

Properties of Non-Differentiable Tax Policies

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Received March 1, 2013; revised April 10, 2013; accepted May 16, 2013

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ABSTRACT

In this study, we reconsider the effect of variable transformations on the redistribution of income. We assume that the density function is continuous. If the theorems should hold for all income distributions, the conditions earlier given are both necessary and sufficient. Different conditions are compared. One main result is that continuity is a necessary condition if one demands that the income inequality should remain or be reduced. In our previous studies, of tax policies the assumption was that the transformations were differentiable and satisfy a derivative condition. In this study, we show that it is possible to reduce this assumption to a continuity condition.

Keywords: Gini Indices; Income Inequality; Income Redistributive Policies; Lorenz Curves; Lorenz Dominance

1. Introduction

It is a well-known fact that variable transformations are valuable in considering the effect of tax and transfer policies on income inequality. The transformation is usually assumed to be positive, monotone increasing and continuous. Under the assumption that the theorems should hold for all income distributions, conditions given earlier are both necessary and sufficient [1,2]. In this study, we reconsider the effect of variable transformations on the redistribution of income. Different versions of the conditions are compared [1,3-5]. One main result is that continuity is a necessary condition if one assumes that income inequality should remain or be reduced. In addition, in our earlier studies of classes of tax policies, the results were based on the assumption that the transformations were differentiable and satisfies a derivative condition [6,7].

2. Basic Properties of Income Transformations

Consider income X with the distribution function $F_X(x)$, the mean μ_X , and the Lorenz curve $L_X(p)$. We assume that X is defined for $x \geq 0$ and that $f_X(x)$ is continuous.

A fundamental theorem concerning the effect of income transformations on Lorenz curves and Lorenz dominance was given by Fellman [3] and Jakobsson [1] and later by Kakwani [4]. We have.

Theorem 1. [1,3,4]. *Let X be an arbitrary non-negative, random variable with the distribution $F_X(x)$, mean μ_X , and Lorenz curve $L_X(p)$. Let $u(x)$ be non-negative, continuous and monotone-increasing and let $\mu_Y = E(u(X))$ exist. Then the Lorenz curve $L_Y(p)$ of $Y = u(X)$ exists and the following results hold*

- 1) $L_Y(p) \geq L_X(p)$
if $\frac{u(x)}{x}$ is monotone decreasing;
- 2) $L_Y(p) = L_X(p)$
if $\frac{u(x)}{x}$ is constant and;
- 3) $L_Y(p) \leq L_X(p)$
if $\frac{u(x)}{x}$ is monotone increasing.

Following Fellman [2], we obtain in 1) a sufficient condition that the transformation $u(x)$ generates a new income distribution which Lorenz dominates the initial one. The analysis should be based on the difference

$$D(p) = L_Y(p) - L_X(p) = \int_0^{x_p} \frac{x}{\mu_Y} \left(\frac{u(x)}{x} - \frac{\mu_Y}{\mu_X} \right) f_X(x) dx \quad (1)$$

where $x_p = F_X^{-1}(p)$; that is, $p = F_X(x_p)$ [2]. Furthermore, $D(0) = D(1) = 0$. In order to obtain Lorenz dominance, the difference $D(p)$ in Equation (1) must start from zero, attain positive values and then decrease back to zero. Consequently, the difference

$$v(x) = \left(\frac{u(x)}{x} - \frac{\mu_Y}{\mu_X} \right) \tag{2}$$

must start from positive (non-negative) values and then change its sign and become negative. If $\frac{u(x)}{x}$ is exceptionally increasing within the interval $a \leq x \leq b$, then a variable X with a distribution $f_X(x)$ defined in the interval $a \leq x \leq b$ exists such that 3) holds and $L_X(p) \geq L_Y(p)$. Consequently, the condition that $\frac{u(x)}{x}$ is decreasing is necessary if the rule holds for all income distributions $F_X(x)$ [1,2]. Analogously, if the other results in Theorem 1 hold for every income distribution, the conditions in 2) and 3) are also necessary.

Hence, the continuity of $u(x)$ is a necessary condition if we demand that the transformed variable should Lorenz dominate the initial variable for every distribution. From this it follows that if the condition in Theorem 1 1) has to be necessary, it implies continuity and hence an explicit statement of continuity can be dropped. Considering the condition in 2), we observe that $u(x) = kx$ and $u(x)$ consequently is continuous.

However, in case 3) discontinuities do not jeopardize the monotone increasing property of the quotient $\frac{u(x)}{x}$ and the result in Theorem 1 3) holds even if the function is discontinuous. Therefore, Fellman [2] dropped the explicit continuity assumption in this case as well.

Summing up, for arbitrary distributions, $F_X(x)$, the conditions 1), 2), and 3) in Theorem 1 are both necessary and sufficient for the dominance relations and an additional assumption about the continuity of the transformation $u(x)$ can be dropped. We obtain the more general theorem [2].

Theorem 2. *Let X be an arbitrary non-negative, random variable with the distribution $F_X(x)$, mean μ_X and Lorenz curve $L_X(p)$, let $u(x)$ be a non-negative, monotone increasing function and let $Y = u(X)$ and $E(Y) = \mu_Y$ exist. Then the Lorenz curve $L_Y(p)$ of Y exists and the following results hold:*

1) $L_Y(p) \geq L_X(p)$

if and only if $\frac{u(x)}{x}$ is monotone-decreasing

2) $L_Y(p) = L_X(p)$

if and only if $\frac{u(x)}{x}$ is constant

3) $L_Y(p) \leq L_X(p)$

if and only if $\frac{u(x)}{x}$ is monotone-increasing.

Remark. It follows from the discussion above that the transformation $u(x)$ can be discontinuous only in case

3).

Hemming and Keen [5] gave an alternative condition for Lorenz dominance. Their condition, with our notations, is that for a given distribution $F_X(x)$, $\frac{u(x)}{x}$

crosses the $\frac{\mu_Y}{\mu_X}$ level once from above. Consequently,

$v(x) = \frac{u(x)}{x} - \frac{\mu_Y}{\mu_X}$ in (2) starts from positive values,

changes its sign once and ends up with negative values. Hence, their condition is equivalent to our condition.

Furthermore, if we assume that $\frac{u(x)}{x}$ is monotone-

decreasing (non-increasing), then $\frac{u(x)}{x}$ satisfies the

condition “crossing once from above for every distribution $F_X(x)$ ”. Hence, both conditions, the Hemming-Keen condition and ours, are also equivalent as necessary conditions. Recently, Fellman [8] obtained limits for the transformed Lorenz curves. These limits are related to the results given by Hemming and Keen.

3. Properties of Tax Policies

If we apply the results above to tax policies, the transformed variable $Y = u(X)$ is the income after the taxation (cf., e.g., [6,7,9,10]). In order to obtain a realistic class of policies, Fellman [6,7] assumed continuous transformations and included the additional restriction $u'(x) \leq 1$. This condition indicates that the tax paid is an increasing function of the income x . In order to generalize the results and allow that the function $u(x)$ is not uniformly differentiable everywhere, we replace the derivative restriction in this study by the more general condition $\Delta u(x) \leq \Delta x$. According to this restriction, the tax is an increasing function of the income x . In fact, the tax is $x - u(x)$ and the increment in the tax is $\Delta x - \Delta u(x)$ and a positive increment Δx yields the restriction $\Delta u(x) \leq \Delta x$. If $u'(x) \leq 1$ holds, it follows that

$$\Delta u(x) = u(x + \Delta x) - u(x) = u'(\xi) \Delta x \leq \Delta x, \tag{3}$$

but the condition $\Delta u(x) \leq \Delta x$ is more general and does not imply uniform differentiability. Both restrictions imply that the transformation $u(x)$ is continuous. We intend to show that the assumption $\Delta u(x) \leq \Delta x$ is sufficient for the whole theory.

Now, the class of tax policies is

$$U: \begin{cases} u(x) \leq x \\ \Delta u(x) \leq \Delta x \\ E(u(X)) = \mu_X - \tau \end{cases} . \tag{4}$$

We consider the extreme policies

$$u_0(x) = \begin{cases} x, & x \leq a_0 \\ a_0, & x > a_0 \end{cases} \tag{5}$$

and

$$u_\infty(x) = \begin{cases} 0, & x \leq c_\infty \\ x - c_\infty, & x > c_\infty \end{cases} \tag{6}$$

It is apparent that while function (5) is not differentiable at point a_0 and (6) at point c_∞ , the condition $\Delta \bar{u}(x) \leq \Delta x$ holds for all x . The Lorenz curve corresponding to (5) is

$$L_{u_0}(p) = \begin{cases} \frac{\mu_X}{\mu_X - \tau} L_X(p), & p \leq p_0 \\ \frac{\mu_X}{\mu_X - \tau} L_X(p_0) + \frac{a_0}{\mu_X - \tau} (p - p_0), & p > p_0 \end{cases}, \tag{7}$$

where $p_0 = F_X(a_0)$ and the Lorenz curve corresponding to (6) is

$$L_{u_\infty}(p) = \begin{cases} 0, & p \leq p_\infty \\ \frac{\mu_X}{\mu_X - \tau} L_X(p) + \frac{c_\infty(1-p) - \tau}{\mu_X - \tau}, & p > p_\infty \end{cases}, \tag{8}$$

where $p_\infty = F_X(c_\infty)$ ([6]).

Policy (5) is optimal, that is, it Lorenz dominates all the policies in class U , and policy (6) is Lorenz dominated by all policies in U [6,7].

In the following, we show how the main result in [7] can be obtained when we replace the restriction $\bar{u}'(x) \leq 1$ by the more general restriction $\Delta \bar{u}(x) \leq \Delta x$. The function $\bar{u}(x)$ may be piecewise differentiable like transformations (5) and (6). We consider post-tax income distributions with the mean $\mu_X - \tau$. Without the restriction $\Delta \bar{u}(x) \leq \Delta x$, the necessary and sufficient condition that a given Lorenz curve $\bar{L}(p)$ of the distribution $\bar{F}_Y(y)$ corresponds to a member of class U is that the initial distribution $F_X(x)$ stochastically dominates $\bar{F}_Y(y)$. The inclusion of the restriction $\Delta \bar{u}(x) \leq \Delta x$ results in the stochastic dominance being only necessary; that is, the transformed distribution $\bar{F}_Y(y)$ must satisfy additional conditions.

Assume a given differentiable Lorenz curve $\bar{L}(p)$ with a continuous derivative. These conditions can be assumed because the corresponding transformation $\bar{u}(x)$ has to continuously satisfy the condition $\Delta \bar{u}(x) \leq \Delta x$. Starting from $\bar{L}(p)$, the connection between $\bar{L}(p)$ and the post-tax distribution $\bar{F}_Y(y)$ with the mean

$\mu_X - \tau$ is that $\bar{F}_Y(y) = M\left(\frac{y}{\mu_X - \tau}\right)$, where $M(\cdot)$ is

the inverse function of $\bar{L}'(p)$. The corresponding transformation is

$$\bar{u}(x) = y = (\mu_X - \tau) \bar{L}'(F_X(x)).$$

The condition $\Delta \bar{u}(x) \leq \Delta x$ can be written as

$$\Delta \bar{u}(x) = (\mu_X - \tau) (\bar{L}'(F_X(x + \Delta x)) - \bar{L}'(F_X(x))) \leq \Delta x, \tag{9}$$

where $p = F_X(x_p)$ and $p + \Delta p = F_X(x_p + \Delta x)$.

On the other hand, we can write

$$\Delta \bar{u}(x) = (\mu_X - \tau) (\bar{L}'(p + \Delta p) - \bar{L}'(p)), \tag{10}$$

and define y_p and $y_{p+\Delta p}$

$$p = \bar{F}_Y(y_p), p + \Delta p = \bar{F}_Y(y_{p+\Delta p}).$$

If we assume that $\bar{u}(x)$ is piecewise differentiable, then $\bar{L}'(p)$ and $\bar{F}_Y(y)$ are piecewise differentiable.

If we assume that the density functions $f_X(x)$ and $\bar{f}_Y(y)$ exist, we obtain

$$\Delta p = F_X(x_p + \Delta x) - F_X(x_p) = f_X(\xi) \Delta x, \tag{11}$$

where $x < \xi < x + \Delta x$ and

$$\begin{aligned} \Delta p &= \bar{F}_Y(y_{p+\Delta p}) - \bar{F}_Y(y_p) = \bar{f}_Y(\eta) (y_{p+\Delta p} - y_p) \\ &= \bar{f}_Y(\eta) (\bar{u}(x_p + \Delta x) - \bar{u}(x_p)), \end{aligned} \tag{12}$$

where $\bar{f}_Y(y) = \bar{F}'_Y(y)$ and $y_p < \eta < y_{p+\Delta p}$.

Consequently, $p = \bar{F}_Y(y_p) = F_X(x_p)$ and $y_p = \bar{F}_Y^{-1}(F_X(x))$.

From $f_X(\xi) \Delta x = \Delta p = \bar{f}_Y(\eta) \Delta \bar{u}(x)$ and from the condition $\Delta \bar{u}(x) \leq \Delta x$ it follows that

$$f_X(\xi) \Delta x = \bar{f}_Y(\eta) \Delta \bar{u}(x) \leq \bar{f}_Y(\eta) \Delta x \tag{13}$$

and, consequently, $\frac{f_X(\xi)}{\bar{f}_Y(\eta)} \leq 1$. If we let $\Delta x \rightarrow 0$, then

$\Delta p \rightarrow 0, \xi \rightarrow x_p$ and $\eta \rightarrow y_p$ and we obtain

$\frac{f_X(x_p)}{\bar{f}_Y(y_p)} \leq 1$ for all p . This condition can also be written

as $\frac{f_X(x)}{\bar{f}_Y(\bar{u}(x))} \leq 1$ or

$$\frac{f_X(x)}{\bar{f}_Y(y)} \leq 1 \tag{14}$$

when $y = \bar{u}(x)$. We can reverse the steps from (14) to (9) and all the results in Fellman [7] still hold, but the proof had to be slightly modified.

4. Conclusions

In this study we reconsidered the effect of variable transformations on the redistribution of income. The aim was to generalise the conditions considered in earlier

papers. We were particularly interested in whether we could drop the assumption of differentiability of the transformations when tax policies are considered. The main result is that with a slight modification of the proof

the additional condition $\frac{f_x(x)}{f_y(y)} \leq 1$ is obtained.

We have also seen that if we demand sufficient and necessary conditions, theorems obtained earlier still hold and the continuity assumption can be included in the general conditions. The main result is that continuity is a necessary condition if one maintains that the income inequality should remain or be reduced.

The study of the class of tax policies indicated that the differentiability assumed earlier, can be dropped but, if one wants to retain the realism of the class, the transformations should still be continuous and satisfy the restriction $\Delta \bar{u}(x) \leq \Delta x$. The previous results in Fellman [6,7] still hold.

Empirical applications of the optimal policies of a class of transfer policies and the class of tax policies considered here have been discussed by Fellman *et al.* [9,10], where “optimal yardsticks” to gauge the effectiveness of given real tax and transfer policies in reducing inequality were developed.

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