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A Study on Lucas' "Expectations and the Neutrality of Money"

Masayuki Otaki

Institute of Social Science, University of Tokyo, Tokyo, Japan
Email: ohtaki@iss.u-tokyo.ac.jp

ABSTRACT

This short article shows that the functional equation on the equilibrium price function is more complicated than that considered by Lucas [1], and that modification is required to complete the proof. Furthermore, we shall provide a sufficient condition that guarantees the uniqueness of the equilibrium price function.

1. Introduction

This study aims to show that some additional condition is necessary for the operator T in Lucas [1] to become the contraction mapping. This is because transformation between the functional equations on the equilibrium price function is not equivalent. We also provide a sufficient condition such that T can become the contraction mapping.

The paper is as follows. Section 2 rewrites the functional equation in the correct form and shows that unlike the original paper, the contraction mapping method cannot be easily applied. A sufficient condition for the uniqueness of the equilibrium price function is provided in Section 3. Section 4 contains brief concluding remarks.

2. Equivalent Transformation

Lucas [2] admits that there is no guarantee that p_0 and z is a one-to-one correspondence, and some reservation is necessary for the conclusion. Furthermore, he also recognizes that the equilibrium price function $p(m, x, \theta)$ should be specified as

$$m\phi\left(\frac{x}{\theta}\right)$$

for determining the unique equilibrium. However, besides these problems, the transformation between functional equations below is not equivalent. The aim of the study is to clarify that fact and show rather restrictive condition for supporting the original result.

Lucas [1] firstly derives the following functional equation as the equilibrium condition of money market:

$$h\left[\frac{x}{\theta\phi\left(\frac{x}{\theta}\right)}\right]\frac{x}{\theta\phi\left(\frac{x}{\theta}\right)}$$
$$= \int V'\left[\frac{xx'}{\theta\xi\phi\left(\frac{x'}{\theta'}\right)}\right]\frac{xx'}{\theta\xi\phi\left(\frac{x'}{\theta'}\right)}\mathrm{d}G\left(\xi, x', \theta'\left|\frac{x}{\theta}\right.\right), \quad (1)$$

where x is the realized value of the increment of money during the current period. θ also denotes the realize value of the population of the young generation. x', θ' are random variables of each exogenous shock during the next period. We must note the existence of the random variable ξ. Although $z \equiv \frac{x}{\theta}$ is an available information through the inverse equilibrium price function, x cannot be directly observed alone by household. Thus, when x singly appears in the functional equation, it should be treated as the random variable ξ.

The right-hand side of (1) means the marginal utility of the current consumption, and the left-hand side implies the expected marginal utility of the future consumption. Namely, functional Equation (1) is the Euler equation in this model. Lucas [1] asserts that (1) is equivalently transformed into

$$h\left[\frac{z}{\phi(z)}\right]\frac{z}{\phi(z)}$$
$$= \int V'\left[\frac{\theta'}{\theta}\frac{z'}{\phi(z')}\right]\frac{\theta'}{\theta}\frac{z'}{\phi(z')}\tilde{H}(z, \theta)H(z', \theta')\mathrm{d}\theta\mathrm{d}z'\mathrm{d}\theta'. \quad (2)$$

However, (1) and (2) is not equivalent. We shall deal with this problem. This transformation assumes $\xi = x$. Nevertheless, as discussed above, x is a realized vale (real number) of the random variable ξ (measurable function). Hence they cannot be cancelled out. The equivalent transformation from (1) to (2) is

$$h\left[\frac{z}{\phi(z)}\right]\frac{z}{\phi(z)} = \int V'\left[\frac{z\theta'}{\xi}\frac{z'}{\phi(z')}\right]\frac{z\theta'}{\xi}\frac{z'}{\phi(z')}dG(\xi, x', \theta'|z). \tag{3}$$

Let us define

$$\Psi(z) \equiv h\left[\frac{z}{\phi(z)}\right]\frac{z}{\phi(z)},$$

$G^{-1}(x) \equiv xh(x)$ and $G_2(x) \equiv V'(x)x$. Using these definitions (1) is transformed into

$$\Psi(z) = \int G_2\left[\frac{z\theta'}{\xi}G_1(\Psi(z'))\right]dG(\xi, x', \theta'|z).$$

Then the correct form of the operator T in the Appendix of Lucas [1] becomes

$$Tf \equiv \ln\int G_2\left[\frac{z\theta'}{\xi}G_1\left(e^{f(z')}\right)\right]dG(\xi, x', \theta'|z). \tag{4}$$

Consequently, inequality ($A.6$) in Lucas' [1] appendix

$$\|Tf - Tg\| \leq \sup_{z, \theta', \theta}\left|\ln G_2\left[\frac{\theta'}{\theta}G_1\left(e^{f(z)}\right)\right] - \frac{\theta'}{\theta}\ln G_2\left[G_1\left(e^{g(z')}\right)\right]\right|(A.6)$$

is modified as

$$\|Tf - Tg\| \leq \sup_z\sup_{z', \theta', \xi}\left|\ln G_2\left[\frac{z\theta'}{\xi}G_1\left(e^{f(z')}\right)\right] - \ln G_2\left[\frac{z\theta'}{\xi}G_1\left(e^{g(z')}\right)\right]\right|. \tag{5}$$

It is noteworthy that z', θ' and ξ' are functions of z in (5). Let us denote those functions as

$$z' = \omega(z), \frac{\theta'}{\xi} = \chi(z). \tag{6}$$

Accordingly, (5) becomes

$$\|Tf - Tg\| \leq \sup_z\left|\ln G_2\left[z\chi(z)G_1\left(e^{f(\omega(z))}\right)\right] - \ln G_2\left[z\chi(z)G_1\left(e^{g(\omega(z))}\right)\right]\right|. \tag{7}$$

Let us define z^*, x_1 and x_2 as

$$z^* \equiv \arg\sup_z\left|\ln G_2\left[z\chi(z)G_1\left(e^{f(\omega(z))}\right)\right] - \ln G_2\left[z\chi(z)G_1\left(e^{g(\omega(z))}\right)\right]\right|.$$

$$x_1 \equiv f\left(\omega(z^*)\right), x_2 \equiv g\left(\omega(z^*)\right).$$

Consequently, (7) is transformed into

$$\|Tf - Tg\| \leq \left|\ln G_2\left[z^*\chi(z^*)G_1\left(e^{x_1}\right)\right] - \ln G_2\left[z^*\chi(z^*)G_1\left(e^{x_2}\right)\right]\right|. \tag{8}$$

Applying the mean value theorem and Lucas' [1] assumptions ($A.2$) and ($A.3$)

$$0 < \frac{xG'_1(x)}{G_1(x)} < 1(A.2), 0 < \frac{xG'_2(x)}{G_2(x)} \leq 1 - a < 1(A.3)$$

to (8), we finally obtain

$$\|Tf - Tg\| \leq (1-a)|x_1 - x_2| \leq (1-a)\|f(\omega(z)) - g(\omega(z))\|. \tag{9}$$

Since $\omega(z) \neq z$ generally, the original paper has not succeeded in proving that the operator T is the contraction mapping.[1] Some additional condition is necessary for completing the proof.

3. A Sufficient Condition

Since the difficulty arises from the fact that (5) explicitly depends on z, we assume that the function G_2 is multiplicatively separable. Namely, suppose that G_2 satisfies

$$G_2(xy) = G_2(x)G_2(y). \tag{10}$$

In this case, (5) is modified as

$$\|Tf - Tg\| \leq \sup_{z', \theta', \xi}\left|\ln G_2\left[\frac{\theta'}{\xi}G_1\left(e^{f(z')}\right)\right] - \ln G_2\left[\frac{\theta'}{\xi}G_1\left(e^{g(z')}\right)\right]\right|. \tag{11}$$

This inequality is essentially identical to ($A.6$), and thus, T becomes the contraction mapping.

Nevertheless, the function G_2, which satisfies the functional Equation (10), is confined to power functions (See Small [3]). Hence,

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[1]It is noteworthy that

$$\|f(\omega(z)) - g(\omega(z))\| \leq \|f(z) - g(z)\|$$

does not necessarily hold. For example, when $\omega(z) = 2z, g < f$ and f is a stricktly increasing function, g is stricktly decreasing, then

$$\|f(2z) - g(2z)\| \geq \|f(z) - g(z)\|$$

holds.

$$V'(x)x = x^\beta \Rightarrow V(x) = x^\beta \beta. \qquad (12)$$

(A.3) also requires $0 < \beta < 1$. To sum up, CRRA (Constant Relative Risk Aversion) family, whose relative risk aversion is located within $(0,1)$, is the only function satisfying the sufficient condition (10).

4. Conclusion

We have shown that the functional equation of the equilibrium price function is more complicated than that considered by Lucas [1]. Hence, some additional condition is necessary to employ the contraction mapping method. This study finds that if V belongs to CRRA family of low relative risk aversion, the uniqueness of the solution is guaranteed. To sum up, Lucas' assertion on the neu-trality of money under uncertainty hold only rather restrictive utility functions than has been considered.

REFERENCES

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