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An Efficient and Concise Algorithm for Convex Quadratic Programming and Its Application to Markowitz's Portfolio Selection Model

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Abstract

This paper presents a pivoting-based method for solving convex quadratic programming and then shows how to use it together with a parameter technique to solve mean-variance portfolio selection problems.

Keywords: Convex Quadratic Programming, Mean-Variance Portfolio Selection Model, Pivoting Algorithm

1. Introduction

There are a variety of algorithms for solving convex quadratic programming (QP). The most well-known one is active set method [1,2]. However this method is difficult to learn for many people due to its operational manner. In this paper we present a pivoting-based method for solving convex QP which is efficient for calculating and concise for understanding.

The Karush-Kuhn-Tucker conditions (KKT conditions) of QP is a system of (in) equalities where all the expressions are linear equalities or inequalities except for complementarity conditions. Like Wolfe's method [3] we solve the linear part by a kind of pivoting operation while maintaining the complementarity conditions in the computational process. However the system computed by our method has a much smaller size because many variables are deleted from the KKT conditions.

In 1952 Harry Markowitz published the milestone paper "Portfolio Selection" in which the portfolio selection problem is formulated as a convex quadratic program with nonnegative variables [4,5]. It is nearly 60 years. However most people still think the model is difficult to solve and few people conduct their investment activities by calculating efficient portfolios for references. We see that every investments textbooks have a figure to show portfolios which are formed by two assets with different correlation coefficients. But we can hardly see a book which has an example to show portfolios formed by three or more assets in the case of no short sale. We will present a pivoting-based algorithm together with a parameter technique to solve Markowitz's portfolio selection model in a very simple form. The most important advantage of our parameter technique is that it can continuously obtain minimal risk portfolios at different levels of return in linear time after a portfolio is established by the pivoting algorithm.

The rest sections of this paper are organized as follows. In Section 2, we introduce a pivoting-based algorithm for solving the system of linear inequalities and the parameter technique for generating new basic solutions which are associated with different right-hand side terms. In Section 3, we introduce basic concepts and operations for solving the convex QP. In Section 4, we discuss how to solve Markowitz's mean-variance portfolio selection model. For brevity, relative theorems and proofs as well as tedious reasoning processes are omitted. For details, the reader refers to [6-8].

2. The System of Linear Inequalities

Consider a system of linear inequalities in the form:

$$\begin{aligned} &a_i x = b_i,\ i = 1, 2, \cdots, l, \\ &a_i x \ge b_i,\ i = l+1, l+2, \cdots, m \end{aligned} \quad (2.1)$$

where $x = (x_1, x_2, \cdots, x_n)^T$, $a_i = (a_{i1}, a_{i2}, \cdots, a_{in})$ and b_i is a scalar, $i = 1, 2, \cdots, m$.

In our algorithm, the following concepts are employed.

A set of maximum linearly independent vectors of a_1, a_2, $\cdots$, a_m is called a basis of (2.1). Vectors in the basis are called basic otherwise called nonbasic. The equalities and inequalities associated with basic vectors are called basic otherwise called nonbasic. The system of basic (in) equalities is called a basic system and whose solution set is called a basic cone. The solution to the system of equations that are corresponding to basic (in) equalities, *i.e.*, the vertex of the basic cone, is called basic solution.

Geometrically, our algorithm begins with a basic cone whose vertex is denoted by $x^{(1)}$. If $x^{(1)}$ lies on the hyper-planes determined by every equalities and lies in the half-spaces determined by every inequalities of (2.1), it is a solution to the entire system. Otherwise there exists an inequality (or equality) such as $a_r x \ge b_r$ so that $a_r x^{(1)} < b_r$ which is called a violating inequality against $x^{(1)}$. If the intersection of the half-space $a_r x \ge b_r$ and the basic cone is empty, there is no solution for (2.1). Otherwise $x^{(1)}$ is projected onto the boundary $a_r x = b_r$ of the half-space where $a_r x \ge b_r$ replaces a basic inequality to constitute a new basic cone. Then the same process is repeated.

Algebraically, a nonbasic (in) equality replacing a basic inequality is accomplished by the following pivoting operation.

Given a basis of (2.1), let I_0, I_1, I_2 and I_3 be the index sets for basic equalities, basic inequalities, nonbasic inequalities and nonbasic equalities respectively. Let $x^{(1)}$ be the basic solution, *i.e.*, $a_j x^{(1)} = b_j$, $j \in I_0 \cup I_1$. Suppose that

$$a_i = \sum_{j \in I_0 \cup I_1} w_{ij} a_j, \; i \in I_3 \cup I_2 . \qquad (2.2)$$

If $w_{rs} \ne 0$ for an $r \in I_3 \cup I_2$ and an $s \in I_1$, from the rth expression of (2.2) we have

$$a_s = \left(1/w_{rs}\right) a_r + \sum_{j \in I_0 \cup I_1 \setminus \{s\}} \left(-w_{rj}/w_{rs}\right) a_j . \qquad (2.3)$$

Substitute it into the other expressions of (2.2) to yield

$$a_i = \left(w_{is}/w_{rs}\right) a_r + \sum_{j \in I_0 \cup I_1 \setminus \{s\}} \left[w_{ij} - \left(w_{is}/w_{rs}\right) w_{rj}\right] a_j , \quad i \in I_3 \cup I_2 \setminus \{r\} \qquad (2.4)$$

Now, we have a new basic cone whose index set is $\{r\} \cup I_0 \cup I_1 \setminus \{s\}$ and the associated basic solution is denoted by $x^{(2)}$. Multiply both sides of (2.4) by $x^{(2)}$ on the right to have

$$\begin{aligned} a_i x^{(2)} &= \left(w_{is}/w_{rs}\right) a_r x^{(2)} \\ &\quad + \sum_{j \in I_0 \cup I_1 \setminus \{s\}} \left[w_{ij} - \left(w_{is}/w_{rs}\right) w_{rj}\right] a_j x^{(2)} \\ &= \left(w_{is}/w_{rs}\right) b_r + \sum_{j \in I_0 \cup I_1} \left[w_{ij} - \left(w_{is}/w_{rs}\right) w_{rj}\right] b_j . \end{aligned}$$

Multiply both sides of (2.2) by $x^{(1)}$ to have

$$a_i x^{(1)} = \sum_{j \in I_0 \cup I_1} w_{ij} b_j$$

Therefore

$$\begin{aligned} a_i x^{(2)} - a_i x^{(1)} &= \left(w_{is}/w_{rs}\right) b_r - \sum_{j \in I_0 \cup I_1} \left(w_{is}/w_{rs}\right) w_{rj} b_j \\ &= \left(w_{is}/w_{rs}\right)\left(b_r - a_r x^{(1)}\right). \end{aligned}$$

Rewrite the above expression as

$$a_i x^{(2)} - b_i = a_i x^{(1)} - b_i - \left(w_{is}/w_{rs}\right)\left(a_r x^{(1)} - b_r\right)$$

and let $\sigma_i = a_i x^{(1)} - b_i$, $\sigma_r = a_r x^{(1)} - b_r$, $\sigma_i' = a_i x^{(2)} - b_i$ to have

$$\sigma_i' = \sigma_i - \left(w_{is}/w_{rs}\right)\sigma_r , \; i \in I_3 \cup I_2 \setminus \{r\} .$$

In the same way, from (2.3) we have

$$a_s x^{(2)} - b_s = -\left(1/w_{rs}\right)\left(a_r x^{(2)} - b_r\right)$$

or

$$\sigma_s{}' = -\sigma_r / w_{rs} ,$$

where $\sigma_s{}' = a_s x^{(2)} - b_s$.

The above operational process is called a pivoting (operation) and w_{rs} is called pivot (element). The row of w_{rs} is called pivot row and the column of w_{rs} is called pivot column. We say a_r enters and a_s leaves the basis and the exchange of these two vectors is denoted by $a_r \leftrightarrow a_s$. The process is simply shown by **Tables 1** and **2**. Where

$$w_{ij}' = w_{ij} - \left(w_{is}/w_{rs}\right) w_{rj} , i \in I_3 \cup I_2 \setminus \{r\}, j \in I_0 \cup I_1 \setminus \{s\};$$

$$\sigma_i' = \sigma_i - \left(w_{is}/w_{rs}\right)\sigma_r , i \in I_3 \cup I_2 \setminus \{r\}.$$

For **Table 1**, $\sigma_i = a_i x^{(1)} - b_i$ is called the deviation of a_i or the associated (in-) equality with respect to $x^{(1)}$. If

$$\sigma_i = a_i x^{(1)} - b_i = 0, i \in I_3 ;$$

$$\sigma_i = a_i x^{(1)} - b_i \ge 0, i \in I_2 ,$$

then $x^{(1)}$ is a solution to system (2.1). If $\sigma_r = a_r x^{(1)} - b_r < 0$ for some $r \in I_2$, $a_r x \ge b_r$ is called

Table 1. Initial table.

	$\boldsymbol{a_s}$	$\boldsymbol{a_j}$	
$\boldsymbol{a_r}$	w_{rs}^*	w_{rj}	σ_r
$\boldsymbol{a_i}$	w_{is}	w_{ij}	σ_i

Table 2. The result of pivoting.

	$\boldsymbol{a_r}$	$\boldsymbol{a_j}$	
$\boldsymbol{a_s}$	$1/w_{rs}$	$-w_{rj}/w_{rs}$	$-\sigma_r/w_{rs}$
$\boldsymbol{a_i}$	w_{is}/w_{rs}	w_{ij}'	σ_i'

violating inequality against $x^{(1)}$ and $|a_r x^{(1)} - b_r| / \|a_r\|$ is called the distance from $x^{(1)}$ to $a_r x \geq b_r$.

The computational steps for the system of linear inequalities are as follows.

Algorithm 2.1. Pivoting algorithm for system (2.1).

Step 1. Construct initial table.

If (2.1) has an inequality in the form of $x_i \geq l_i$, $x_i \geq l_i$ is an initial basic inequality; otherwise introduce $x_i \geq -M$ into (2.1) and let $x_i \geq -M$ be the initial basic inequality, i = 1, 2, $\cdots$, n. Where M is a number large enough, $x_i \geq -M$ is called artificial inequality and whose coefficient vector is called artificial vector. The initial basic solution is $x^{(0)} = \left(x_1^{(0)}, \cdots, x_n^{(0)}\right)^T$ where $x_i^{(0)} = l_i$ or $-M$, i = 1, 2, $\cdots$, n. The initial basic vector is $\boldsymbol{e}_i$ which is the ith row of the identity matrix of order n, i = 1, 2, $\cdots$, n. Other (in) equalities of (2.1) and their coefficient vectors are nonbasic whose deviation with respect to $x^{(0)}$ is $\sigma_i = a_i x^{(0)} - b_i$. Thus we have an initial table as shown by **Table 1**.

Step 2. Preprocessing.

Let I_3 be the index set of equalities, *i.e.*, I_3 = {1, $\cdots$, l}.

1) If $I_3 = \varnothing$, go to step 3. Otherwise, for an $r \in I_3$ if the deviation σ_r of $\boldsymbol{a}_r$ is negative, positive or zero, go to 2), 3) and 4) respectively.

2) If there is no positive element in the row of a_r that is corresponding to the basic inequality, (2.1) has no solution; otherwise carry out a pivoting on one of the positive elements, let $I_3 = I_3 \setminus \{r\}$ and return to 1).

3) If there is no negative element in the row of a_r that is corresponding to the basic inequality, (2.1) has no solution, otherwise carry out a pivoting on one of the negative elements, let $I_3 = I_3 \setminus \{r\}$ and return to 1).

4) If all of the elements in the row of a_r that are corresponding to basic inequalities are zeros, let $I_3 = I_3 \setminus \{r\}$ and return to 1); otherwise carry out a pivoting on one of the non-zero elements, let $I_3 = I_3 \setminus \{r\}$ and return to 1).

Step 3. Main iterations.

1) If all of the deviations of non-basic vectors are nonnegative, the current basic solution is a solution to (2.1). Otherwise

2) select a non-basic vector with negative deviation to enter the basis. If there is no positive element in the row of entering vector that is corresponding to the basic inequality, (2.1) has no solution, otherwise carry out a pivoting on one of the positive elements and return to 1).

Note that step 1 of Algorithm 2.1 is just a simplified statement for constructing the initial table. In fact, if (2.1) has no inequality $x_i \geq l_i$ but has $x_i \leq u_i$ for a finite u_i, $-x_i \geq -u_i$ can serve as an initial basic inequality and no need to introduce the artificial $x_i \geq -M$.

From steps 2 and 3 of Algorithm 2.1 we see that basic equalities never leave the basic system. Because if (2.1) has a solution, it must satisfy all the equalities. Therefore, once a non-basic equality enters the basic system, the corresponding column can be eliminated.

Sometimes, a system of inequalities has inequalities in the form of $b_i' \geq a_i x \geq b_i$, especially $u_i \geq x_i \geq l_i$, which are formally written as $a_i x \geq b_i$ and $-a_i x \geq -b_i'$ in our method. Let $\bar{x}$ be the basic solution, then deviations of a_i and $-a_i$ are $a_i \bar{x} - b_i$ and $-a_i \bar{x} + b_i'$ respectively and with a sum of $b_i' - b_i$. Since a_i and $-a_i$ are linearly dependent, they cannot be basic simultaneously. If one of them is basic, the deviation of the other one is $b_i' - b_i > 0$, hence can be ignored. If both of a_i and $-a_i$ are non-basic, they can share one row in the table since the coefficients in the expressions of a_i and $-a_i$ in terms of basic vectors have reversed signs.

It should be noted that even though the basic solution is not explicitly presented in the table, it can be easily obtained from the table. Let $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)^T$ be the basic solution which is obtained as follows. If $x_i \geq l_i$ or $-M$ is basic, $\bar{x}_i = l_i$ or $-M$; otherwise $\bar{x}_i$ equals to the deviation of this inequality plus l_i or $-M$. Or if $-x_i \geq -u_i$ is basic, $\bar{x}_i = u_i$; otherwise $\bar{x}_i$ equals to u_i minus the deviation of $-x_i \geq -u_i$.

There are many rules for selecting a vector to enter or leave the basis. We will give several ones which are used in the main iterations stage.

For a given basis of (2.1) let I_0, I_1 and I_2 be index sets for basic equalities, basic inequalities and non-basic inequalities respectively. Let $x^{(1)}$ be the current basic solution and

$$a_i = \sum_{j \in I_0 \cup I_1} w_{ij} a_j, \sigma_i = a_i x^{(1)} - b_i, i \in I_2.$$

Rule 1. (The smallest deviation rule). Among all the non-basic vectors with negative deviations, select a vector a_r having the smallest deviation to enter the basis. That is if $\sigma_r = \min\{\sigma_i : \sigma_i < 0, i \in I_2\}$, then a_r enters the basis.

Rule 2. (The largest distance rule). Among all the non-basic vectors with negative deviations, select a vector a_r whose associated inequality is farthest away from the current basic solution to enter the basis. That is if $\sigma_r / \|a_r\| = \min\left\{\sigma_i / \|a_i\| : \sigma_i < 0, i \in I_2\right\}$, then a_r enters the basis.

Rule 3. (Rule of the farthest distance along an edge). Among all the non-basic vectors with negative deviations, select a vector a_r whose associated inequality is farthest away from the current basic solution along an edge of the basic cone to enter the basis. That is if σ_r / w_{rs} = $\min\left\{\sigma_i / w_{is} : \sigma_i < 0 \text{ and } w_{is} > 0, i \in I_2\right\}$ for an $s \in I_1$, then a_r enters the basis.

Rule 4. (The smallest index rule). Among all the non-basic vectors with negative deviations, select a vector a_r having the smallest index to enter the basis; and among all the basic vectors to leave the basis, select a vector a_s

having the smallest index to leave the basis. That is if $r = \min\{i \in I_2 : \sigma_i < 0\}$, then a_r enters the basis; and if $s = \min\{j \in I_1 : w_{rj} > 0\}$, then a_s leaves the basis.

Like Bland's anti-cycling rule [9], it can be proved that when the smallest index rule is used to solve system (2.1), cycling doesn't occur.

Now let us consider the parameter technique by which a new basic solution can be obtained from the current table by changing the right-hand side terms of some basic (in) equalities.

For a basis of (2.1), let I_B and I_N denote the index sets of basic and nonbasic vectors respectively and $\bar{x}$ be the basic solution. Suppose

$$a_i = \sum_{j \in I_B} w_{ij} a_j, i \in I_N.$$

Multiply both sides by $\bar{x}$, since $a_j \bar{x} = b_j, j \in I_B$, to have

$$a_i \bar{x} = \sum_{j \in I_B} w_{ij} b_j.$$

Therefore the deviation of the nonbasic a_i is

$$\sigma_i = a_i \bar{x} - b_i = \sum_{j \in I_B} w_{ij} b_j - b_i, i \in I_N.$$

If b_j is changed to $b_j + \delta_j$, $j \in I_B$, the deviation of a_i becomes

$$\sigma_i' = \sum_{j \in I_B} w_{ij}\left(b_j + \delta_j\right) - b_i = \sigma_i + \sum_{j \in I_B} w_{ij}\delta_j, i \in I_N. \quad (2.5)$$

The above operation is denoted by $a_j^{+\delta_j}$, $j \in I_B$.

Usually we just change the right-hand side of one basic (in) equality such as changing b_s to $b_s + \Delta$ for an $s \in I_B$. Then the deviation of a_i becomes

$$\sigma_i' = \sigma_i + w_{is}\Delta, i \in I_N. \quad (2.6)$$

It indicates that when the right-hand side of a basic (in) equality is increased by Δ, deviations of nonbasic vectors with respect to the new basic solution can be obtained by multiplying the column of this basic (in) equality by Δ, and then add it to the last column (the deviation column). We will use $a_s^{+\Delta}$ to denote such an operation which gives the result (2.6).

3. Convex Quadratic Programming

3.1. Fundamental Concepts and Algorithm

Consider quadratic programming in this form:

$$\begin{aligned} &\min f(x) = \frac{1}{2}x^T H x + cx \\ &\text{s.t. } a_i x = b_i, i = 1,2,\cdots,l, \\ &\quad a_i x \ge b_i, i = l+1, l+2,\cdots,m, \\ &\quad x_i \ge l_i, i = 1,2,\cdots,n. \end{aligned} \quad (3.1)$$

where $x = \left(x_1, x_2 \cdots, x_n\right)^T$, $H = \left(h_{ij}\right)_{n\times n}$ is positive semi-definite, $c = \left(c_1, c_2, \cdots, c_n\right)$, $a_i = \left(a_{i1}, a_{i2}, \cdots, a_{in}\right)$, b_i and l_i are real numbers.

Let λ_i be the Lagrange multiplier associated with $a_i x = b_i$ or $a_i x \ge b_i$, μ_i be the Lagrange multiplier associated with $x_i \ge l_i$, and e_i be the ith row of the identity matrix of order n. The KKT conditions for (3.1) are as follows.

$$\begin{aligned} &Hx + c^T = \sum_{i=1}^m \lambda_i a_i^T + \sum_{i=1}^n \mu_i e_i^T, \\ &\lambda_i \ge 0, \lambda_i(a_i x - b_i) = 0, i = l+1, l+2,\cdots,m, \\ &\mu_i \ge 0, \mu_i\left(x_i - l_i\right) = 0, i = 1,2,\cdots,n, \\ &a_i x = b_i, i = 1,2,\cdots,l, \\ &a_i x \ge b_i, i = l+1, l+2,\cdots,m, \\ &x_i \ge l_i, i = 1,2,\cdots,n. \end{aligned}$$

Since

$$Hx - \sum_{i=1}^m \lambda_i a_i^T + c^T = \sum_{i=1}^n \mu_i e_i^T = \left(\mu_1, \mu_2, \cdots, \mu_n\right)^T \ge 0,$$

eliminate all the μ_i from above KKT conditions to have

$$\begin{aligned} &h_{i1}x_1 + \cdots + h_{in}x_n - a_{1i}\lambda_1 - \cdots - a_{mi}\lambda_i \ge -c_i, i = 1,2,\cdots,n, \\ &a_{i1}x_1 + \cdots + a_{in}x_n = b_i, i = 1,2,\cdots,l \\ &a_{i1}x_1 + \cdots + a_{in}x_n \ge b_i, i = l+1, l+2,\cdots,m \\ &x_i \ge l_i, i = 1,2,\cdots,n, \\ &\lambda_i \ge 0, i = l+1, l+2,\cdots,m, \\ &\left(h_{i1}x_1 + \cdots + h_{in}x_n - a_{1i}\lambda_1 - \cdots - a_{mi}\lambda_i + c_i\right)\left(x_i - l_i\right) = 0 \\ &\qquad i = 1,2,\cdots,n, \\ &\left(a_{i1}x_1 + \cdots + a_{in}x_n - b_i\right)\lambda_i = 0, i = l+1, l+2,\cdots,m. \end{aligned} \quad (3.2)$$

We will solve the linear part of (3.2) by Algorithm 2.1 while maintaining all of the complementarity conditions.

(3.2) has $n + m$ variables where $\lambda_1, \lambda_2, \cdots, \lambda_l$ are free. In order to initiate the computation, we introduce artificial inequalities

$$\lambda_1 \ge -M, \lambda_2 \ge -M, \cdots, \lambda_l \ge -M$$

into (3.2). The coefficient vectors of first $n + m$ (in) equalities of (3.2) are

$$h_i = \left(h_{i1}, \cdots, h_{in}, -a_{1i} \cdots - a_{mi}\right), i = 1,2,\cdots,n,$$

$$a_i = \left(a_{i1}, \cdots, a_{in}, 0, \cdots, 0\right), i = 1,2,\cdots,m,$$

and the coefficient vectors of last $n + m$ inequalities are denoted by $\boldsymbol{e}_i$ which is the ith row of the identity matrix of order $n + m$, $i = 1, 2, \cdots, n + m$.

In system (3.2),

$$h_{i1}x_1+\cdots+h_{in}x_n-a_{1i}\lambda_1-\cdots-a_{mi}\lambda_i\ge -c_i \quad\text{and}\quad x_i\ge l_i,$$
$$\lambda_i\ge 0 \quad\text{and}\quad a_{i1}x_1+\cdots+a_{in}x_n\ge b_i$$

are called complementary inequalities, and their coefficient vectors are called complementary vectors in which

$$\boldsymbol{h}_1,\boldsymbol{h}_2,\cdots,\boldsymbol{h}_n,\boldsymbol{e}_{n+1},\boldsymbol{e}_{n+2},\cdots,\boldsymbol{e}_{n+m}$$

are called Lagrange vectors and

$$\boldsymbol{e}_1,\boldsymbol{e}_2,\cdots,\boldsymbol{e}_n,\boldsymbol{a}_1,\boldsymbol{a}_2,\cdots,\boldsymbol{a}_m$$

are called constraint vectors, and the associated inequalities are called Lagrange inequalities and constraint (in) equalities respectively.

According to the definition of basic solution of the system of linear inequalities, if one of the complementary inequalities is basic, the corresponding complementarity condition is satisfied. Hence we will keep one of them basic and the other one nonbasic in the computational process.

The initial basic system for solving (3.2) is

$$x_1\ge l_1,\cdots,x_n\ge l_n,$$
$$\lambda_1\ge -M,\cdots,\lambda_l\ge -M,\lambda_{l+1}\ge 0,\cdots,\lambda_m\ge 0,$$

i.e., $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n, \boldsymbol{e}_{n+1}, \cdots, \boldsymbol{e}_{n+m}$ are initial basic vectors, and

$$y^{(0)}=\left(l_1,\cdots,l_n,-M,\cdots,-M,0,\cdots,0\right)^T$$

is initial basic solution; $\boldsymbol{h}_i$ and $\boldsymbol{a}_i$ are initial nonbasic vectors with deviations

$$\sigma_i=h_iy^{(0)}+c_i=h_{i1}l_1+\cdots+h_{in}l_n+\left(a_{1i}+\cdots+a_{li}\right)M+c_i,$$
$$i=1,2,\cdots,n,$$

and

$$\sigma_{n+i}=a_iy^{(0)}-b_i=a_{i1}l_1+\cdots+a_{in}l_n-b_i,$$
$$i=1,2,\cdots,m.$$

The representation of nonbasic vectors in terms of basic ones and their deviations are given in **Table 3**.

Since $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_l$ are coefficient vectors of equalities, we first put them into the basis as many as possible. Without loss of generality, suppose that $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_l$ are linearly independent and suppose l pivoting operations can be carried out along the diagonal of the sub-matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & & \vdots \\ a_{l1} & \cdots & a_{ll} \end{pmatrix}.$$

In order to maintain the complementarity conditions, another l pivoting operations are carried out along the diagonal of

$$\begin{pmatrix} -a_{11} & \cdots & -a_{l1} \\ \vdots & & \vdots \\ -a_{1l} & \cdots & -a_{ll} \end{pmatrix}$$

which translate the l basic artificial vectors out of the basis. The above $2l$ pivoting operations are called preprocessing. Since the basic solution must satisfy every equality constraints, once an equality enters the basic system, it is never selected to leave the basic system. For this reason, the columns of basic equalities and the associated rows of artificial inequalities can be deleted from the table. It implies that the upper bound M of the artificial variable may be any number. For convenience of computation, we always take $M = 0$.

The computational process following the preprocessing is called main iterations where each table is called general (pivoting) table, as shown by **Table 4**.

Where h_N and h_B are a partition of the Lagrange vectors $\{h_1,\cdots,h_n\}$, e_{n+N} and e_{n+B} are a partition of the Lagrange vectors $\{e_{n+1},\cdots,e_{n+m}\}$, e_N and e_B as well as a_N and a_B are partitions of the constraint vectors $\{e_1,\cdots,e_n\}$ and $\{a_1,\cdots,a_m\}$ respectively, and σ_h, σ_λ, σ_e, σ_a in the last column are vectors formed by deviations of the corresponding nonbasic vectors.

We say the principal sub-matrix associated with nonbasic Lagrange vectors and basic constraint vectors to be Lagrange matrix and denote it by L, and say the principal sub-matrix associated with nonbasic constraint vectors and basic Lagrange vectors to be constraint matrix and denote it by Γ. For the initial table, $L = H$ and $\Gamma = O$; for **Table 4**,

$$L=\begin{pmatrix} L_{11} & L_{12} \\ L_{12}^{\mathrm{T}} & L_{22} \end{pmatrix} \quad\text{and}\quad \Gamma=\begin{pmatrix} D_{11} & D_{12} \\ D_{12}^{T} & D_{22} \end{pmatrix}.$$

Table 3. Initial table for (3.2).

	$\boldsymbol{e}_1$	$\cdots$	$\boldsymbol{e}_n$	$\boldsymbol{e}_{n+1}$	$\cdots$	$\boldsymbol{e}_{n+m}$	
$\boldsymbol{h}_1$	h_{11}	$\cdots$	h_{1n}	$-a_{11}$	$\cdots$	$-a_{m1}$	σ_1
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\boldsymbol{h}_n$	h_{n1}	$\cdots$	h_{nn}	$-a_{1n}$	$\cdots$	$-a_{mn}$	σ_n
$\boldsymbol{a}_1$	a_{11}	$\cdots$	a_{1n}	0	$\cdots$	0	σ_{n+1}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\boldsymbol{a}_m$	a_{m1}	$\cdots$	a_{mn}	0	$\cdots$	0	σ_{n+m}

Table 4. The general table.

	e_B	a_B	h_B	e_{n+B}	
h_N	L_{11}	L_{12}	$-E_{11}^T$	$-E_{21}^T$	σ_h
e_{n+N}	L_{12}^T	L_{22}	$-E_{12}^T$	$-E_{22}^T$	σ_λ
e_N	E_{11}	E_{12}	D_{11}	D_{12}	σ_e
a_N	E_{21}	E_{22}	$D_{12}{}^T$	D_{22}	σ_a

It can be proved that both Lagrange matrix and constraint matrix are positive semi-definite when the Hessian matrix H of (3.1) is positive semi-definite. A positive semi-definite matrix has the property that if the ith diagonal entry is zero, then all the entries in the ith row and in the ith column are zeros.

Main iterations involve two kinds of operations. One is called principal pivoting which is carried out on a positive diagonal element of the table. The other one is called double pivoting which is carried out on a pair of symmetric off-diagonal elements successively while the diagonal element is zero. The $2l$ pivoting operations in the preprocessing stage are l double pivoting operations.

The basic solution associated with **Table 4** is

$$x_B = l_B, x_N = \sigma_e + l_N, \lambda_B = 0, \lambda_N = \sigma_\lambda .$$

Besides, we have $\mu_B = 0$ and $\mu_N = \sigma_h$. Where l_B and l_N are vectors formed by the right-hand side terms of basic and nonbasic $x_i \ge l_i$, λ_B and λ_N are vectors formed by basic and nonbasic multipliers of $\lambda_1, \lambda_2, \cdots, \lambda_m$, μ_B and μ_N are vectors formed by basic and nonbasic multipliers of $\mu_1, \mu_2, \cdots, \mu_n$ respectively.

Computational steps for the convex QP are as follows.

Algorithm 3.1. Computational steps for (3.2).

Step 1. Initial step.

Let

$$x_1 \ge l_1, \cdots, x_n \ge l_n, \lambda_1 \ge 0, \cdots, \lambda_m \ge 0$$

be the initial basic system and construct initial table given by **Table 3**.

Step 2. Preprocessing.

For $i = 1, 2, \cdots, l$, when the deviation of a_i is less than, greater than, or equal to 0, go to 1), 2) and 3) respectively.

1) If there is no positive elements in the row of a_i that are corresponding to basic inequalities, there is no feasible solution, stop; otherwise carry out a pivoting on one of the positive elements and then carry out a pivoting on the symmetric element.

2) If there is no negative elements in the row of a_i that are corresponding to basic inequalities, there is no feasible solution, stop; otherwise carry out a pivoting on one of the negative elements and then carry out a pivoting on the symmetric element.

3) If all the elements in the row of a_i that are corresponding to basic inequalities are zero, delete the row of a_i and then delete the column of e_{n+i}; otherwise carry out a pivoting on one of the nonzero elements and then carry out a pivoting on the symmetric element.

Step 3. Main iterations.

1) If all the deviations of nonbasic vectors (except for nonbasic Lagrange vectors $e_{n+1}, \cdots, e_{n+l}$ that are corresponding to equality constraints) are non-negative, the current basic solution is the solution to (3.2), stop; otherwise select a nonbasic vector with negative deviation to enter the basis.

2) If there is no positive element in the row of the entering vector that is corresponding to the basic inequality, (3.2) has no solution, stop. Otherwise

3) if the diagonal element in the row of the entering vector is positive, carry out a pivoting on that diagonal element, return to 1); otherwise carry out a pivoting on the largest off-diagonal element in the row of the entering vector, and then carry out a pivoting on the symmetric element, and return to 1).

We will prove that the smallest index rule can prevent cycling even though it may involve more pivoting operations than the smallest deviation rule does. Here each pair of complementary vectors has the same index. For example, h_i and e_i have index i for $i = 1, 2, \cdots, n$, and a_i and e_{n+i} have index $n + i$ for $i = 1, 2, \cdots, m$.

If cycling occurs, some vectors would enter and leave the basis repeatedly. Let s be the largest index of such vectors and consider two tables in the computational process. In one table the sth Lagrange vector enters and the sth constraint vector leaves the basis, and in the other table just reverse. Let P and Q be the principal sub-matrices associated with complementary nonbasic vectors of these two tables, and σ_P and σ_Q be deviations associated with P and Q respectively. Then $Q = P^{-1}$, $\sigma_Q = -Q\sigma_P$ and

$$\sigma_P^T \sigma_Q = -\sigma_P^T Q \sigma_P \le 0$$

where Q can be partitioned into a 2×2 matrix as shown in **Table 4** or positive definite. It contradicts the assumption that the components of σ_P and σ_Q with index s are negative and other components are nonnegative.

3.2. Convex QP with Upper Bounded Variables

Now let us consider the quadratic programming where variables are bounded from above:

$$\begin{aligned} &\min f(x) = \frac{1}{2} x^T H x + cx \\ &\text{s.t. } a_i x = b_i, i = 1, 2, \cdots, l, \\ &\quad a_i x \ge b_i, i = l+1, l+2, \cdots, m, \\ &\quad u_i \ge x_i \ge l_i, i = 1, 2, \cdots, n. \end{aligned} \tag{3.3}$$

Let

$$h_i(x, \lambda) = h_{i1} x_1 + \cdots + h_{in} x_n - a_{1i} \lambda_1 - \cdots - a_{mi} \lambda_i + c_i, \quad i = 1, \cdots, n$$

and let I_1 and I_2 be a partition of $\{1, \cdots, n\}$, *i.e.*, $I_1 \cup I_2 = \{1, \cdots, n\}$ and $I_1 \cap I_2 = \varnothing$, but empty I_1 or I_2 is allowed. It can be verified that if $\bar{x}$ is a solution to the system

$$h_i(x,\lambda)\ge 0,\ x_i\ge l_i,\ h_i(x,\lambda)(x_i-l_i)=0,\ i\in I_1,$$
$$-h_i(x,\lambda)\ge 0,\ -x_i\ge u_i,\ -h_i(x,\lambda)(-x_i+u_i)=0,\ i\in I_2,$$
$$a_i x\ge b_i,\ \lambda_i\ge 0,\ \lambda_i(a_i x-b_i)=0,\ i=l+1,\cdots,m,$$
$$a_i x=b_i,\ i=1,2,\cdots,l, \tag{3.4}$$

and every components of $\bar{x}$ satisfy $-x_i\ge -u_i\ (i\in I_1)$ and $x_i\ge l_i\ (i\in I_2)$, then $\bar{x}$ is the optimal solution for (3.3).

The solution of (3.4) is as follows.

Firstly, let $I_1 = \{1, 2, \cdots, n\}$ and $I_2 = \varnothing$, *i.e.*, ignoring all the upper bounds u_i, to solve (3.4) where the first two steps of Algorithm 3.1 are applied. In the following computational process we will make the basic solution satisfy not only $x_i\ge l_i$ but also $-x_i\ge -u_i$. That is we should make deviations of both e_i and $-e_i$ non-negative. In other words, if the deviation of e_i or $-e_i$ is negative, then it is a candidate to enter the basis.

If $-e_i$ is entering the basis but it does not appear in the table, we change the table as follows. Replace the deviation σ_i of e_i with $u_i - l_i - \sigma_i$, reverse signs of other entries in the row of e_i, and then substitute $-e_i$ for e_i; reverse signs of entries in the column of h_i and then substitute $-h_i$ for h_i. The above operation is called a vector substitution for e_i. It amounts to replacing the nonbasic $x_i\ge l_i$ with $-x_i\ge -u_i$ and replacing the basic $h_i(x,\lambda)\ge 0$ with $-h_i(x,\lambda)\ge 0$. Similarly, if e_i is entering the basis but it does not appear in the table, we update the table by conducting a vector substitution for $-e_i$.

Another problem in solving (3.4) is that when a Lagrange vector h_i, $-h_i$ or e_{n+i} is entering the basis, all the entries in this row are non-positive. In this situation, the change of the table is stated in the following 1) and 2).

1) Suppose that the Lagrange vector h_i is entering the basis but all the entries in this row are non-positive. In this case we conduct a vector substitution for h_i, *i.e.*, reverse signs of all the entries in the row of h_i and then substitute $-h_i$ for h_i; reverse signs of entries in the column of e_i and then substitute $-e_i$ for e_i. It amounts to replacing the nonbasic $h_i(x,\lambda)\ge 0$ with $-h_i(x,\lambda)\ge 0$ and replacing the basic $x_i\ge l_i$ with $-x_i\ge -u_i$. The associated equalities of the last two inequalities are $x_i=l_i$ and $x_i=u_i$, therefore the change of the right-hand side of $x_i=l_i$ is $u_i - l_i$. Considering signs of all the entries in the column of $-e_i$ had been reversed, by (2.6) we multiply the column of $-e_i$ by $l_i - u_i$, and then add it to the deviation column. Similarly, if $-h_i$ is entering the basis but all the entries in this row are non-positive, we conduct a vector substitution for $-h_i$

2) Suppose that the Lagrange vector e_{n+i} ($i = l + 1, \cdots, m$) is entering the basis but all the entries in this row are non-positive. We see that $\boldsymbol{e}_{n+i}$ is a $n + m$ dimensional unit vector with 1 at the $(n + i)$ th position, $\boldsymbol{h}_i$ and $-\boldsymbol{h}_i$ are $n + m$ dimensional vector whose last m components are not all zeros, and the last m components of constraint vectors are zeros. It implies that the expression of e_{n+i} in terms of basic vectors must contain $\boldsymbol{h}_j$ or $-\boldsymbol{h}_j$. Suppose that w_{ij} is a negative off-diagonal entry in the row of e_{n+i} and in the column of $\boldsymbol{h}'_j$ (= $\boldsymbol{h}_j$ or $-\boldsymbol{h}_j$). We conduct a vector substitution for $\boldsymbol{e}_{n+i}$ in this way: reverse signs of entries in the column of $\boldsymbol{h}'_j$ and then substitute $-\boldsymbol{h}'_j$ for $\boldsymbol{h}'_j$; replace the deviation σ_j of $\boldsymbol{e}'_j$ (= $\boldsymbol{e}_j$ or $-\boldsymbol{e}_j$) with $u_j - l_j - \sigma_j$, reverse signs of other entries in the row of $\boldsymbol{e}'_j$, and then substitute $-\boldsymbol{e}'_j$ for $\boldsymbol{e}'_j$. This substitution transforms w_{ij} into a positive number so that a regular pivoting $\boldsymbol{e}_{n+i} \leftrightarrow -\boldsymbol{h}'_j$ can be carried out.

Now let us give the algorithm for solving (3.3). In this algorithm, the non-basic vectors include not only those listed in the table but also those that are not listed in the table where $\boldsymbol{e}'_i$ = $\boldsymbol{e}_i$ or $-\boldsymbol{e}_i$ and $\boldsymbol{h}'_i$ = $\boldsymbol{h}_i$ or $-\boldsymbol{h}_i$.

Algorithm 3.2. Computational steps for (3.4).

Step 1. Initial step, see step 1 of Algorithm.3.1.

Step 2. Preprocessing, see step 2 of Algorithm 3.1.

Step 3. Main iterations.

1) If all the deviations of non-basic vectors except for $e_{n+1}, \cdots, e_{n+l}$ are non-negative, the current basic solution is optimal for (3.3), stop. Otherwise

2) select a non-basic vector with negative deviation to enter the basis. If the entering vector is $\boldsymbol{a}_i$, $\boldsymbol{e}'_i$, $\boldsymbol{h}'_i$ or $\boldsymbol{e}_{n+i}$, go to a), b), c), d) respectively.

a) If all the entries in the row of a_i that are corresponding to basic inequalities are non-positive, (3.3) has no feasible solution, stop. If the diagonal entry in the row of a_i is positive, carry out a pivoting on that entry, return to 1); otherwise carry out a pivoting on the largest entry in the row of a_i and then carry out a pivoting on the symmetric entry, return to 1).

b) If $\boldsymbol{e}'_i$ is not listed in the table, conduct a vector substitution for $-\boldsymbol{e}'_i$. In the row of the entering vector $\boldsymbol{e}'_i$, if all the entries that are corresponding to basic inequalities are non-positive, (3.3) has no feasible solution, stop; otherwise if the diagonal entry is positive, carry out a pivoting on that entry, return to 1); otherwise carry out a pivoting on the largest entry and then carry out a pivoting on the symmetric entry, return to 1).

c) If the diagonal entry in the row of $\boldsymbol{h}'_i$ is positive, carry out a pivoting on that entry, return to 1); otherwise if there is a positive entry in the row of $\boldsymbol{h}'_i$, carry out a pivoting on the largest entry and then carry out a pivoting on the symmetric entry, return to 1); otherwise conduct a vector substitution for $\boldsymbol{h}'_i$, return to 1).

d) If the diagonal entry in the row of $\boldsymbol{e}_{n+i}$ is positive, carry out a pivoting on that entry, return to 1); otherwise if there is a positive entry in the row of $\boldsymbol{e}_{n+i}$, carry out a pivoting on the largest entry and then carry out a pivoting on the symmetric entry, return to 1); otherwise conduct a vector substitution for $\boldsymbol{e}_{n+i}$, return to 1).

4. Mean-Variance Portfolio Optimization

4.1. Markowitz's Portfolio Selection Model

Suppose that there are n assets for selection. The rates of return of these n assets are $R_1, \cdots, R_n$ (random variables) whose means are $r_1, \cdots, r_n$ respectively and the covariance matrix is $H = (\sigma_{ij})_{n\times n}$ where $\sigma_{ij} = COV(R_i, R_j)$. Let $x_1, x_2, \cdots, x_n$ be fractions of n assets. The problem is to determine the portfolio $(x_1, x_2, \cdots, x_n)$ so that it has a minimal variance risk at a certain level of expected return rate. It is formulated as follows.

$$\begin{aligned} &\min \frac{1}{2}x^T Hx \\ &\text{s.t. } r_1x_1 + \cdots + r_nx_n = r_p, \\ &\quad x_1 + \cdots + x_n = 1, \\ &\quad x_1, \cdots, x_n \ge 0, \end{aligned} \quad (4.1)$$

where $x = (x_1, x_2 \cdots, x_n)^T$ and r_p is the expected return rate of the investor. Obviously, r_p should satisfy

$$\min\{r_1, \cdots, r_n\} \le r_p \le \max\{r_1, \cdots, r_n\}$$

to ensure the feasibility.

The KKT conditions of model (4.1) are

$$\begin{aligned} &\sigma_{i1}x_1 + \cdots + \sigma_{in}x_n - \lambda_1 - \lambda_2 r_i \ge 0, i = 1, \cdots n, \\ &x_1 + \cdots + x_n = 1, \\ &r_1x_1 + \cdots + r_nx_n = r_p, \\ &x_1, \cdots, x_n \ge 0, \\ &\left(\sigma_{i1}x_1 + \cdots + \sigma_{in}x_n - \lambda_1 - \lambda_2 r_i\right)x_i = 0, i = 1, \cdots, n. \end{aligned} \quad (4.2)$$

Add up the n complementarity conditions to have $x^T Hx - \lambda_1 - \lambda_2 r_p = 0$, thus

$$x^T Hx = \lambda_1 + \lambda_2 r_p .$$

This is another way to compute the variance risk of the portfolio.

(4.1) is a special case of (3.1) mainly in that (4.1) has no general inequality constraints and has feasible solutions. It makes the computation of (4.1) much easy.

Algorithm 4.1. Pivoting algorithm for (4.2).

Step 1. Initial step.

Let

$$x_1 \ge 0, \cdots, x_n \ge 0, \lambda_1 \ge 0, \lambda_2 \ge 0$$

be the initial basic system to construct initial table as shown by **Table 5**.

Table 5. Initial table.

	$\boldsymbol{e}_1$	...	$\boldsymbol{e}_n$	$\boldsymbol{e}_{n+1}$	$\boldsymbol{e}_{n+2}$	
$\boldsymbol{h}_1$	σ_{11}	...	σ_{1n}	-1	$-r_1$	0
...	...	...	...	...	...	...
$\boldsymbol{h}_n$	σ_{n1}	...	σ_{nn}	-1	$-r_n$	0
$\boldsymbol{e}$	1	...	1	0	0	-1
$\boldsymbol{r}$	r_1	...	r_n	0	0	$-r_p$

Where $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n, \boldsymbol{e}_{n+1}, \boldsymbol{e}_{n+2}$ are coefficient vectors of the initial basic system which are the $n + 2$ rows of the identity matrix of order $n + 2$, $\boldsymbol{h}_i = (\sigma_{i1}, \cdots, \sigma_{in}, -1, -r_i)$, $\boldsymbol{e} = (1, \cdots 1, 0, 0)$ and $\boldsymbol{r} = (r_1, \cdots, r_n, 0, 0)$.

Step 2. Preprocessing.

Suppose that $r_1 = \min\{r_1, \cdots, r_n\}$, $r_2 = \max\{r_1, \cdots, r_n\}$ and $r_1 < r_2$. Carry out four pivoting operations as follows:

$$e \leftrightarrow e_1,\ h_1 \leftrightarrow e_{n+1},\ r \leftrightarrow e_2,\ h_2 \leftrightarrow e_{n+2}.$$

Step 3. Main iterations.

1) If all the deviations of nonbasic vectors except for e_{n+1} and e_{n+2} are nonnegative, the current basic solution is the solution of (4.2), stop. Otherwise

2) select a nonbasic vector (except for e_{n+1} and e_{n+2}) with the most negative deviation to enter the basis. If the diagonal entry in the row of entering vector is positive, carry out a pivoting on that entry, return to 1); otherwise carry out two pivoting operations first on the largest entry in the entering vector row and then carry out a pivoting on the symmetric entry, return to 1).

Denote *rank* (H) by the rank of H. It can be proved that among n basic vectors $e_1, \cdots, e_n$ at most *rank* (H) + 2 ones may be pivoted out of the basis. It implies that at most *rank* (H) + 2 components of $(x_1, \cdots, x_n)$ may be nonzeros.

Let $r_{i1}, \cdots, r_{iT}$ be the realization of R_i in T periods, then $r_i = \sum_{t=1}^{T} r_{it}/T$. Let

$$D = \begin{pmatrix} r_{11} - r_1 & r_{21} - r_2 & \cdots & r_{n1} - r_n \\ r_{12} - r_1 & r_{22} - r_2 & \cdots & r_{n2} - r_n \\ \cdots & \cdots & \cdots & \cdots \\ r_{1T} - r_1 & r_{2T} - r_2 & \cdots & r_{nT} - r_n \end{pmatrix},$$

then the covariance matrix $H = D^T D/(T-1)$. Since D is an $T \times n$ matrix, *rank* (H) is no more than T. Therefore each portfolio obtained by Algorithm 4.1 contains no more than $T + 2$ assets no matter how larger n is.

An experiment was conducted by using algorithm 4.1 combining with the parameter formula (2.6) for 70 weekend closed prices of 1072 stocks. Here $n = 1072$ and $T = 70 - 1 = 69$. It experiences about 80 pivoting operations to obtain one efficient portfolio and 314 pivoting operations to obtain 20 minimal variance portfolios with different values of r_p which are evenly ranging from the minimal mean to the maximal mean of the 1072 stocks. Each pivoting requires about 1074 × 1075 multiplications and additions. Therefore the total amounts of computation are 80 × 1074 × 1075 and 314 × 1074 × 1075 multiplications and additions respectively. We know that it requires about $1072^3/3$ multiplications and additions to solve a system of 1072 linear equations in 1072 variables by Gaussian elimination.

Now let us give a small example to show how to solve the portfolio problem by using Algorithm 4.1 and the parameter technique introduced in section 2.

Example 1. An investor is interested in three assets. The means of these three assets are 0.05, 0.11 and 0.08, and the covariance matrix is $\begin{pmatrix} 0.54 & 0.11 & 0.09 \\ 0.11 & 0.32 & 0.02 \\ 0.09 & 0.02 & 0.21 \end{pmatrix}$. Solve a portfolio with the minimal variance risk at expected return rate $r_p = 0.07$, 0.08, 0.09 or 0.1.

Solution. The model of this problem is

$$\begin{aligned} &\min \frac{1}{2} x^T H x \\ &\text{s.t. } x_1 + x_2 + x_3 = 1, \\ &\quad 0.05x_1 + 0.11x_2 + 0.08x_3 = r_p, \\ &\quad x_1, x_2, x_3 \geq 0, \end{aligned}$$

where $x = (x_1, x_2, x_3)^T$, H is the covariance matrix and $r_p = 0.07$, 0.08, 0.09 or 0.1.

The initial table for $r_p = 0.07$ is given by **Table 6**.

Since the minimal mean is in the first column, we carry out a double pivoting $(e \leftrightarrow e_1, h_1 \leftrightarrow e_4)$ to have **Table 7**.

Since the maximal mean is in the second column of the initial table, we carry out a double pivoting $(r \leftrightarrow e_2, h_2 \leftrightarrow e_5)$ to have **Table 8**.

In **Table 8**, ignoring the deviations of e_4 and e_5 since they are associated with equalities, there is only one negative deviation –0.2217 taken by h_3. Therefore h_3 enters the basis. Since the diagonal entry 0.37 in the row of h_3 is positive, carry out a principal pivoting on 0.37 to yield **Table 9** ignoring the last three columns.

From the column $r_p = 0.07$ we see that all the deviations of nonbasic vectors except for e_4 and e_5 are positive, therefore the minimal risk portfolio for $r_p = 0.07$ is (x_1, x_2, x_3) = (0.3671, 0.0338, 0.5991), and $\lambda_1 = 0.4165$ and $\lambda_2 = -3.2117$. The variance risk of this portfolio is

$$\lambda_1 + \lambda_2 r_p = 0.4165 - 3.2117 \times 0.07 = 0.1916.$$

Now consider the portfolio with $r_p = 0.08$. since $\Delta = 0.08 - 0.07 = 0.01$, by (2.6), we multiply the column of r by 0.01 and then add it and the column of $r_p = 0.07$ to get the column of $r_p = 0.08$. We see that the last three numbers of the column of $r_p = 0.08$ are still positive. Therefore the minimal risk portfolio for $r_p = 0.08$ is (x_1, x_2, x_3) = (0.2095, 0.2095, 0.5811), and $\lambda_1 = 0.2607$ and $\lambda_2 = -1.4459$. The variance risk of this portfolio is

$$\lambda_1 + \lambda_2 r_p = 0.2607 - 1.4459 \times 0.08 = 0.1451.$$

For $r_p = 0.09$, since $\Delta = 0.09 - 0.08 = 0.01$, we multiply the column of r by 0.01 and then add it and the column of $r_p = 0.08$ to get the column of $r_p = 0.09$. We see that the last three numbers of the column of $r_p = 0.09$ are positive also. Therefore the minimal risk portfolio for $r_p = 0.09$ is (x_1, x_2, x_3) = (0.0518, 0.3851, 0.5631), and $\lambda_1 = 0.1050$ and $\lambda_2 = 0.3198$. The variance risk is

$$\lambda_1 + \lambda_2 r_p = 0.1050 + 0.3198 \times 0.09 = 0.1338.$$

In the same way, we can get the column of $r_p = 0.1$. We see that the deviation of e_1 is negative and the diagonal entry is positive. Carry out a pivoting $e_1 \leftrightarrow h_1$ to have **Table 10**.

From **Table 10** we see that the last three deviations are positive. Therefore the minimal risk portfolio for $r_p = 0.1$ is (x_1, x_2, x_3) = (0, 0.6667, 0.3333), and $\lambda_1 = -0.2811$ and $\lambda_2 = 4.5556$. The variance risk of this portfolio is

$$\lambda_1 + \lambda_2 r_p = -0.2811 + 4.5556 \times 0.1 = 0.1744.$$

4.2. Portfolio of Upper Bounded Assets

For institutional investors some asset such as a stock is not allowed greater than a limited ratio of the total capital. In this situation the model with upper bounded variables is required. Let u_i be the upper bound for asset x_i that satisfies $u_1 + \cdots + u_n \geq 1$ and $0 < u_i \leq 1$ for $i = 1, \cdots, n$. This problem is formulated as

Table 6. Initial table for $r_p = 0.07$.

	e_1	e_2	e_3	e_4	e_5	
h_1	0.54	0.11	0.09	-1^*	–0.05	0
h_2	0.11	0.32	0.02	–1	–0.11	0
h_3	0.09	0.02	0.21	–1	–0.08	0
e	1^*	1	1	0	0	–1
r	0.05	0.11	0.08	0	0	–0.07

Table 7. Result of first double pivoting.

	e	e_2	e_3	h_1	e_5	
e_4	0.54	–0.43	–0.45	–1	–0.05	0.54
h_2	–0.43	0.64	0.36	1	-0.06^*	–0.43
h_3	–0.45	0.36	0.57	1	–0.03	–0.45
e_1	1	–1	–1	0	0	1
r	0.05	0.06^*	0.03	0	0	–0.02

Table 8. Result of second double pivoting.

	e	r	e_3	h_1	h_2	
e_4	1.7011	–16.0556	–0.2683	–1.8333	0.8333	0.5722
e_5	–16.0556	177.7778	0.6667	16.6667	–16.6667	–3.6111
h_3	–0.2683	0.6667	0.37*	0.5	0.5	–0.2217
e_1	1.8333	–16.6667	–0.5	0	0	0.6667
e_2	–0.8333	16.6667	–0.5	0	0	0.3333

Table 9. Result of first principal pivoting.

	e	r	h_3	h_1	h_2	$r_p = 0.07$	$r_p = 0.08$	$r_p = 0.09$	$r_p = 0.1$
e_4	1.5065	–15.5721	–0.7252	–1.4707	1.1959	0.4165	0.2607	0.1050	–0.0507
e_5	–15.5721	176.5766	1.8018	15.7658	–17.5676	–3.2117	–1.4459	0.3198	2.0856
e_3	0.7252	–1.8018	2.7027	–1.3514	–1.3514	0.5991	0.5811	0.5631	0.5450
e_1	1.4707	–15.7658	–1.3514	0.6757*	0.6757	0.3671	0.2095	0.0518	–0.1059
e_2	–1.1959	17.5676	–1.3514	0.6757	0.6757	0.0338	0.2095	0.3851	0.5608

Table 10. Result of principal pivoting for r_p = 0.1.

	e	r	h_3	e_1	h_2	
e_4	4.7078	–49.8889	–3.6667	–2.1767	2.6667	–0.2811
e_5	–49.8889	544.4445	33.3333	23.3333	–33.3333	4.5556
e_3	3.6667	–33.3333	0	–2	0	0.3333
h_1	–2.1767	23.3333	2	1.48	–1	0.1567
e_2	–2.6667	33.3333	0	1	0	0.6667

$$\begin{aligned} &\min \frac{1}{2} x^T H x \\ &\text{s.t. } r_1 x_1 + \cdots + r_n x_n = r_p, \\ &\quad x_1 + \cdots + x_n = 1, \\ &\quad 0 \le x_i \le u_i, i = 1, \cdots, n. \end{aligned} \tag{4.3}$$

where $r_{\min} \le r_p \le r_{\max}$ to ensure the feasibility. The minimal value of r_p is obtained as follows. Suppose that $r_1 \le r_2 \le \cdots \le r_n$. Distribute 1 to $x_1, x_2, \cdots, x_n$ sequentially such that $x_1 = u_1,\ x_2 = u_2, \cdots,\ x_{k-1} = u_{k-1},\ x_k = v$ where $u_1 + u_2 + \cdots + u_{k-1} + v = 1$ and $0 < v \le u_k$. Then

$$r_{\min} = r_1 u_1 + r_2 u_2 + \cdots + r_{k-1} u_{k-1} + r_k v .$$

Similarly, we can obtain the maximal value $r_{\max}$ of r_p. The algorithm for (4.3) is as follows.

Algorithm 4.2. Computational steps for (4.3).

Step 1. Initial step, see step 1 of Algorithm 4.1.

Step 2. Preprocessing, see step 2 of Algorithm 4.1.

Step 3. Main iterations.

1) If all the deviations of non-basic vectors except for e_{n+1} and e_{n+2} are non-negative, the current basic solution is optimal for (4.3), stop. Otherwise

2) select a non-basic vector with negative deviation to enter the basis. If this vector is not listed in the table, conduct a vector substitution. If the diagonal entry in that row is positive, carry out a pivoting on that entry, return to 1); otherwise carry out a pivoting on the most positive entry in that row and then carry out a pivoting on the symmetric entry, return to 1).

Example 2. In Example 1, each asset is not allowed to exceed 50%. Solve a minimal risk portfolio with r_p = 0.09.

Solution. The initial form is given by **Table 11**.

As done for Example 1, carrying out 4 pivoting operations $e \leftrightarrow e_1$, $h_1 \leftrightarrow e_4$, $r \leftrightarrow e_2$, $h_2 \leftrightarrow e_5$, and then carrying out a principal pivoting $h_3 \leftrightarrow e_3$, we have **Table 12** where the columns of e and r and the rows of e_4 and e_5 are deleted.

We see that the third asset 0.5631 is greater than 0.5. Therefore we let $-e_3$ enter the basis. But $-e_3$ is not listed

Table 11. Initial table for r_p = 0.09.

	e_1	e_2	e_3	e_4	e_5	
h_1	0.54	0.11	0.09	–1	–0.05	0
h_2	0.11	0.32	0.02	–1	–0.11	0
h_3	0.09	0.02	0.21	–1	–0.08	0
e	1	1	1	0	0	–1
r	0.05	0.11	0.08	0	0	–0.09

Table 12. Result of 5 pivoting operations.

	h_3	h_1	h_2	
e_3	2.7027	–1.3514	–1.3514	0.5631
e_1	–1.3514	0.6757	0.6757	0.0518
e_2	–1.3514	0.6757	0.6757	0.3851

Table 13. Result of vector substitution for e_3.

	$-h_3$	h_1	h_2	
$-e_3$	2.7027*	1.3514	1.3514	–0.0631
e_1	1.3514	0.6757	0.6757	0.0518
e_2	1.3514	0.6757	0.6757	0.3851

Table 14. Final table.

	$-e_3$	h_1	h_2	
$-h_3$	0.37	–0.5	–0.5	0.0233
e_1	0.5	0	0	0.0833
e_2	0.5	0	0	0.4167

in the current table. For this we conduct a vector substitution for e_3: replace the deviation 0.5631 of e_3 with 0.5 – 0.5631 = –0.0631, reverse signs of other entries in the row of e_3, and then substitute $-e_3$ for e_3; reverse signs of entries in the column of h_3 and then substitute $-h_3$ for h_3. It results in **Table 13**.

In **Table 13** we carry out a pivoting on the diagonal entry 2.7027 to yield **Table 14**.

Now all the deviations are positive and no assets are greater than 0.5. Hence we get the required portfolio. Since $-e_3$ is basic, $x_3 = u_3 = 0.5$, and $x_1 = 0.0833$, $x_2 = 0.4167$. The variance risk $x^T Hx = 0.1353$.

An experiment was conducted by using the same data with $n = 1072$ and $T = 69$. For $u_i = 0.1$, $i = 1, 2, \cdots, n$, it experiences about 95 pivoting operations to obtain one efficient portfolio and 347 pivoting operations to obtain 20 minimal variance portfolios with different values of r_p. Each pivoting requires about 1074 × 1075 multiplications and additions. Therefore the total amounts of computation are 95 × 1074 × 1075 and 347 × 1074 × 1075 multiplications and additions respectively.

5. Conclusions

In this paper we proposed a series of pivoting-based algorithms for solving the following problems:

- the system of linear inequalities;
- convex quadratic programming;
- mean-variance portfolio selection problems.

These algorithms are concise for understanding and efficient for computing as shown by the numerical examples and computer experiments for 1072 stocks. We also proved the convergence of the smallest index rule for convex QP therefore for mean-variance portfolio optimization for the first time.

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