end{aligned}
\tag{115}
$$

Since $\exp[i(\pi,\pi)\cdot\vec{x}_i]=(-1)^{P_i}$, (115) indicates

$$
F_{NG}^{S,zz}(i,j)=F_{NG}^{S,xx}(i,j). \tag{116}
$$

For $F_{NG}^{S,zz}(i,j)$, the magnon with the wave vector $\sim(\pi,\pi)$ plays the same role as the magnon with the wave vector $\sim(0,0)$ for $F_{NG}^{S,xx}(i,j)$.

When the distance between i and j is large, the summation on $\vec{k}$ can be dominated by the contributions by the low energy. Then we can replace the energy by

$$
\omega_{\vec{k}}\to\frac{1}{\sqrt{2}(1+\bar{g}_2)}\sqrt{\tau^2+\vec{k}^2},\quad \tau\equiv\sqrt{2\bar{g}_2(2+\bar{g}_2)}. \tag{117}
$$

Replacing the summation by the integral we obtain the correlation function at the large distance r [25],

$$
F_{NG}^{S,xx}(i,j)\to\sqrt{2}S(1+\bar{g}_2)\frac{1}{2\pi r}\mathrm{e}^{-\tau r},\quad r=|\vec{x}_i-\vec{x}_j|. \tag{118}
$$

8. Cluster Property

In this section we discuss the violation of the cluster property in the spin correlation function for the antiferromagnet on the square lattice with SU(2) symmetry.

Each spin correlation function $\Delta F^{\alpha\alpha}(i,j)$ is the sum of the function due to the quasi-degenerate states and the one due to Nambu-Goldstone mode, which have been calculated in the previous sections. Let us collect our results. First we consider $\Delta F_{DS}^{\alpha\alpha}(i,j)$. Using expressions of Δs and Δt in (57) the results for $\Delta F_{DS}^{xx}(i,j)$ given in (81) and for $\Delta F_{DS}^{zz}(i,j)$ in (97) become

$$\Delta F_{DS}^{xx}(i,j) = F_{DS}^{xx}(i,j) = \sqrt{\frac{(m^+)^3}{2}}\sqrt{\frac{g_1}{g_2 N}} + (-1)^{P_i+P_j}\frac{3}{2}\sqrt{\chi_0 m^+}\sqrt{\frac{g_2}{N^2}}.$$

$$\Delta F_{DS}^{zz}(i,j) = F_{DS}^{zz}(i,j) = \frac{3}{2}\sqrt{\frac{(m^+)^3}{\chi_0}}\sqrt{\frac{1}{g_2 N^2}} + (-1)^{P_i+P_j}\sqrt{\frac{m^+}{8}}\sqrt{\frac{g_2}{g_1 N^3}}. \quad (119)$$

Since the lattice size N is fixed to be 10^{20}, the parameters in (119) are the strength g_1 and the strength g_2 of the additional interactions $\hat{V}_1$ and $\hat{V}_2$. In addition to the condition $g_2 \ll 1$, we impose the following conditions on g_1 and g_2, so that the conditions (62) are also satisfied,

$$g_1 \gg \frac{1}{N}, \quad g_2 \gg \frac{g_1}{N}. \quad (120)$$

With (120) it is easy to see that we can neglect the second term in both expressions of (119). As a result $\Delta F_{DS}^{\alpha\alpha}(i,j)$ becomes independent of the site. Therefore we use the symbol $F_{DS}^{\alpha\alpha}$ instead of $\Delta F_{DS}^{\alpha\alpha}(i,j)$, which are defined by

$$F_{DS}^{xx} \equiv \sqrt{\frac{(m^+)^3}{2}}\sqrt{\frac{g_1}{g_2 N}}, \quad F_{DS}^{zz} \equiv \frac{3}{2}\sqrt{\frac{(m^+)^3}{\chi_0}}\sqrt{\frac{1}{g_2 N^2}}. \quad (121)$$

We see that these correlation functions do not depend on the distance and these values are non-zero as far as N is finite.

As for the contribution from Nambu-Goldstone mode we conclude that, based on discussions in Section 7,

$$F_{NG}^{xx}(i,j) = F_{NG}^{zz}(i,j) \to F_{NG}(r) \equiv \frac{\sqrt{2}}{4\pi r}\mathrm{e}^{-\tau r}, \quad r = \left|\vec{x}_i - \vec{x}_j\right| \gg 1. \quad (122)$$

Here we obtain $F_{NG}(r)$ by putting $S = 1/2$ and $1+\bar{g}_2 = 1+g_2/2 \sim 1$ in (118). Also we have $\tau \sim \sqrt{2g_2}$ from (117).

From the final results of (121) and (122) we conclude that the cluster property violates in the correlation functions due to the quasi-degenerate states, and that the functions due to Nambu-Goldstone mode exponentially decrease with distance. Although this conclusion is the same as that for U(1) symmetry [25], there are several characteristic properties newly found for SU(2) symmetry.

- We have two kinds of correlation functions, F_{DS}^{xx} and F_{DS}^{zz}, which violate the cluster property. The number "two" originates from the two kinds of the independent generators of SU(2) symmetry.

- It is noticeable that both F_{DS}^{xx} and F_{DS}^{zz} show the same dependency on g_2. The reason is that the additional interaction $\hat{V}_2$ induces the same effect on both variables *s* and *t* in (56).
- We also should note that the correlation function $F_{NG}(r)$ does not depend on the value of g_1, since the interaction with g_1 is expressed by the magnon operators with the zero-wave vector as is shown in (101). The correlation function F_{DS}^{zz} is independent of g_1, too. The correlation function F_{DS}^{xx}, on the other hand, strongly depends on the value of g_1.

Let us numerically examine F_{DS}^{xx} and F_{DS}^{zz}. We fix $N = 10^{20}$. As values of m^+ and χ_0 on the quite large lattice, we use $m^+ = 0.31$ [26] [34] and $\chi_0 = 0.063$ [26] [34] [35]. Expressions in (121) indicate that F_{DS}^{xx} is proportional to $\sqrt{g_1}$ for fixed g_2 and N while F_{DS}^{zz} does not contain g_1. We therefore plot F_{DS}^{xx} with four typical values of g_1, which are $g_1 = 10^{-15}, 10^{-10}, 10^{-5}$ and 1, together with F_{DS}^{zz}. Figure 3 shows these results as a function of g_2. Note that, from the condition (120), we should focus the region where $g_2 \gg 10^{-20} g_1$. We see that F_{DS}^{xx} with above values of g_1 is larger than F_{DS}^{zz}.

Next let us discuss whether this violation in $F_{DS}^{\alpha\alpha}$ is measurable or not. To answer this question it is necessary to compare $F_{DS}^{\alpha\alpha}$ with $F_{NG}(r)$ given by (122). In the range of the distance r where $F_{DS}^{\alpha\alpha} \geq F_{NG}(r)$ we would be able to observe the violation in experiments. In Figure 4 we plot, as a function of g_2, the distance r_c where $F_{DS}^{\alpha\alpha} = F_{NG}(r)$ holds. We see that all curves in the figure

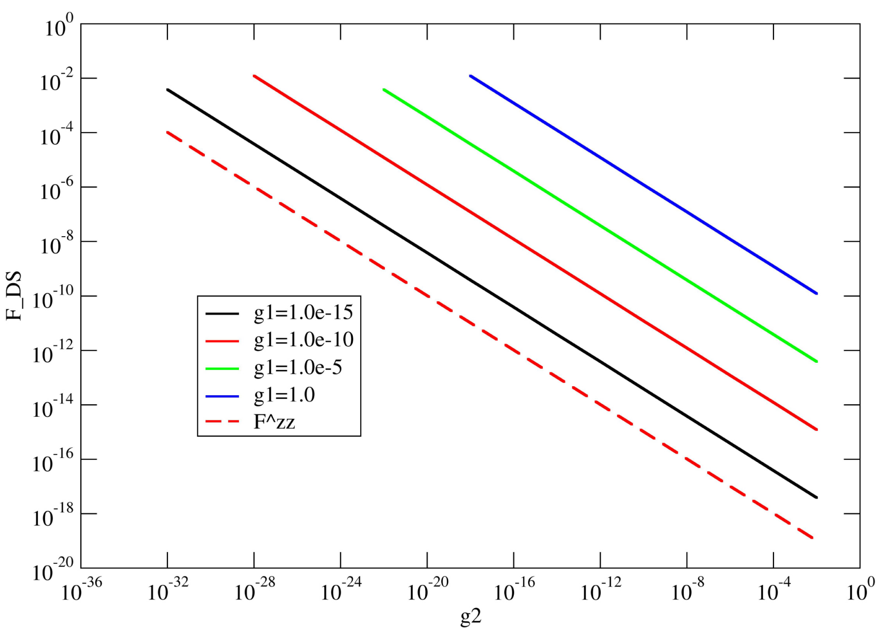

Figure 3. The correlation functions F_{DS}^{xx} and F_{DS}^{zz} defined in (121) with the lattice size $N = 10^{20}$. They are plotted as a function of g_2 which is the strength of the symmetry breaking interaction $\hat{V}_2$ defined by (33). The solid lines are the results of F_{DS}^{xx} for $g_1 = 10^{-15}, 10^{-10}, 10^{-5}$ and 1. The result of F_{DS}^{zz}, which is independent of g_1, is plotted by the red-dashed line.

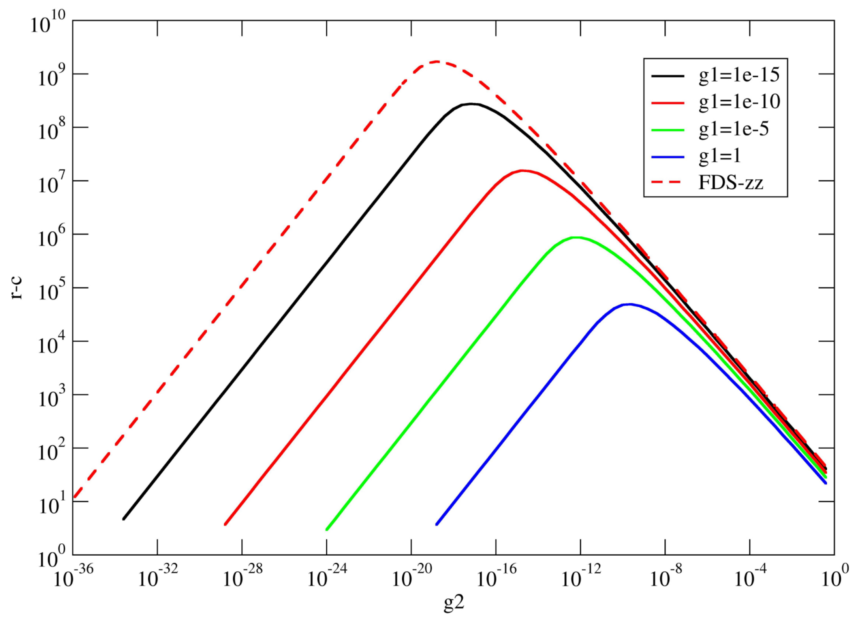

Figure 4. The distance r_c where the magnitude of $F_{NG}(r)$ agrees with $F_{DS}^{\alpha\alpha}$, as a function of g_2. The solid curves show the results for F_{DS}^{xx} with $g_1 = 10^{-15}, 10^{-10}, 10^{-5}$ and 1. The result for F_{DS}^{zz} is plotted by the red-dashed curve. In the region above these curves $F_{DS}^{\alpha\alpha}$ is larger than $F_{NG}(r)$. The lattice size is $N = 10^{20}$.

resemble each other's in shape. As g_2 increases the value of r_c on each curve increases monotonously until it reaches its maximum value, which we denote $r_c^{\max}$, and then decreases monotonously. The location of the maximum point shifts depending on α and, also on the values of g_1 for $\alpha = x$. Since $F_{DS}^{\alpha\alpha} > F_{NG}(r)$ holds in the area above these curves, where $r > r_c$, we have a possibility to observe the violation of the cluster property there. We see that the value of g_2 which maximizes r_c, say $g_2^{\max}$, is larger when g_1 is larger. We also see that $r_c^{\max}$ is smaller for larger g_1.

Hereafter we focus our attention to curves of F_{DS}^{xx} in order to make our discussion simple. In the area to the left of the maximum point, namely for $g_2 < g_2^{\max}$, the region where $r > r_c$ holds becomes wide as g_1 grows. When $g_2 > g_2^{\max}$, on the other hand, values of r_c scarcely depend on g_1. The results in Figure 4 suggest in what region the observation can be expected. In Figure 5 we plot $F_{NG}(r)$ versus r with two typical values of g_2, which we choose $g_2 = 10^{-16}$ and $g_2 = 10^{-4}$, in order to make it easy to understand the measurable area. Then we compare values of F_{DS}^{xx} with $F_{NG}(r)$ for several values of g_1. The value of r_c in Figure 4 is now indicated by the cross point where the horizontal line of F_{DS}^{xx} and the curve of $F_{NG}(r)$ meet. Our results show that F_{DS}^{xx} would be observable in a wide range of r and g_1 provided that g_2 is moderately small.

9. Summary and Discussions

In this paper, we investigated the violation of the cluster property for the spin 1/2

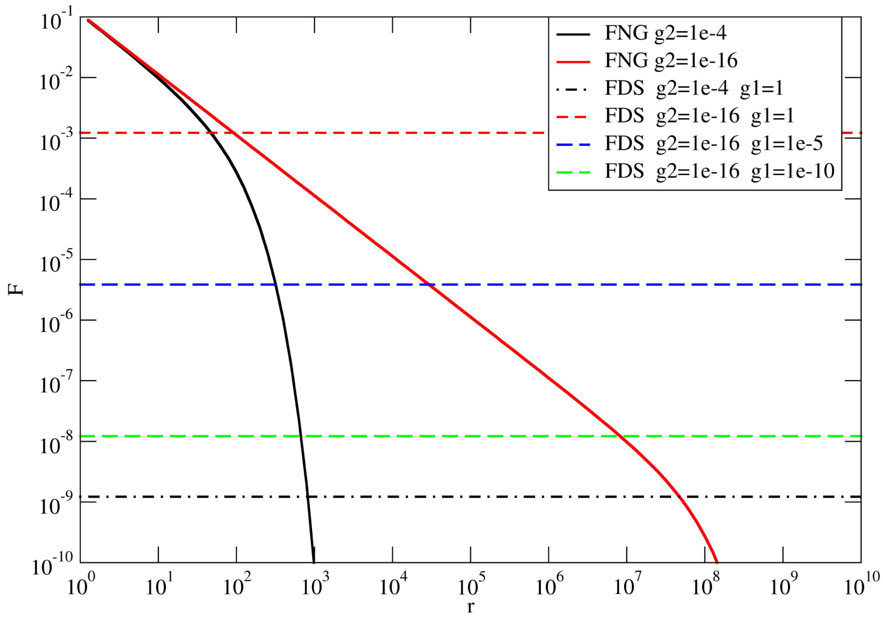

Figure 5. $F_{NG}(r)$ and F_{DS}^{xx} shown as a function of the distance r. The solid curves plot the values of $F_{NG}(r)$ for $g_2 = 10^{-16}$ and $g_2 = 10^{-4}$, while the horizontal lines indicate the values of F_{DS}^{xx}. The dashed lines are results for $g_2 = 10^{-16}$ and, $g_1 = 1, 10^{-5}$ and 10^{-10}. The dashed-dotted line is the result for $g_2 = 10^{-4}$ and $g_1 = 1$. The lattice size is $N = 10^{20}$.

Heisenberg antiferromagnet with SU(2) symmetry on the square lattice, following the previous paper [25] that has shown the violation in the correlation function for the spin 1/2 XXZ antiferromagnet with U(1) symmetry on the square lattice.

Spontaneous symmetry breaking for the system can be explained by the model where the ferromagnets of A and B sub-lattices couple to form the effective Hamiltonian [26]. In this model, we could clearly define the quasi-degenerate states and could calculate the ground state including the interactions that explicitly and completely break SU(2) symmetry. Note that, in the concrete calculation of the ground state, we have kept the lattice size large but finite.

In order to find the violation of the cluster property, we calculated the spin correlation functions. We examined two kinds of correlation functions which consist of the function due to the degenerate states and the one due to Nambu-Goldstone mode. We calculated the former functions $F_{DS}^{\alpha\alpha}$ using the quasi-degenerate states. As for the latter functions $F_{NG}^{\alpha\alpha}(r)$, we obtained it by means of linear spin wave theory.

We see that both F_{DS}^{xx} and F_{DS}^{zz} violate the cluster property in Heisenberg spin systems with SU(2) symmetry. Also when there is the interaction $\hat{V}_1 = g_1 \left(\hat{S}^z\right)^2$, F_{DS}^{xx} becomes large since it is proportional to $\sqrt{g_1}$. By these enhancements of the correlation functions, we find the wide region in the parameter space where it would be possible to observe the violation of cluster property. This fact encourages us to search for the violation in experiments.

Let us now discuss experiments to observe the violation of the cluster property. As described in the previous paper [25], it is required to measure the correlation length [36] [37] [38]. The experiments on the material $Sr_2CuO_2Cl_2$ [39] [40] found that the correlation length ξ is ~200 at temperature $T = 300\ \text{K}$. Also they confirmed that the dependence of ξ on T is $\log(\xi) \propto 1/T$, which is derived by the nonlinear sigma model [41]. This means that ξ becomes huge even when the temperature is not so small. As an example, consider the case in which the temperature T_0 is 50 K. Then the correlation length at this temperature is $\xi_0 = \xi^{T/T_0} \sim 10^{13}$, which would be large enough when we explore the region of $r\left(\le \sqrt{N} = 10^{10}\right)$ in **Figure 4** and **Figure 5**. The results shown in these figures, we believe, are worth examining in experiments to find the violation of the cluster property.

Finally, let us make several comments on our calculations and our results.

- In this work we studied the effect from the additional interaction $\hat{V}_1$ which is defined by the squared spin operators. Since this term was introduced from theoretical point of view it might be difficult to include it in the ordinary magnetic materials. For the experimental study to realize this interaction in the spin systems it is valuable to notice Bose condensation of the atom in the double well potential [42]. This system is described by the effective spin system, where the squared spin is included.
- The second comment is about other interactions of the squared spin operators to break the symmetry. When, for example, we include $\hat{V}_3 \equiv g_3\left\{\left(\hat{S}^x\right)^2 + \left(\hat{S}^y\right)^2\right\}$ instead of $\hat{V}_2$, we have $\hat{V}_3\left|\Psi_{J,M}\right\rangle = \left|\Psi_{J,M}\right\rangle\left\{J\left(J+1\right) - M^2\right\}$ for the state $\left|\Psi_{J,M}\right\rangle$. By the positive value of g_3, the difference between $J\left(J+1\right)$ and M^2 becomes small. Then we cannot apply our approximation adopted in this paper to obtain the ground state. Therefore we would need further extensive work on the ground state with this interaction.
- The third comment is about the more complicated systems. For example, it is quite interesting to study the Heisenberg antiferromagnet on the triangular lattice. In this system we suppose that the effective Hamiltonian is constructed by the three magnets on three sub-lattices. In spite of the complicated algebraic calculations for three magnetic states, it is quite interesting to study this system since the large amount of differences between the quantum spin system on the triangular lattice and those on the square lattice has been found [26].

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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