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Existence of a Walrasian Equilibrium in Locally Convex Topological Vector Spaces, with Interdependent Utility Functions

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Abstract

In this paper we extend the well known result of existence of Walrasian equilibrium given in Debreu (1952) to economies modeled over topological vector spaces. Our work is closely related to the previous results of Yannelis-Pranhakar (1983), and Tan and Yuan (1999), but unlike these papers we prove the existence of a Walrasian equilibrium in infinite dimensional economies with interdependent and price-dependent utilities.

Keywords

Pseudo Games, Generalized Economies, Topological Vector Spaces, Walrasian Equilibrium

1. Introduction

Looking for commodities as physical goods which may differ on time or in the states of the world in which they become available, and allowing infinite variation in these contingents, the generalization of the classical model of General Equilibrium to models of economies with infinitely many commodities looks natural. Since the consumption sets of these economies are subsets of infinite dimensional vector spaces they are usually called, infinite dimensional economies.

The purpose of this paper is to prove the existence of Walrasian equilibrium in economies whose consumption sets are subsets of a locally convex topological vector space with a lattice structure. Utilities are, interdependent, (*i.e.*, may depend on the choices of others) and prices dependent. The set of consumers is finite.

Our work generalizes the seminal work of G. Debreu (1952) (see [1]) to economies whose consumptions sets are subsets of locally convex topological vector spaces. Several problems arise when we try to extend well known results in finite dimensional economies to infinite dimensional economies. In the specialized literature are considered finite economies those whose consumption sets are subsets of finite dimensional vector spaces like $\Re^n$ and infinite dimensional economies those whose consumption sets are subsets of infinite dimensional vector spaces, like, for instance, Banach spaces, the space of bounded functions, the set of square integrable functions, the set of bounded sequences, etc...

The main difficulty in extending to infinite-dimensional economies the well known results for finite dimensional economies, lies in the non-uniqueness of possible linear[1] Hasudorff[2] topologies to be chosen for a same infinite dimensional vector spaces, while in finite-dimensional vector spaces there are (essentially) a single topology combining these two properties (Hausdorff and linearity). More precisely, every linear Hausdorff topology on a finite dimensional space, define the same family of open sets. In infinite dimensional vector spaces this claim is no longer true. In this case we must choose a topology. The generality of the results obtained in these models depends largely on the chosen topology. Recall that when we choose the topology we are choosing the family of open sets and consequently the continuous functions and also the compact sets, the convergent nets, etc.

The existence of a continuous utility function and a compact budget constraints is a sufficient condition for the existence of the individual demand function. If we consider in a vector space, a weaker topology than another (*i.e.*, a topology with less open subsets), some functions continuous in the initial topology will no longer be continuous in the weaker one. The less open sets a topology has, the less continuous functions there are, but more compact sets there are. Note that Hausdorff topology must have a significant number of open sets. A vector space with a topology associate is called a topological vector space.

Given that utility functions represent preferences, is clear that we must be careful when elucidating the topology. We must try not to lose generality and lose the fewest possible utility functions to be considered in our analysis. On the other hand, recall that prices are a subset of the linear and continuous real functions (functionals) on a given topological space then, we need choose a topology such that linear functionals remain being continuous even when we consider a weaker topology. The weakest topology verifying this property is called the weak topology. For a survey on this topic see [2].

Our results are close to the previous work of Yannelis and Prabhakar (1983) (see [3]). But unlike what was done in this and in others subsequent works, where only the existence of an equilibrium in the sense of Shafer-Sonnenschein[3] is proved, we show the existence of Walrasian equilibrium. That is, we show the existence of a pair (p,x) conformed by prices and an allocation, in a dual pair (E,F) (see below), such that every consumer maximizes his preferences (that may depends on the choices of the others and/or on prices) in their budget sets. Here, by E we denote a locally convex topological vector space and by F we denote a total subspace of E^* the algebraic dual of E *i.e.*; the set of all linear functionalities defined on E We consider that the

[1]A topology defined in a vector space is said to be linear if with respect to it, the vectors sum and the multiplication of a vector by a scalar are both continuous operations.

[2]A topology is Hausdorff or T_2 if for distinct points there are disjoint open set.

[3]see [4].

consumption set of each consumer is the positive cone of a locally convex topological vector space. In order to prove the existence of the Walrasian equilibrium we assume the existence of a countable and increasing family of bounded and closed subsets of E which cover E As we shall see this property is not to strong and it is verified by a wide range of topological vector spaces.

Unlike [5] we do not consider the hypothesis of 0-diagonality quasi-concave utility functions, however we assume the weak continuity of such functions. This condition appears naturally in different situations. In particular in finance models, where any condition with respect to the continuity of utilities should be considered to ensure the existence of equilibrium, see for instance [6]. Generally this condition is known as *consumer impatience* and it is equivalent to the weak continuity of utilities. Unlike de G. Chichilnisky and A. Mas-Colell (see [7] and [8]) and others authors, we do not consider the hypothesis of properness for the utility functions.

The rest of the paper is organized as follows: In Section 2 we consider some topological facts which are necessary to understand the rest of the work. In Section 3 we introduce the model and in Section 4, we show the existence of a Social Equilibrium for a generalized game. In Section 5 we consider a pure exchange economy whose utilities are interdependent and prices dependent The consumption set, for this economy is the positive cone of a locally convex topological vector space and we prove the existence of a Walrasian equilibrium, without any additional condition on the consumption set. In section 8 we analyze the main characteristics of the utility functions considered in this work. And finally we give some conclusions.

2. Some Topological Facts

Suppose that E and F are linear spaces, and that $\langle \cdot,\cdot \rangle$ is a bilinear form on $E\times F$, in which case E and F are paired spaces. In this case any ny $y\in F$ determines a linear functional on E, namely $x\to \langle x,y\rangle$. We say that a topology τ on E is a topology of the pair (E,F) if F is the topological dual (i.e; the set of continuous linear functions defined on E), In this case $F=(E,\tau)'$. We denote by $\sigma(E,F)$ the weakest topology in E which makes continuous all the linear functionals, $x\to \langle x,y\rangle$.

If for each nonzero $x\in E$ there exists $y\in F$ such that $\langle x,y\rangle \neq 0$ then F is said total or that distinguishes points of E The analogous meaning is attached to E distinguishes points of F If each vector space distinguishes points of the other, then we call (E,F) a dual pair. The weakest topology defined on a vector space E, such that F represent the set of continuous linear functions, is called the weak topology on E defined by F and will be denoted by the symbol $\sigma(E,F)$.

The following result is useful (see [9]).

Theorem 1. *For a pair* (E,F) *the following are equivalent:*

1. E distinguishes points of F.
2. $\sigma(E,F)$. is Hausdorff.

Given a dual pair: (E,F) a locally convex topology[4] τ for X is a topology of

[4]A topology defined on E is said locally convex if each point $x\in E$ has a local base of neighborhoods consisting of convex sets.

the pair or is consistent with the pairing if $F=(E,\tau)'$. So, two topologies τ and σ in E are said consistent with the dual pair (E,F) if $(E,\tau)'=(E,\sigma)'=F$.

In what follows we consider a dual pair (E,F) and the weak topology $\sigma(E,F)$. We recall the following properties and definitions of weak-closed and bounded susbset.

1. According to the Mackey's Theorem, all locally convex topologies consistent with a given dual pair, have the same collection of bounded sets and the same collection of closed convex sets. This property is known as the Permanence of closed convex sets.

2. A subset $A\subset E$ is weakly bounded if it is pointwise bounded. That is $A\subset E$ is weakly bounded if and only if for every $y\in F$ the set $\{\langle x,y\rangle : x\in A\}$ is bounded in $\mathbb{R}$.

Analogously, $B\subset F$ is $\sigma(F,E)$-bounded if and only if for every $x\in X$ the set $\{\langle x,y\rangle : y\in B\}$ is bounded in $\mathbb{R}$.

For a locally convex topological vector space,(l.c.t.v.s) (E,τ) by E^* we denote its algebraic dual, *i.e.*, the vector space of all linear functional on E (continuous or not) and by E' we denote the vector spaces of all continuous linear functionals on E.

Let F be a subspace of E^* such that distinguishes points, or equivalently F is a total subspace. It follows that,

1. (E,F) is a dual pair. Given the dual pair (E,F), some times to make less cumbersome notation, we shall denote the weak topology $\sigma(E,F)$, just as w.

2. The topology $\sigma(E,F)$ is Hasudorff and locally convex.

3. Note that $x_\alpha \to^{\sigma(E,F)} x$ in E if and only if $\langle x_\alpha, x'\rangle \to \langle x,y\rangle$ for all $y\in F$.

4. The locally convex Hasudorff topology $\sigma(F,E)$ is known as the weak* topology (or simply the w^*-topology) on F.

5. $x'_\alpha \to^{\sigma(F,E)} x'$ in F if and only if

$$\langle x, x'_\alpha\rangle \to \langle x, x'\rangle \quad \text{for all } x\in E.$$

By Alaoglu's Theorem we know that:

If $V\subset E$ is a neighborhood of zero for some locally convex consistent topology with the dual (E,F) then, its polar

$$V^0=\{x'\in F : |\langle x,x'\rangle|\le 1 \text{ for all } x\in V\}$$

is $\sigma(F,E)$-compact. Similarly, if W is a neighborhood of zero for a consistent locally convex topology on F then its polar

$$W^0=\{x\in E : |\langle x,x'\rangle|\le 1 \quad \text{for all} \quad x'\in W\}$$

is $\sigma(F,E)$-compact.

Note also that the Alaoglu's theorem provides something like a Heine-Borel property for the w^*-topology, namely that closed an bounded subset of E' are w^*-compact.

Next we describe hemicontinuity of correspondences in terms of nets.

Definition 1. *Let (X,σ) and (Y,τ) be two locally convex vectorial topological spaces. Let $\phi: X\to Y$ be a correspondence between these spaces, Let $x\in X$ then:*

1. ϕ is **upper hemicontinuous** at x if and only if the following property holds: Let $\{x_\alpha\}$ be a net such that $x_\alpha\to x$ and let $y_\alpha\in\phi(x_\alpha)$, then the net $\{y_\alpha\}$ has a limit point in $\phi(x)$.

2. ϕ is **lower hemicontinuous** at x if and only if the following property holds: Let $\{x_\alpha\}$ be a net such that $x_\alpha \to x$ and let $y \in \phi(x)$ then there exists a subnet x_{α_λ} and elements $y_\lambda \in \phi(x_{\alpha_\lambda})$ such that the net $\{y_{\alpha_\lambda}\}$ converges to y.
3. ϕ is **continuous** at x if and only is upper and lower hemicontinuous.

3. The Model

We consider a locally convex, topological vector space (E, w) equipped with a partial order $\geq$. That is $\geq$ is a transitive reflexive and antisymmetric binary relation, compatible with the algebraic structure of E in the sense that, the following properties are verified:

1. $x \geq y$ implies $x + z \geq y + z$.
2. $x \geq y$ implies $\alpha x \geq \alpha y$ for each $\alpha \geq 0$.

The positive cone is the pointed convex set $E_+ = \{x \in E : x \geq 0\}$. Given x and y in E $x \geq y$ if and only if $x - y \in E_+$.

In order to prove the existence of the Walrasian equilibrium, we assume the existence of a increasing fundamental sequence $\{D_n\}_{n\in N}$ of $\sigma(E,F)$-closed, convex, balanced and bounded subsets of E, such that $E \subseteq \bigcup_{n\in N} D_n$ and verifying that $0 \in D_n, \forall n \in N$. Note that this property is verified by a wide range of topological vector spaces. For instance all Banach spaces even with any compatible topology and all strong dual of Fréchet spaces, among others.

In which follows we assume that if (X, w) and (Y, μ) are topological vector spaces then $X \times Y$ is doted with the product topology, *i.e.*, $(X \times Y, w \times \sigma)$.

Following [1], we introduce the definition of Abstract Economy or Generalized Game.

Definition 2. (Abstract Economy or Generalized Game) *Consider a set of* n *agents indexed by* $I = \{1, \cdots, n\}$. *An Abstract Economy is a set of* n *triads* $J_i = \{X_i, A_i, v_i\}; i \in I$ *where:*

1. X_i is a collection of subsets of E. For each $i \in I$, X_i represents the set of pure strategies of the *i*-th agent.
2. $v_i : X_i \times X_{-i} \to \mathbb{R}$, is the utility function of the *i*-th player. Where:

(a) $X_{-i} = \Pi_{j\neq i} X_j = X_1 \times, \cdots, \times X_{i-1} \times X_{i+1} \times, \cdots, \times X_n$ thus $x_{-i} \in X_{-i}$ if and only if $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots x_n)$.

(b) We denote by $X = \Pi_{i=1}^n X_i$ the set of strategies profiles.

(c) $A_i : X_{-i} \to 2^{X_i}$ is a correspondence, such that for each $x_{-i} \in X_{-i}$ we have $A_i(x_{-i}) \subset X_i$.

Note that each $x_{-i} \in X_{-i}$ represents a list of actions chosen by each player except for the *i*-th and X_i represents the set of strategies of the *i*-th player.

We consider that each agent chooses $x_i^* \in X_i$ in order to solve the maximization problem $\max_{x_i \in A_i(x_{-i})} v_i(x_i, x_{-i})$

4. The Social Equilibrium Existence

This background makes the following formal definition Social Equilibrium.

Definition 3. (Social Equilibrium) *A strategy profile* $x^* \in X_i \times X_{-i}$ *is a Social Equilibrium if for each* $i \in I$ *it follows that* $x_i^* \in A_i(x_{-i}^*)$ *and*

$v_i\left(x_i^*, x_{-i}^*\right) = \max\left\{v_i\left(x_i, x_{-i}^*\right), x_i \in A_i\left(x_{-i}^*\right)\right\}$.

In that follows we assume that for each $i \in I$,

1. X_i is a nonempty, weak compact and convex subset of E. That is X_i is a $\sigma(E, F)$-compact and convex subset of E.

2. $v_i : X_i \times X_{-i} \to X_i$ is $\sigma(E, F)$ continuous, and quasi concave in the first variable, *i.e.*, $v_i(\cdot, x_{-i}) : X_i \to \mathbb{R}$ is quasi concave for each $x_i \in X_i$

3. The correspondence $A_i : X_{-i} \to X_i$ is nonempty weak continuous and weak compact valued.

The following theorem establishes the existence of a social equilibrium for an abstract economy.

Theorem 2. (Existence of the Social equilibrium) *Let* (E, w) *be a topological vector space, where* w *is a locally convex topology on* E *consistent with the dual pair* (E, F).

Let $X_i \subset E, i \in I$ be a finite collection of nonempty, convex, $\sigma(E, F)$-compact subsets of E and let $v_i : X_i \times X_{-i} \to \mathbb{R}$ be a family of weak continuous functions and quasi-concave in the first variable. Let $A_i : X_{-i} \to X_i$ be a $\sigma(E, F)$-continuous correspondence and weak compact valued. Then there exists a Social Equilibrium for the abstract economy.

In order to prove this theorem we need to proceed step by step. Some previous definitions, theorems and its corollaries are necessary.

Let $\phi : X \to Y$ be a correspondence. We denote by $Gr(\phi)$ the graph of ϕ, *i.e.*; the subset of $X \times Y$

$$Gr(\phi) = \left\{(x, y) \in X \times Y : y \in \phi(x)\right\}.$$

We shall use the following theorem [10].

Theorem 3. (Berge's Maximun Theorem) *Let* X *and* Y *be locally convex, topological vector spaces and* $\phi : X \to Y$ *be a continuos correspondence with nonempty and compact values. Suppose that* $f : X \times Y \to \mathbb{R}$ *is a continuous function. Then*

1. $m(x) = \max_{y \in \phi(x)} f(x, y)$ is a continuous function.

2. The correspondence $\mu(x) = \left\{y \in \phi(x); y \in \arg\max f(x, y)\right\}$ is upper hemicontinuous correspondence with compact values.

The following corollary is straightforward.

Corollary 1. (Weak Berge's Theorem) *Let* (E, w) *be a locally convex topological vector space, and* w *a consistent topology with the dual pair* (E, F). *Let* (Y, τ) *be a Hausdorff locally convex vector space where* τ *is a compatible topology with de dual pair* (Y, Y'). *Let* X *be a nonempty,* $\sigma(E, F)$*-compact and convex subset of* E *and* $\phi : X \to Y$ *a* $(\sigma(E, F), \tau)$*-continuous correspondence with nonempty* τ*-compact and convex values, and suppose that* $f : X \times Y \to \mathbb{R}$ *is a* $(\sigma(E, F) \times \tau)$*-continuous function. Then*

1. $m(x) = \max_{y \in \phi(x)} f(x, y)$ is a $\sigma(E, F)$-continuous function.

2. The correspondence $\mu(x) = \left\{y \in \phi(x); y \in \arg\max f(x, y)\right\}$ is an upper weak hemicontinuous correspondence.

The following fundamental fixed point theorem is due to Kakutani-Fan-Glicksberg, see [11].

Theorem 4. (Kakutani-Fan-Glicksberg) *Let* K *be a nonempty compact convex subset of a locally convex Hausdorff space and let the correspondence* $\phi : K \to K$ *be upper hemicontinuous, compact and convex valued. Then the set of fixed points of* ϕ *is compact and nonempty.*

We have the next straightforward corollary.

Corollary 2. (Weak Kakutani-Fan-Glicksberg) *Let* K *be a* $\sigma(E,F)$*- compact subset of* E *and let the correspondence* $\phi : K \to K$ *be* $\sigma(E,F)$ *-upper hemicontinuous,* $\sigma(E,F)$*- comoact valued. Then the set of fixed points of* ϕ *is* $\sigma(E,F)$*- compact and nonempty.*

Now we give the proof of the existence of the Social equilibrium.

Proof: Consider the correspondence

$$\sigma(E,F)\,\mu_i(x_{-i}) = \left\{z \in A_i(x_{-i}) : v_i(z, x_{-i}) \geq v_i(x_i, x_{-i}) \forall x_i \in A_i(x_{-i})\right\}.$$

Since $A_i : X_{-i} \to X_i$ is a weak-continuous correspondence, weak-compact and convex valued and v_i is weak continuous function, then by the corollary (1), it follows that μ_i is an upper weak hemicontinuous correspondence. Since in addition, v_i is quasi-concave in the first variable, then μ_i is convex valued.

The product correspondence $\mu : X \to X$ given by

$$\mu(x) = \mu_1(x_{-1}) \times \cdots \times \mu_n(x_{-n})$$

is jointly upper $\sigma(E,F)-$hemicontinuous, compact and convex valued correspondence. Recall that the product of a family of topological spaces is compact in the product topology if and only if each factor is compact. Finally, since the $\sigma(E,F)-$ topology is a Hausdorff topology then, by the corollary (2), there exists a fixed point $x^* \in X$ such that $x^* \in \mu(x^*)$.•

5. "Generalized" Economies and Pseudo-Games

Consider the locally convex topological vector space (E, w), partially ordered by $\geq$. Consequently the positive cone is given by

$$E_+ = \{x \in E : 0 \leq x\},$$

i.e.; the subset of all of element of E lower bounded by 0.

Let $V(0) \subset E$ be a neighborhood of zero. The polar $V(0)^*$ of $V(0)$ will be denoted by P *i.e.*;

$$P = \left\{p \in F : \left|\langle x, p\rangle\right| \leq 1, \forall x \in V(0)\right\}.$$

In general we can consider the $r-$polars.

$$V^r = \left\{p \in F : \left|\langle x, p\rangle\right| \leq r\right\}, r \geq 0.$$

Note that

$$\left|\langle x, p\rangle\right| \leq r \Leftrightarrow \left|\langle x, p/r\rangle\right| \leq 1.$$

By the Alaoglu's theorem P and P^r are $\sigma(F,E)$- compact.

We symbolize by P_+ the subset of P given by,

$$P_+ = \left\{ p \in P : \left\langle p, x \right\rangle \geq 0, \forall x \in E_+ \right\}.$$

Recall the next proposition (see for instance [12]).

Proposition 5. *Let* E *a locally convex topological vector space. If the space* F *distinguishes points in* E *and* C *and* K *are non empty* $\sigma(E,F)$*- ccompact and disjoint subsets of* E *then there exists* $y' \in F$ *that strongly separates* C *and* K *i.e.*:

$$y'(z) \leq r < s \leq y'(w); \forall z \in C, \text{ and } w \in K.$$

The proposition follows because the topology $\sigma(E,F)$ is Hausdorff C y K are $\sigma(E,F)$- compact, and since F is the dual of $(E, \sigma(E,F))$ then by the strong separation theorem there exist a nonzero $y' \in F$ strongly separating them.

In our case $C = P_+$ is $\sigma(F,E)$- compact, and by K take a point $p \in F/P_+$. So, K and C are $\sigma(E,F)$- closed disjoint and convex subset of F

We consider a pure exchange economy $\mathcal{E} = \left\{ X_i, u_i, w_i, i \in I \right\}$, where

1. I is a finite set of indexes, $I = \{1, \cdots, n\}$ one for each consumer.
2. $X_i = E_+$ is the consumption subset of the i^{-th} consumer. By X_{-i} we symbolize the cartesian product $X = \Pi_{j \neq i} X_j = (E_+)^{n-1}$.
3. By $v_i : E_+ \times (E_+)^{n-1} \times P_+ \to \mathbb{R}$ we denote the utility function for the *i*-th consumer. In addition we assume that utilities are $\sigma(F,E)$ -continuous and strictly quasi concave functions in the first variable.
4. By w_i we symbolizes the initial endowment of the *i*-th consumer.

We shall use a pseudo-game as a mathematical tool that allow us to prove the existence of a Walrasian Equilibrium in interchange economies. We consider consumers as the players of a pseudo game, and the budget set is the set of available strategies for each consumer. These players are looking to maximize their utility functions. A fictional player that corresponds to the market, is introduced. His objective is to choose a set of prices $p \in P_+$ maximizing $v_f(p,x) = p \sum_{i=1}^n (x_i - w_i)$ given the allocation $x = (x_1, \cdots, x_n)$.

Recall that the evaluation map $(p,x) \to \langle p, x \rangle$ restricted to $P \times E$ is jointly continuous in the $\sigma(F,E) \times \sigma(E,F)$- topology.

Now, we represent an interchange economy in terms of pseudogames.

1. Let $A_i : X_{-i} \times P_+ \to X_i$ be the budget correspondence for the *i*-th agent, given by,

$$A_i(x_{-i}, p) = B_i(p) = \left\{ x \in X_i : \langle p, x \rangle \leq \langle p w_i \rangle \right\} \subset X_i.$$

2. Note that the budget correspondence is $(\sigma(F,E), \sigma(E,F))$- upper hemicontinuous (moreover is constant in x_{-i}) and convex valued. But no need to be lower hemicontinuos.
3. For each $i \in I$ the function $v_i : X_i \times X_{-i} \times P_+ \to \mathbb{R}$ given by $v_i(x_i, x_{-i}, p)$, is the utility function of the *i*-th agent. These are $\sigma(E,F)$ continuous functions and strictly quasi concave in the first variable *i.e.*, $v_i(\cdot, x_{-i}, p) : E_+ \to \mathbb{R}$ is strictly quasi concave for each $(x_{-i}, p) \in E_+^{n-1} \times P_+$.
4. Each consumer looks to solve the maximization problem $\max_{x_i \in A_i(x_{-i}, p)} v_i(x_i, x_{-i}, p),.$
5. For the fictional player the correspondence $v_f : X \times P_+ \to P_+$ given by

$$v_f(x,p) = p\left(\sum_{i=1}^{n} x_i - w_i\right)..$$

6. This player looks to solve the maximization problem $\max_{p\in P_+} v_f(x,p)$.

To prove the existence of a Walrasian equilibrium we begin considering a restricted economy:

$$R\mathcal{E} = \{K_i, v_i, w_i, i \in I\}$$

where $w_i \neq 0$ and K_i is a $\sigma(E,F)$-compact and convex subset of E_+ for all $i \in I = \{1,\cdots,n\}$. By $B(K_i)$ we denote the budget set of the *i*-th agent corresponding to the restricted economy.

For some $\epsilon > 0$ we define the subset $P_\epsilon \subseteq P_+ \subset F$ by

$$P_\epsilon = \{p \in P_+ : \epsilon \leq \langle p, w_i\rangle \forall i \in \{1,\cdots,n\}\}.$$

Consider the correspondence $B_{K_i\epsilon}; P_\epsilon \to K_i$ defined by

$$B_{K_i\varepsilon}(p) = \{x \in K_i : \varepsilon \leq \langle p, x\rangle \leq \langle p, w_i\rangle\}.$$

Note that this is a continuous correspondence between the $\sigma(F,E)$-compact set P_ϵ and X_i with the $\sigma(E,F)$ topology.

We consider for each $p \in P_\epsilon$ the *i*-th restricted budget set

$$B_{K_i\epsilon}(p) = A_i(x_{-i}, p) \cap K_i \subset E$$

of the restricted economy. For each $p \in P_\epsilon$ $B_{K_i\epsilon}(p)$ is a $\sigma(E,F)$-compact subset of E. Note that $B_{K_i\epsilon}(p)$ can be written as

$$B_{K_i\epsilon}(p) = \left\{x \in K_i : \epsilon \leq \left\langle \frac{p}{\langle p, w_i\rangle}, x\right\rangle \leq 1\right\},$$

and from the Alaoglu theorem is a $\sigma(F,E)$-compact subset of F.

Consider the set $B_{K_i\epsilon}(p), i = 1,\cdots,n$ as the strategy set of each consumer and in addition, consider a fictional player with P_ϵ as his strategy set. While consumers choose their strategy in order to maximize their utility function, taking into account what others do, the fictional player chooses his strategy trying to maximize the function $v_f(x,p) = p\left(\sum_{i=1}^{n} x_i - w_i\right)$, given x, the choice of consumers.

- Given x_{-i} and p, the *i*-th consumer looks to solve the optimization problem

$$\max_{x_i \in B_{K_i}(p)} v_i(x_i, x_{-i}, p)$$

given $x_{-i} \in X_{-i}$ and $p \in P_\epsilon$.

- Given, $x \in X$ the fictional player looks to solve

$$\max_{p\in P_\epsilon} v_f(x,p) = p\left(\sum_{i=1}^{n} x_i - w_i\right)$$

Now consider

$$\mu_i(x_{-i}, p) = \left\{x_i^* \in B_{K_i}(p) : v_i(\overline{x}_i, x-i, p) \geq v_i(x_i, x_{-i}, p) \forall x_i \in B_{K_i,\epsilon}\right\}$$

$$= \left\{x_i^* \in B_{K_i\epsilon}(p) : u_i(\overline{x}_i, p) \geq u_i(x_i, p) \forall x_i \in B_{K_i}(p)\right\}, i \in \{1,\cdots,n\},$$

$$\mu_f(x) = \{\overline{p} \in P_+ : v_f(x, \overline{p}) \geq v_f(x,p) \text{for all } p \in P_\epsilon\}.$$

where by f we denote the fictional player.

Let $B_{K\epsilon} = B_{K_1\epsilon} \times \cdots \times B_{K_n\epsilon} \times P_\epsilon$ be the product of the budgets correspondence and $\mu : B_{K\epsilon} \times P_\epsilon \to B_{K\epsilon} \times P_\epsilon$ be the correspondence product given by

$$\mu(x, p) = \mu_1(x_{-1}, p) \times \cdots \times \mu_n(x_{-n}, p) \times \mu_f(x).$$

Given that we are in the conditions of the theorem (2) then, there exists a fixed point for the correspondence μ, *i.e.*; $(\bar{x}, \bar{p}) \in B_{K\epsilon}(\bar{p}) \times P_+$ such that $(\bar{x}, \bar{p}) \in \mu(\bar{x}, \bar{p}) \subset K^n \times P_\epsilon$. This fixed point is a Social equilibrium for the $\mathcal{RE}$-economy with the restricted budget set.

As it is easy to see, this Social equilibrium is a Walrasian equilibrium for the restricted economy.

Let $(\bar{x}, \bar{p}) \in (B_{K,\epsilon})^n \times P_\epsilon$ be a Nash equilibrium, then the following inequality holds,

$$v_i(\bar{x}_i, \bar{x}_{-i}, \bar{p}) \geq v_i\left(z, \bar{x}_{-i}, \bar{p} \text{ for all } z_i \in B_{K_i,\epsilon.}\right)$$

Since for all $p \in P_\epsilon$. the inequality $p\sum_{i=1}^n(\bar{x}_i - w_i) \leq \bar{p}\sum_{i=1}^n(\bar{x}_i - w_i) \leq 0$ holds then, we have that $\sum_{i=1}^n(\bar{x}_i - w_i) = 0.$

By contradiction, if $\sum_{i=1}^n(\bar{x}_i - w) > 0,$ then there exists $q \in P_\epsilon$. such that $q\left(\sum_{i=1}^n(x_i^* - w_i\right) > 0.$ This contradicts the fact that $v_f(\bar{x}, \bar{p}) \geq v_f(\bar{x}, p)$ for all $p \in P_\epsilon$.

Next we show that there exists a restricted economy, such that its Walrasian equilibrium is a Walrasian equilibrium for the economy.

6. The Walrasian Equilibrium

We shall choose an useful weak compact subset $\mathcal{K}_h \subset E_+$ verifying that $K_i = \mathcal{K}_h \forall i \in I$ such that if, $(\bar{x}, \bar{p})$ is Walrasian equilibrium of the restricted economy $\mathcal{RE}$, then it is a Walrasian equilibrium for the original or unrestricted economy $\mathcal{E} = \{E_i, w_i, v_i, I\}$.

In order to do this, we begin considering the set of feasible allocations for the economy, *i.e.*; the set

$$\mathcal{F} = \left\{x \in (E_+)^n : \sum_{i=1}^n w_i - \sum x_i \in E_+\right\}.$$

For each $j \in \{1, \cdots, n\}$ it follows that $0 \leq x_j \leq \sum_{i=1}^n x_i \leq \sum_{i=1}^n w_i = w.$

Let $\{D_h\}_{h\in N}$ be the fundamental sequence of $\sigma(E, F)$-closed, convex, balanced and bounded subsets of E, already defined in section (3), verifying that $D_h \subset D_{h+1} \forall h \in N$ and that covers E i.e.; $E \subset \bigcup_{h\in N} D_h$.

Choose h^* large enough that $w \in E_+ \cap D_{h^*}$. Let $\mathcal{K}_{h^*}$ be a $\sigma(E, F)$- compact subset of the the order interval $[0, w]$ given by

$$\mathcal{K}_{h^*} = \left\{x \in E_+ \cap D_{h^*} : \langle p, x\rangle \leq \langle p, w\rangle \forall p \in P_\epsilon\right\}.$$

The $\sigma(E, F)$- compactness property of $\mathcal{K}_{h^*}$ follows from the Alaoglu theorem.

From the previous section we know that there exists a Walrasian equilibrium for the restricted economy $\mathcal{RE}$.

Let $(\bar{p}, \bar{x})$ be the Walrasian equilibrium for the restricted economy $\mathcal{RE} = \{K_i, v_i, w_i, i = 1, \cdots, n\}$, where for all i, $K_i = \mathcal{K}_{h^*}$. We shall show, that $(\bar{p}, \bar{x})$

is a Walrasian equilibrium for the (unrestricted) economy $\mathcal{E} = \{E_+, w_i, v_i, I\}$.

In order to see this we consider $(\bar{x}, \bar{p})$ a Walrasian equilibrium of the economy $\mathcal{RE}$. Then it follows that:

1. $\sum_{i=1}^{n} \bar{x}_i \le \sum_{i=1}^{n} w_i$

2. For all $i \in \{1, \cdots, n\}, \bar{x}_i$. maximizes $v_i(z_i, x_{-i}, p)$ in the budget set $B_i(\bar{p}) = \{z \in E_+ : \langle \bar{p}, z \rangle \le \langle \bar{p} w_i \rangle\}$.

The claim 1, is straightforward from the definition. To prove the claim 2, we argue by contradiction.

(a) Suppose that there exist some $z \in (E_+)^n$ such that $z_i \ge \bar{x}_i$ and $\langle \bar{p}, z_i \rangle \le \langle \bar{p}, w_i \rangle$.

(b) Let $L[z, \bar{x}]$ be the segment linking $\bar{x}_i$ with z, *i.e.*,

$$L[z, \bar{x}] = \{x \in E_+ : x = \lambda z + (1-\lambda)\bar{x}_i, 0 \le \lambda \le 1\}.$$

For every $0 \le \lambda \le 1$ it follows that

$$\langle \bar{p}, x \rangle = \lambda \langle \bar{p}, z \rangle + (1-\lambda) \langle \bar{p}, \bar{x}_i \rangle \le \langle \bar{p}, w_i \rangle.$$

So, $x \in B_{K_i} = B(\mathcal{K}_h)$.

(c) Since preferences are strictly convex, we have that then

$$x = \lambda z + (1-\lambda)\bar{x}_i > \bar{x}_i.$$

But if $x > \bar{x}_i$ then $\langle \bar{p}, x \rangle > \langle \overline{p x_i} \rangle = \langle p, w_i \rangle$ contradicting the previous inequality.

Thus $(\bar{x}, \bar{p})$ is a Walrasian equilibrium for the economy $\mathcal{E}$.

Remark 1. *Note that we have shown the existence of a Walrasian equilibrium* (p, x) *where* $x \in E$ *and* $p \in F$ *with the only restriction that* (E, F) *is a dual pairing.*

Remark 2. *Let* (p^*, x^*) *be a Walrasian equilibrium for the economy* $\mathcal{E}$. *Note that if the i-th consumer prefers more than less, then an equilibrium prices* p^* *can not satisfy* $\langle p^*, x \rangle = 0$ *for some* $x \ne 0 \in E_+$, *because in this case*

$$\langle p^*, x_i^* + x \rangle = \langle p^*, x_i^* \rangle = \langle p^*, w_i \rangle$$

and $x_i^* + x >_i x_i^*$, contradicting the assumption that x^* is a Walrasian allocation.

7. A Consideration about the Hypothesis of Weak Continuity

Note that the assumption of weak continuity of utility functions is not too strong. In fact, in [6] is shown that assuming continuity of preferences with respect to topologies stronger than the Mackey topology, it is possible to obtain a large class of economies without Pareto optimal allocations and then, without equilibrium. Moreover in the opus cited is shown that the Mackey topology is the strongest topology for which upper semi continuous preferences are impatient.

On the other hand in [13] is shown that impatient behavior is enough to guarantee the existence of Pareto optimal allocations.

Example

The existence theorem has an immediate application in economies without temporal horizon, for instance in finance theory weak continuity of the utilities functions is a natural and necessary hypothesis.

Classically, these economies are modeled in l_2. This set is a reflexive Banach space, where $x = \{x_1, \cdots x_n, \cdots\} \in l_2$ if and only if $\sum_{i_1}^{\infty} x_i^2 < \infty$. The coordinate $x_j, j = 1, \cdots$ denote the consumption on time j.

In this case, the weak continuity of the utility functions is equivalent to discount the future (some kind of impatient behavior). Meaning that if $x, y \in l_{2+}$ are allocations verifying that x is strictly preferred to y then for each $z \in l_{2+}$ there exists $n_0(z)$ such that x is strictly preferred to $y + z(n_0)$, where

$$z(n_0)_i = \begin{cases} 0 & \text{for all } i \le n_0 \\ z_i > 0 & \text{for all } i > n_0. \end{cases}$$

This is an habitual assumption in macroeconomic and in finance theory.

8. Considerations on the Utilities, Revisited

Note that following this approach, we have shown that there exist the Walrasian equilibrium even when individual preferences depend on the action of the others, or they are prices dependent.

Prices dependent preferences are considered for instance in [14], [15] and in [16]. Preferences for goods may depend on prices because, for instance, people judge quality from price. This procedure can be supported by a rational strategy under a framework of uncertainty, but it looses a part of this value when, even in a framework of uncertainty, some peoples are considered as expert. In this case the preference is influenced by the action of the others. This dependence is an important factor for decisions in several circumstances. For instance when people imite the action of their pairs, in particular when individuals look to fashion.

In Game Theory the rational individual choice depends on that the others are doing, so in this field the literature considering this dependence is natural and abundant. However, in economics the literature looking for the dependence of individual preferences on that others are doing is very sparse. Moreover generally, preferences are considered part of the foundations of the economy and therefore immutable. In our approach utility functions can be considered as prices depending, and the set of alternatives can change according with the choice of the others consumers and even under these circumstances, provided that $A_i : X_{-i} \to X_i$ is a weak compact correspondence, the Walrasian Equilibrium exists. The utilities are symbolized by

$$u_i : X_i \times X_{-i} \times P_+ \to \mathbb{R}, \quad i = 1, \cdots, n$$

and the consumer problem is to solve the following optimization problem:

$$\max_{x_i \in B_i(\bar{p}) \cap A_i(\bar{x}_{-i})} u_i(x_i, \bar{x}_{-i}, \bar{p}) \text{ for each } (\bar{x}_{-i}, \bar{p}) \in X_{-i} \times P_+.$$

9. Conclusions

In this work we have shown that it is possible to extend to economies of infinite dimension, the demonstration of the existence of the Walrasian equilibrium made for finite economies. Without significant loss of generality in the choice of utility functions and consequently in the possible preferences to be considered in the model. Moreover, we show that Walrasian equilibrium exists, even though these utilities defined in very general sets (locally convex topological vector spaces) also depend on the choice of others and prices. In order to do this, we use techniques well known in Functional Analysis. This extension shows the strength of concepts such as the Walraisian equilibrium and its permanence when we generalize the economic models.

We conclude saying that the techniques of Functional Analysis are of great utility for solving in an unified way, several economic problems corresponding to different areas of the economic theory. For instance, for growth theory in which the vector space of continuous functions is the set in which utilities or profits are maximized, or for finance theory in which the vector spaces L_2 of square integrable functions is widely used.

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