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Further Results on Convergence for Nonlinear Transformations of Fractionally Integrated Time Series

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ABSTRACT

This paper presents some new results for the nonlinear transformations of the fractional integration process. Specifically, this paper reviews the weight fractional integration process with the Hurst parameter, $3/2 > d > 5/6$, and investigates the asymptotics of asymptotically homogeneous functional transformations of weight fractional integration process. These new results improve upon the earlier research of Tyurin and Phillips [1].

Keywords: Long Memory; Fractional Brownian Motion; Tanaka Formula; Nonlinearity

1. Introduction

Since the breakthrough papers of Park and Phillips [2, 3], the research on the nonlinear cointegration has generated a lot of interest in recent years. In traditional research on cointegration, econometricians always adopt linear models. Using a linear cointegration model, econometricians can derive large sample properties easily. However, these settings have a serious drawback: there are many nonlinear relationships between dependent variables and independent variables in the cointegration model. Thus, it is a subjective process to set cointegration as a linear form in advance. A nonlinear regression model may improve this problem in a cointegration system. Although nonlinear regressions have obvious merit for cointegration models, it is difficult to derive the asymptotics for their estimated parameters and test statistics. Park and Phillips [2, 3] were the first to use local time to obtain asymptotics under nonlinear transformations of the I(1) process. Pötscher [4] and de Jong and Wang [5] later extended to these results to more flexible assumptions. The asymptotics of nonlinear transformations for nonstationary time series consistently concentrated on the I(1) process in early nonlinear cointegration research. Tyurin and Phillips [1] extended their method to the nonstationary I(d) process. Jeganathan [6] investigated the asymptotics of nonlinear transformations for generalized fractional stable motions. Although they presented some new results for the nonlinear transformations of the nonstationary fractionally integrated process, they only concentrated on integrable functions.

This paper uses a weight nonstationary fractionally integrated process instead of the standard nonstationary fractionally integrated process. This paper extends the results of Tyurin and Phillips [1] to asymptotically homogeneous functions. Specifically, this paper uses the fractional Brownian motion Tanaka formula to obtain the asymptotics of nonlinear transformations for the nonstationary fractional integration process. The results of this paper address the shortcomings of Tyurin and Phillips [1].

2. Assumptions and Basic Results

Consider the following fractional integration processes:

$$(1-L)^d x_t = \varepsilon_t \tag{1}$$

where ε_t is an $i.i.d.(0,\sigma^2)$ and $3/2 > d > 5/6$. x_t is called a nonstationary fractionally integrated processes. In addition to the definition of nonstationary fractionally integrated processes, This paper uses the following additional assumptions.

Assumption 1. *For some* $q > 2p > 2\max(1/H, 2)$, $E|\varepsilon_k|^q < \infty$ *and* $E(\varepsilon_t^2) < \infty$ *,where* H *is the Hurst exponent,* $H = d - 1/2$, and $p \geq 2$

Assumption 2.

1) $Var\left(\sum_{t=1}^{n} x_t\right) = n^{2H} M(n)$, where $M(n)$ is a slowly varying function.

2) The distribution of ε_k, $k = 0, \pm 1, \pm 2, \cdots$, is absolutely continuous with respect to the Lebesgue measure and has characteristic function $\psi(t) = E\left(e^{it\varepsilon_k}\right)$ for which $\lim_{t\to\infty} t^{\eta}\psi(t) = 0$ for some $\eta > 0$.

Based on these assumptions, we can obtain the fractional central limit theorem for the nonstationary I(d) processes.

Theorem 1. *Consider the process defined by*

$$x_t = (1-L)^{-d}\,\varepsilon_t,$$

where $3/2 > d > 5/6$. $\{\varepsilon_t\}$ satisfies $E(\varepsilon_t)=0$ and Assumption (1). Then the process

$$n^{-(d-1/2)}x_{[nr]} \Rightarrow \sigma(n)B_H(r), \qquad (2)$$

where $[.]$ is Gauss sign and $r\in[0,1]$. $B_H(r)$ is a fractional Brownian motion with Hurst exponent defined by stochastic integral.

$$B_H(r) = \frac{1}{\Gamma(1/2+H)}\int_{-\infty}^{r}(r-s)^{H-1/2}\,\mathrm{d}W(s), \qquad (3)$$

where W(.) is a Brownian Motion on $[0,1]$. $\sigma(n)$ is a slowly varying function with $\sigma(n)=\sqrt{M(n)}$.

Proof of Theorem 1:

See Akonom and Gourieroux [7] □

To obtain the asymptotics of the transformed fractionally integrated series, it is necessary to use the local time $L(t,s)$, which is generally defined as quadratic variation finite in the stochastic process literature. Quadratic variation is infinite when we use fractional Brownian motion instead of Brownian motion. To prevent this problem, this paper adopts the fractional Brownian motion Tanaka formula from Coutin, Nualart and Tudor [8].

$$|B_H(t)-s| = |B_H(0)-s| + \int_0^r \operatorname{sgn}(B_H(t)-s)\,\mathrm{d}B_H(t) + L(t,s)$$

where $\operatorname{sgn}(z)=1,0,-1$ as $z>0,=0,<0$ and $1>H>1/3$. H is a Hurst exponent.

Remark 1. (*Occupation time formula*) *Let* f *be locally integrable and* $3/2>H>5/6$. *Then*

$$\int_0^n f(B_H(t))\,\mathrm{d}t = \int_{-\infty}^{\infty} f(s)L(t,s)\,\mathrm{d}s. \qquad (4)$$

for all $t\in\mathbb{R}$

If we set $f(x)=1\{|x-s|<u\}$, we can obtain fractional local time formula.

$$L(t,s) = \lim_{u\to 0}\frac{1}{2u}\int_0^t 1\{|x-s|<u\}\,\mathrm{d}r \qquad (5)$$

3. Asymptotically Homogeneous Functions

Park and Phillips [2] considered the transformation of asymptotically homogeneous functions for I(1) process.

$$T(\lambda x) = \nu(\lambda)F(x) + R(x,\lambda), \qquad (6)$$

where $F(x)$ is locally integrable function and $R(...)$ is a reminder. This paper defines the notion of an asymptotically homogeneous function following de Jong and Wang [4]:

Definition 1. *A function* $T(.)$ *is called asymptotically homogeneous if for all* $K>0$ *and some function* $F(.)$,

$$\lim_{\lambda\to\infty}\int_{-K}^{K}\left|\nu(\lambda)^{-1}T(\lambda x)-F(x)\right|\mathrm{d}x = 0. \qquad (7)$$

If

$$\nu(\lambda)^{-1}T(\lambda x)\to F(x) \quad \text{as} \quad \lambda\to\infty \qquad (8)$$

and $\left|\nu(\lambda)^{-1}T(\lambda x)\right|\le G(x)$ for a locally integrable function $G(.)$, then $T(.)$ is asymptotically homogeneous. In addition to Definition 1, $T(x)$ must satisfy monotonic regular. Use the following Lemma to prove the asymptotics:

Lemma 1. *Under Assumption, for any* $K>0$,

$$n^{-1}\sum_{t=1}^{n} I\left(n^{-(d-1/2)}x_t\le x\right) \Rightarrow \int_0^1 I\left(B_H(r)\le x\right)\mathrm{d}r, \qquad (9)$$

where "$\Rightarrow$" denotes weak convergence in $D[-K,K]$ (*i.e.* the space of functions that are continuous on $[0,1]$ except for a finite number of discontinuities) and H is a Hurst parameter.

Proof of Lemma 1:

From Jeganathan [6], because slowly varying function, $\sigma(n)$, will not affect our proof, we set $\sigma(n)=1$. Pointwise in x, the result follows from Remark 3.5 of Tyurin and Phillips [1], and therefore it suffices to show stochastic equicontinuity of

$$n^{-1}\sum_{t=1}^{n} I\left(n^{-(d-1/2)}x_t\le x\right).$$

By the Skorokhod representation, we can assume that

$$\sup_{r\in[0,1]}\left|n^{-(d-1/2)}x_{[rn]} - B_H(r)\right| \overset{as}{\to} 0.$$

Then for *n* large enough,

$$\sup_{r\in[0,1]}\left|n^{-(d-1/2)}x_{[rn]} - B_H(r)\right| \le \delta$$

almost surely, implying that for n large enough

$$\begin{aligned}
&\sup_{|x|\le K}\ \sup_{x':x<x'<x+\delta}\left|n^{-1}\sum_{t=1}^{n}\left(I\left(n^{-(d-1/2)}x_t\le x\right)-I\left(n^{-(d-1/2)}x_t\le x'\right)\right)\right| \\
&\le \sup_{|x|\le K} n^{-1}\sum_{t=1}^{n} I\left(x\le n^{-(d-1/2)}x_t\le x+\delta\right) \\
&\le \sup_{|x|\le K}\int_0^1 I\left(x-\delta\le B_H(r)\le x+2\delta\right)\mathrm{d}r \qquad (10)\\
&= \sup_{|x|\le K}\int_{x-\delta}^{x+2\delta} L(1,s)\,\mathrm{d}s \le 3\delta\sup_{|s|\le K}\left|L(1,s)\right|
\end{aligned}$$

where the equality follows from the occupation times formula (see Tyurin and Phillips [1]) and because $\sup_{|s|\le K}\left|L(1,s)\right|$ is a well-defined random variable. The above chain of inequalities establishes stochastic equicontinuity of

$$n^{-1}\sum_{t=1}^{n} I\left(n^{-(d-1/2)}x_t \le x\right),$$

which completes the proof. □

Theorem 2. *Suppose Assumption holds. Also assume that* $T(.)$ *is asymptotically homogeneous. In addition, assume that* $F(.)$ *is continuous and* $T(.)$ *is monotone regular. Then, for* $1/3 > \alpha \ge 0$, $3/2 \ge d \ge 5/6$ *and* $d - \alpha - 1/2 > 0$,

$$\nu\left(n^{d-1/2-\alpha}\right)^{-1} n^{-1}\sum_{t=1}^{n} T\left(n^{-\alpha}x_t\right) \xrightarrow{d} \int_0^1 F\left(B_H(r)\right)\mathrm{d}r = \int_{-\infty}^{\infty} F(s)L(1,s)\mathrm{d}s. \quad (11)$$

Proof of Theorem 2:

To simplify our proof, we assume $\sigma(n) = 1$. Because $\sup_{1\le t\le n} n^{-(d-1/2)}|x_t| = O_p(1)$, it now suffices to show that for any $K > 0$,

$$\nu\left(n^{d-1/2-\alpha}\right)^{-1} n^{-1}\sum_{t=1}^{n} T\left(n^{-\alpha}x_t\right) I\left(\left|n^{-(d-1/2)}x_t\right| \le K\right) \xrightarrow{d} \int_0^1 F\left(B_H(r)\right) I\left(\left|B_H(r)\right| \le K\right)\mathrm{d}r = \int_{-K}^{K} F(s)L(1,s)\mathrm{d}s. \quad (12)$$

Now, by Lemma 1,

$$n^{-1}\sum_{t=1}^{n} I\left(n^{-(d-1/2)}x_t \le x\right) \Rightarrow \int_0^1 I\left(B_H(r) \le x\right)\mathrm{d}r.$$

By the Skorokhod representation theorem, we can assume without loss of generality that

$$\left|n^{-1}\sum_{t=1}^{n} I\left(n^{-(d-1/2)}x_t \le x\right) - \int_0^1 I\left(B_H(r) \le x\right)\mathrm{d}r\right| = c_n \xrightarrow{as} 0.$$

Now for all $\delta > 0$, let

$$\begin{aligned} S_{1n\delta} &= S_{1n} \\ &= \nu\left(n^{d-1/2-\alpha}\right)^{-1} n^{-1}\sum_{t=1}^{n} T\left(n^{-\alpha}x_t\right) I\left(\left|n^{-(d-1/2)}x_t\right| \le K\right) \\ &= \nu(n^{d-1/2-\alpha})^{-1}\int_{-K/\delta}^{K/\delta-1} n^{-1}\sum_{t=1}^{n} T\left(n^{-\alpha}x_t\right) \times I\left(j\delta \le n^{-(d-1/2)}x_t \le (j+1)\delta\right)\mathrm{d}j, \end{aligned} \quad (13)$$

$$S_{2n\delta} = \nu(n^{d-1/2-\alpha})^{-1}\int_{-K/\delta}^{K/\delta-1} T(n^{d-1/2-\alpha}j\delta)n^{-1} \times \sum_{t=1}^{n} I(j\delta \le n^{-(d-1/2)}x_t \le (j+1)\delta)dj, \quad (14)$$

$$S_{3n\delta} = \nu\left(n^{d-1/2-\alpha}\right)^{-1}\int_{-K/\delta}^{K/\delta-1} T\left(n^{d-1/2-\alpha}j\delta\right) \times \int_0^1 I\left(j\delta \le B_H(r) \le (j+1)\delta\right)\mathrm{d}r\,\mathrm{d}j, \quad (15)$$

$$\begin{aligned} S_{4n\delta} &= \nu\left(n^{d-1/2-\alpha}\right)^{-1}\int_{-K/\delta}^{K/\delta-1} T\left(n^{d-1/2-\alpha}j\delta\right)\delta L(1, j\delta)\mathrm{d}j \\ &= \nu(n^{d-1/2-\alpha})^{-1}\int_{-K}^{K-\delta} T(n^{d-1/2-\alpha}s) \times L(1,s)\mathrm{d}s, \end{aligned} \quad (16)$$

$$\begin{aligned} S_{5n\delta} = S_5 &= \int_{-K}^{K} F(s)L(1,s)\mathrm{d}s \\ &= \int_0^1 F\left(B_H(r)\right) I\left(\left|B_H(r)\right| \le K\right)\mathrm{d}r. \end{aligned} \quad (17)$$

We will show that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\left|S_{jn\delta} - S_{j+1,n\delta}\right| = 0$$

almost surely for $j = 1,\cdots,4$. By the monotone regular condition, we can act as if $T(.)$ is monotone without loss of generality. For $\left|S_1 - S_{2n\delta}\right|$ we then have (See the Equation [18] below) and as $\delta \to 0$, the last term disappears because of continuity of $F(.)$, the second inequality follows from monotonicity of $T(.)$, and the third by our definition of an asymptotically homogeneous function. To show that $\lim_{\delta\to 0}\limsup_{n\to\infty}\left|S_{2n\delta} - S_{3n\delta}\right| = 0$ almost surely, note that (See the Equation [19] below)

$$\begin{aligned} \limsup_{n\to\infty}\left|S_1 - S_{2n\delta}\right| &\le \limsup_{n\to\infty}\nu\left(n^{d-1/2-\alpha}\right)^{-1}\int_{-K/\delta}^{K/\delta-1} n^{-1}\sum_{t=1}^{n}\left|T\left(n^{-\alpha}x_t\right) - T\left(n^{d-1/2-\alpha}j\delta\right)\right| \times I\left(j\delta \le n^{-(d-1/2)}x_t \le (j+1)\delta\right)\mathrm{d}j \\ &\le \limsup_{n\to\infty}\nu\left(n^{d-1/2-\alpha}\right)^{-1}\int_{-K/\delta}^{K/\delta-1} n^{-1}\sum_{t=1}^{n}\left|T\left(n^{d-1/2-\alpha}(j+1)\delta\right) - T\left(n^{d-1/2-\alpha}j\delta\right) I\left(j\delta \le n^{-(d-1/2)}x_t \le (j+1)\delta\right)\mathrm{d}j\right| \\ &\le \limsup_{n\to\infty}\int_{-K/\delta}^{K/\delta-1}\left|\nu\left(n^{d-1/2-\alpha}\right)^{-1} T\left(n^{d-1/2-\alpha}(j+1)\delta\right) - \nu\left(n^{d-1/2-\alpha}\right)^{-1} T\left(n^{d-1/2-\alpha}j\delta\right) - F\left((j+1)\delta\right) + F(j\delta)\right|\mathrm{d}j \\ &\quad + \int_{-K/\delta}^{K/\delta-1}\left|F\left((j+1)\delta\right) - F(j\delta)\right|\mathrm{d}j = \int_{-K}^{K-\delta}\left|F(x+\delta) - F(x)\right|\mathrm{d}x, \end{aligned} \quad (18)$$

$$\begin{aligned} &\left|\nu\left(n^{d-1/2-\alpha}\right)^{-1}\int_{-K/\delta}^{K/\delta-1} T\left(n^{d-1/2-\alpha}j\delta\right)\left(n^{-1}\sum_{t=1}^{n} I\left(j\delta \le n^{-(d-1/2)}x_t \le (j+1)\delta\right) - \int_0^1 I\left(j\delta \le B_H(r) \le (j+1)\delta\right)\mathrm{d}r\right)\mathrm{d}j\right| \\ &\le 2c_n\nu\left(n^{d-1/2-\alpha}\right)^{-1}\int_{-K/\delta}^{K/\delta-1}\left|T\left(n^{d-1/2-\alpha}j\delta\right)\right|dj \\ &\le 2c_n\delta^{-1}\int_{-K}^{K}\left|\nu\left(n^{d-1/2-\alpha}\right)^{-1} T\left(n^{d-1/2-\alpha}x\right) - F(x)\right|dx + 2c_n\delta^{-1}\int_{-K}^{K}\left|F(x)\right|\mathrm{d}x = o(1) \end{aligned} \quad (19)$$

almost surely under our assumptions and by the definition of c_n. For $\left|S_{3n\delta}-S_{4n\delta}\right|$ we have

$$\begin{aligned}&\left|S_{3n\delta}-S_{4n\delta}\right|\\ &\le \nu\left(n^{d-1/2-\alpha}\right)^{-1}\int_{-K/\delta}^{K/\delta}\delta\left|T\left(n^{d-1/2-\alpha}j\delta\right)\right|\\ &\quad\times\left(\delta^{-1}\int_0^1 I\left(j\delta\le B_H\left(r\right)\le\left(j+1\right)\delta\right)\mathrm{d}r-L\left(1,j\delta\right)\right)\mathrm{d}j \\ &\le \nu\left(n^{d-1/2-\alpha}\right)^{-1}\int_{-K/\delta}^{K/\delta}\delta\left|T\left(n^{d-1/2-\alpha}j\delta\right)\right|\mathrm{d}j\\ &\quad\times\sup_{|x|\le K}\left|\delta^{-1}\times\int_0^1 I\left(x\le B_H\left(r\right)\le x+\delta\right)\mathrm{d}r-L\left(1,x\right)\right|.\end{aligned} \tag{20}$$

By the earlier argument,

$$\sup_{n\ge1}\sup_{\delta>0}\nu\left(n^{d-1/2-\alpha}\right)^{-1}\int_{-K/\delta}^{K/\delta}\delta\left|T\left(n^{d-1/2-\alpha}j\delta\right)\right|\mathrm{d}j<\infty, \tag{21}$$

and therefore it suffices to show that as $\delta\to0$,

$$\sup_{|x|\le K}\left|\delta^{-1}\int_0^1 I\left(x\le B_H\left(r\right)\le x+\delta\right)\mathrm{d}r-L\left(1,x\right)\right|\to0. \tag{22}$$

By the occupation times formula, the above expression satisfies

$$\begin{aligned}&\sup_{|x|\le K}\left|\delta^{-1}\int_x^{x+\delta}L(1,s)\mathrm{d}s-L\left(1,x\right)\right|\\ &=\sup_{|x|\le K}\left|\delta^{-1}\int_x^{x+\delta}\left(L\left(1,s\right)-L\left(1,x\right)\right)\mathrm{d}s\right| \\ &\le\sup_{|x|\le K}\sup_{s\in[x,x+\delta]}\left|L\left(1,s\right)-L\left(1,x\right)\right|\to0\ \text{ as }\delta\to0\end{aligned} \tag{23}$$

by uniform continuity of $L(1,.)$ on $[-K,K]$. Finally, for $\left|S_{4n\delta}-S_5\right|$, we have

$$\begin{aligned}&\lim_{n\to\infty}\left|\int_{-K}^{K}\left(\nu\left(n^{d-1/2-\alpha}\right)^{-1}T\left(n^{d-1/2-\alpha}s\right)-H\left(s\right)\right)L\left(1,s\right)\mathrm{d}s\right|\\ &\le\sup_{|s|\le K}\left|L\left(1,s\right)\right|\lim_{n\to\infty}\int_{-K}^{K}\left|\nu\left(n^{d-1/2-\alpha}\right)^{-1}T\left(n^{d-1/2-\alpha}s\right)-F\left(s\right)\right|\mathrm{d}s\\ &=0\end{aligned} \tag{24}$$

by the definition of an asymptotically homogeneous function, which completes the proof.

Theorem (2) expands the transformations asymptotically homogeneous functions to scaled nonstationary I(d) processes. This new result can be used to obtain the asymptotics of the nonlinear fractional cointegration.

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