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On the Use of Second and Third Moments for the Comparison of Linear Gaussian and Simple Bilinear White Noise Processes

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Abstract

The linear Gaussian white noise process (LGWNP) is an independent and identically distributed (*iid*) sequence with zero mean and finite variance with distribution $N(0,\sigma^2)$. Some processes, such as the simple bilinear white noise process (SBWNP), have the same covariance structure like the LGWNP. How can these two processes be distinguished and/or compared? If $X_1, X_2, \cdots, X_n$ is a realization of the SBWNP. This paper studies in detail the covariance structure of $X_t^d, t \in Z; d = 1,2,3$. It is shown from this study that; 1) the covariance structure of $X_t^2, t \in Z$ is non-normal with distribution equivalent to the linear ARMA(2, 1) model; 2) the covariance structure of $X_t^3, t \in Z$ is *iid*; 3) the variance of $X_t^3, t \in Z$ can be used for comparison of SBWNP and LGWNP.

Keywords

White Noise Process, Normality, Stationarity, Invertibility, Covariance Structure

1. Introduction

A stochastic process $X_t, t \in Z$, where $Z = \{\cdots, -1, 0, 1, \cdots\}$ is called a white noise or purely random process, if with finite mean and finite variance, all the autocovariances are zero except at lag zero. In many applications, $X_t, t \in Z$ is assumed to be normally distributed with mean zero and variance, $\sigma^2 < \infty$, and the series is called a linear Gaussian white noise process with the following properties [1]-[7].

$$E(X_t) = \mu \quad (1.1)$$

$$R(0) = \text{var}(X_t) = E(X_t - \mu) = \sigma^2 \quad (1.2)$$

$$R(k) = \text{cov}(X_t, X_{t+k}) = E\left[(X_t - \mu)(X_{t+k} - \mu)\right] = \begin{cases} \sigma^2, & k = 0 \\ 0, & \text{otherwise} \end{cases} \quad (1.3)$$

$$\rho(k) = \text{corr}(X_t, X_{t+k}) = \frac{R(k)}{R(0)} = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases} \quad (1.4)$$

$$\phi_{kk} = \text{corr}\left(X_t, X_{t+k} / X_{t+1}, X_{t+2}, \cdots, X_{t+k-1}\right) = 0 \quad \forall k \quad (1.5)$$

where $R(k)$ is the autocovariance function at lag k, ρ_k is the autocorrelation function at lag k and ϕ_{kk} is the partial autocorrelation function at lag k.

In other words, a stochastic process $X_t, t \in Z$ is called a linear Gaussian white noise if $X_t, t \in Z$ is a sequence of independent and identically distributed (*iid*) random variables with finite mean and finite variance. Under the assumption that the sample $X_1, X_2, \cdots, X_n$ is an *iid* sequence, we compute the sample autocorrelations as

$$\hat{\rho}_X(k) = \frac{\sum_{t=1}^{n}(X_t - \bar{X})(X_{t+k} - \bar{X})}{\sum_{t=1}^{n}(X_t - \bar{X})^2} \quad (1.6)$$

where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_t \quad (1.7)$$

The *iid* hypothesis is always tested with the Ljung and Box [8] statistic

$$Q_{LB}(m) = n(n+2)\sum_{k=1}^{m}\left(\frac{\left[\hat{\rho}_X(k)\right]^2}{n-k}\right) \quad (1.8)$$

where $Q_{LB}(m)$ is asymptotically a chi-squared random variable with m degree of freedom.

Several values of m are often used and simulation studies suggest that the choice of $m \approx \ln(n)$ provides better power performance [9].

If the data are *iid*, the squared data $X_1^2, X_2^2, \cdots, X_n^2$ are also *iid* [10]. Another portmanteau test formulated by Mcleod and Li [10] is based on the same statistic used for the Ljung and Box [8]

$$Q_{ML}(m) = n(n+2)\sum_{k=1}^{m}\left(\frac{\left[\hat{\rho}_{X^2}(k)\right]^2}{n-k}\right) \quad (1.9)$$

where the sample autocorrelations of the data are replaced by the sample autocorrelations of the squared data, $\hat{\rho}_{X^2}(k)$.

As noted by Iwueze *et al.* [11], a stochastic process $X_t, t \in Z$ may have the covariance structure (1.1) through (1.5) even when it is not the linear Gaussian white noise process. Iwueze *et al.* [11] provided additional properties of the linear Gaussian white noise process for proper identification and characterization from other processes with similar covariance structure (1.1) through (1.5).

Let $Y_t = X_t^d, d = 1,2,3,\cdots$ where $X_t, t \in Z$, be the linear Gaussian white noise process, the mean $\left[E(Y_t) = E(X_t^d)\right]$, the variance $\left[\operatorname{var}(Y_t) = \operatorname{var}(X_t^d)\right]$, autocovariances $\left[R_y(k) = \operatorname{cov}(Y_t Y_{t-k}) = \operatorname{cov}(X_t^d X_{t-k}^d)\right]$ were obtained to be [11]

$$E(Y_t) = E(X_t^d) = \begin{cases} \sigma^{2m}(2m-1)!!, \; d = 2m, m = 1,2,\cdots \\ 0, \; d = 2m+1, m = 0,1,2,\cdots \end{cases} \tag{1.10}$$

$$Var(Y_t) = Var(X_t^d) = \begin{cases} \sigma^{4m}\left[\prod_{k=1}^{2m}(2k-1) - \left(\prod_{k=1}^{m}(2k-1)\right)^2\right], d = 2m \\ \sigma^{2(2m+1)}\prod_{k=1}^{2m+1}(2k-1), d = 2m+1 \end{cases} \tag{1.11}$$

$$R_Y(k) = R_{X_t^d}(\ell) = \begin{cases} \sigma^{4m}\left[\prod_{k=1}^{2m}(2k-1) - \left(\prod_{k=1}^{m}(2k-1)\right)^2\right], d = 2m, \; \ell = 0 \\ \sigma^{2(2m+1)}\prod_{k=1}^{2m+1}(2k-1), d = 2m+1, \; \ell = 0 \\ 0, \; \ell \neq 0 \end{cases} \tag{1.12}$$

where

$$(2m-1)!! = \prod_{k=1}^{m}(2k-1) \tag{1.13}$$

It is clear from (1.12) that when $X_t, t \in Z$ are *iid*, the powers $Y_t = X_t^d, d = 1,2,3,\cdots$ of $X_t, t \in Z$ are also *iid*. Iwueze *et al.* [11] also showed the probability density function (pdf) of $Y_t = X_t^2$ to be the pdf of a gamma distribution with parameters

$\alpha = \frac{1}{2}, \beta = 2\sigma^2$. That is, $Y_t = X_t^2 \sim G(\alpha, \beta), \alpha = \frac{1}{2}, \beta = 2\sigma^2$.

when $X_t \sim N(0, \sigma^2)$ and [11] concluded that all powers of a linear Gaussian white noise process are *iid* but not normally distributed.

Using the coefficient of symmetry and kurtosis, Iwueze *et al.* [11] confirmed the non-normality of $Y_t = X_t^d, d = 2,3,\cdots$. **Table 1** gives the mean, variance, the coefficient of symmetry (β_1) and kurtosis (β_2) defined as follows

$$\beta_1 = \frac{\mu_3(d)}{(\mu_2(d))^{3/2}} \tag{1.14}$$

$$\beta_2 = \frac{\mu_4(d)}{(\mu_2(d))^2} \tag{1.15}$$

where

$$\mu_2(d) = E\left[\left(X_t^d - E(X_t^d)\right)^2\right] = \operatorname{var}(X_t^d) \tag{1.16}$$

Table 1. Mean, Variance, Coefficient of symmetry (β_1) and kurtosis (β_2) for $Y_t = X_t^d$, $d = 1,2,3,\cdots,6$, when $X_t \sim N(0,\sigma^2)$.

d	Y_t	$E(Y_t)$ (μ_y)	$\mu_2(d)$ $(\mathrm{var}(Y_t))$	$\mu_3(d)$	$\mu_4(d)$	β_1	β_2
1	X_t	0	σ^2	0	$3\sigma^4$	0	3.000
2	X_t^2	σ^2	$2\sigma^4$	$8\sigma^6$	$60\sigma^8$	2.828	15.000
3	X_t^3	0	$15\sigma^6$	0	$10395\sigma^{12}$	0	46.200
4	X_t^4	$3\sigma^4$	$96\sigma^8$	$9504\sigma^{12}$	$1907712\sigma^{16}$	10.104	207.00
5	X_t^5	0	$945\sigma^{10}$	0	$654729075\sigma^{20}$	0	733.159
6	X_t^6	$15\sigma^6$	$10170\sigma^{12}$	$33998400\sigma^{18}$	$3.142\times10^{11}\sigma^{24}$	33.150	3037.836

Source: Iwueze *et al.* (2017).

$$\mu_3(d) = E\left[\left(X_t^d - E\left(X_t^d\right)\right)^3\right] \tag{1.17}$$

$$\mu_4(d) = E\left[\left(X_t^d - E\left(X_t^d\right)\right)^4\right] \tag{1.18}$$

Using the standard deviations when $\sigma^2 = 1$ and the kurtosis of $Y_t = X_t^d, d = 1,2,3,\cdots$, Iwueze *et al.* [11] determined the optimal value of d to be three ($d = 3$). Hence, for effective comparison of the linear Gaussian white noise process with any stochastic process with similar covariance structure, $Y_t = X_t^d, d = 1,2,3$ must be used.

The most commonly used white noise process is the linear Gaussian white noise process. The process is one of the major outcomes of any estimation procedure which is used in checking the adequacy of fitted models. The linear Gaussian white noise process also plays significant role as a basic building block in the construction of linear and non-linear time series models. However, the major problem is that there are many non-linear processes that exhibit the same covariance structure (Equation (1.1) through Equation (1.5)) as the linear Gaussian white noise process. One of such non-linear models is the bilinear models.

The study of bilinear models was introduced by Granger and Andersen [12] and Subba Rao [13]. Granger and Andersen [14] established that all series generated by the simple bilinear model

$$X_t = \beta X_{t-k} e_{t-j} + e_t,\ k > j \tag{1.19}$$

appear to be second order white noise where β is a constant and $e_t, t \in Z$ is an independent identically distributed normal random variable with $E(e_t) = 0$, $E(e_t^2) = \sigma^2 < \infty$. Guegan [15] studied the existence problem of a simple bilinear process $X_t, t \in Z$ satisfying

$$X_t = \beta X_{t-2} e_{t-1} + e_t \tag{1.20}$$

Martins [16] obtained the autocorrelation function of the process $X_t^2, t \in Z$ for the simple bilinear model defined by (1.19) when $e_t, t \in Z$ is *iid* with a Gaussian distribution. Again, Martins [16] studied the third order moment structure of (1.19) with non-independent shocks. Recently, properties of the simple bilinear model (1.19) were addressed by Malinski and Bielinska [17], Malinski and Figwer [18] and Malinski [19]. Iwueze [20] studied the more general bilinear white noise model

$$X_t = \left(\sum_{j=1}^{m} \beta_j X_{t-q-j} \right) e_{t-q} + e_t \tag{1.21}$$

where $e_t, t \in Z$ is as defined in (1.19). Iwueze [20] was able to show the following.

1) The series $X_t, t \in Z$ satisfying (1.21) is strictly stationary, ergodic and unique.

2) The series $X_t, t \in Z$ satisfying (1.21) is invertible.

3) The series $X_t, t \in Z$ satisfying (1.21) has the same covariance structure as the linear Gaussian white noise processes.

4) Obtained the covariance structure of (1.21) to be

$$\mu = E\left(X_t\right) = 0 \tag{1.22}$$

$$R(k) = \begin{cases} \dfrac{\sigma^2}{1 - \sum_{j=1}^{m} \sigma^2 \beta_j^2}, & k = 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.23}$$

5) The series satisfying (1.21) is invertible if

$$2\sum_{j=1}^{m} \beta_j^2 \sigma^2 < 1 \tag{1.24}$$

For the simple bilinear model (1.19), it follows that

$$R(k) = \begin{cases} \dfrac{1}{1 - \sigma^2 \beta^2}, & \sigma^2 \beta^2 < 1 \\ 0, & \text{otherwise} \end{cases} \tag{1.25}$$

and the invertibility condition is

$$\sigma^2 \beta^2 < \frac{1}{2} \tag{1.26}$$

It is worthy to note that the stationarity condition

$$\sigma^2 \beta^2 < 1 \tag{1.27}$$

is structure (k, n) independent [19] for model (1.19) and our study in this paper will concentrate on model (1.20). The purpose of this paper is to meet the following goals for the simple bilinear model satisfying (1.20).

1) Determine $Var\left(X_t^d\right), d = 2,3$ for the simple bilinear model (1.20).

2) Determine the covariance structure of $X_t^d, d = 2,3$, when $X_t, t \in Z$ satisfies (1.20).

3) Determine for what values of β the simple bilinear white noise process will be identified as a Linear Gaussian white noise process.

4) Determine for what values of β the simple bilinear model will be normally distributed.

This paper is further divided into four sections in order to establish and achieve these goals. Section 2 discusses the covariance structure of $Y_t = X_t^d, d = 1,2,3$ when $X_t = \beta X_{t-2}e_{t-1} + e_t$, $e_t \sim iid\, N\left(0,\sigma^2\right)$, Section 3 presents the methodology, Section 4 is the results and discussion while, Section five is the conclusion.

2. Covariance Structure of $Y_t = X_t^d, d = 1,2,3$, When $X_t = \beta X_{t-2}e_{t-1} + e_t$, $e_t \sim iid\, N\left(0,\sigma^2\right)$

Theorem 2.1.

Let $e_t, t \in Z$ be the linear Gaussian white noise process with $E\left(e_t\right) = 0$ and $E\left(e_t^2\right) = \sigma^2 < \infty$. Suppose there exists a stationary and invertible process $X_t, t \in Z$ satisfying $X_t = \beta X_{t-2}e_{t-1} + e_t$ for every $t \in Z$ for some constant β, then $Y_t = X_t^2$ has the following properties:

$$E\left(Y_t\right) = \mu_Y = \frac{\sigma^2}{1-\sigma^2\beta^2};\ \sigma^2\beta^2 < 1 \tag{2.1}$$

$$R_Y\left(k\right) = \operatorname{cov}\left(Y_t, Y_{t-k}\right) = \begin{cases} \dfrac{2\sigma^4}{\left(1-\sigma^2\beta^2\right)^2\left(1-3\sigma^4\beta^4\right)}, & \sigma^2\beta^2 < \dfrac{1}{\sqrt{3}}, k = 0 \\ \dfrac{2\sigma^6\beta^2}{\left(1-\sigma^2\beta^2\right)^2}, & \sigma^2\beta^2 < 1,\ k = 1 \\ \sigma^2\beta^2 R_Y\left(k-2\right), & k = 2,3,\cdots \end{cases} \tag{2.2}$$

$$\rho_Y\left(k\right) = \frac{R_Y\left(k\right)}{R_Y\left(0\right)} = \begin{cases} 1, & k = 0 \\ \sigma^2\beta^2\left(1-3\sigma^4\beta^4\right), & k = 1 \\ \sigma^2\beta^2\rho_Y\left(k-2\right), & k = 2,3,\cdots \end{cases} \tag{2.3}$$

$Y_t = X_t^2, t \in Z$ has the same covariance structure as the linear ARMA(2, 1) process (2.4)

$$X_t = \lambda + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \theta_1 a_{t-1} + a_t,\ \phi_1 = 0 \tag{2.4}$$

where a_t is the sequence of independent and identically distributed random variable with $E\left(a_t\right) = 0$ and $Var\left(a_t\right) = \sigma_1^2 < \infty$.

Proof:

Let

$$Y_t = X_t^2 = \left(\beta X_{t-2}e_{t-1} + e_t\right)^2 = \beta^2 X_{t-2}^2 e_{t-1}^2 + e_t^2 + 2\beta X_{t-2}e_{t-1}e_t$$

$$E\left(Y_t\right) = E\left(X_t^2\right) = \beta^2 E\left(X_{t-2}^2\right)E\left(e_{t-1}^2\right) + E\left(e_t^2\right) + 2\beta E\left(X_{t-2}\right)E\left(e_{t-1}\right)E\left(e_t\right)$$

$$E\left(Y_t\right) = E\left(X_t^2\right) = \beta^2 E\left(X_t^2\right)E\left(e_t^2\right) + E\left(e_t^2\right) = \sigma^2\beta^2 E\left(X_t^2\right) + \sigma^2$$

$$\left(1-\sigma^2\beta^2\right)E\left(X_t^2\right) = \sigma^2$$

$$\mu_Y = E\left(X_t^2\right) = \frac{\sigma^2}{1-\sigma^2\beta^2};\ \sigma^2\beta^2 < 1 \tag{2.5}$$

$$Var\left(Y_t\right) = Var\left(X_t^2\right) = E\left(X_t^4\right) - \left[E\left(X_t^2\right)\right]^2$$

$$X_t^4 = \beta^4 X_{t-2}^4 e_{t-1}^4 + 4\beta^3 X_{t-2}^3 e_{t-1}^3 e_t + 6\beta^2 X_{t-2}^2 e_{t-1}^2 e_t^2 + 4\beta X_{t-2} e_{t-1} e_t^3 + e_t^4$$

$$E\left(X_t^4\right) = 3\sigma^4\beta^4 E\left(X_t^4\right) + 6\sigma^4\beta^2 E\left(X_t^2\right) + 3\sigma^4$$

$$\left(1-3\sigma^4\beta^4\right) E\left(X_t^4\right) = \frac{6\sigma^6\beta^2}{1-\sigma^2\beta^2} + 3\sigma^4$$

$$\Rightarrow E\left(X_t^4\right) = \frac{3\sigma^4\left(1+\sigma^2\beta^2\right)}{\left(1-\sigma^2\beta^2\right)\left(1-3\sigma^4\beta^4\right)},\ \sigma^4\beta^4 < \frac{1}{\sqrt{3}} \tag{2.6}$$

Now,

$$\begin{aligned} Var\left(Y_t\right) &= Var\left(X_t^2\right) = E\left(X_t^4\right) - \left[E\left(X_t^2\right)\right]^2 \\ &= \frac{3\sigma^4\left(1+\sigma^2\beta^2\right)}{\left(1-\sigma^2\beta^2\right)\left(1-3\sigma^4\beta^4\right)} - \left(\frac{\sigma^2}{1-\sigma^2\beta^2}\right)^2 \\ &= \frac{3\sigma^4\left(1+\sigma^2\beta^2\right)\left(1-\sigma^2\beta^2\right) - \sigma^4\left(1-3\sigma^4\beta^4\right)}{\left(1-\sigma^2\beta^2\right)^2\left(1-3\sigma^4\beta^4\right)} \end{aligned} \tag{2.7}$$

Hence,

$$R_Y\left(0\right) = Var\left(Y_t\right) = Var\left(X_t^2\right) = \frac{2\sigma^4}{\left(1-\sigma^2\beta^2\right)^2\left(1-3\sigma^4\beta^4\right)},\ \sigma^2\beta^2 < \frac{1}{\sqrt{3}} \tag{2.8}$$

$$R_Y\left(k\right) = E\left[Y_t Y_{t-\ell}\right] - \mu_y^2 = E\left[X_t^2 X_{t-\ell}^2\right] - \mu_x^2, k = 0,1,2,\cdots$$

$$Y_t Y_{t-1} = X_t^2 X_{t-1}^2 = \beta^2 X_{t-2}^2 X_{t-1}^2 e_{t-1}^2 + 2\beta X_{t-2} X_{t-1}^2 e_{t-1} e_t + X_{t-1}^2 e_t^2$$

$$E\left[Y_t Y_{t-1}\right] = \beta^2 E\left[X_{t-2}^2 X_{t-1}^2 e_{t-1}^2\right] + \sigma^2 E\left(X_{t-1}^2\right)$$

$$E\left[Y_t Y_{t-1}\right] = \beta^2 E\left[X_{t-1}^2 X_t^2 e_t^2\right] + \sigma^2 E\left(X_t^2\right)$$

$$X_{t-1}^2 X_t^2 e_t^2 = X_{t-1}^2\left(\beta^2 X_{t-2}^2 e_{t-1}^2 + 2\beta X_{t-2} e_{t-1} e_t + e_t\right) e_t^2$$

$$X_{t-1}^2 X_t^2 e_t^2 = \beta^2 X_{t-2}^2 X_{t-1}^2 e_{t-1}^2 e_t^2 + 2\beta X_{t-2} X_{t-1}^2 e_{t-1} e_t^3 + X_{t-1}^2 e_t^4$$

By the assumption of stationarity,

$$E\left[X_{t-1}^2 X_t^2 e_t^2\right] = \sigma^2\beta^2 E\left[X_{t-1}^2 X_t^2 e_t^2\right] + 3\sigma^4 E\left(X_t^2\right)$$

$$\left(1-\sigma^2\beta^2\right) E\left[X_{t-1}^2 X_t^2 e_t^2\right] = 3\sigma^4\left(\frac{\sigma^2}{1-\sigma^2\beta^2}\right)$$

$$E\left[X_{t-1}^2 X_t^2 e_t^2\right] = \frac{3\sigma^6}{\left(1-\sigma^2\beta^2\right)^2}, \sigma^2\beta^2 < 1 \tag{2.9}$$

$$E\left[Y_t Y_{t-1}\right] = \beta^2\left[\frac{3\sigma^6}{\left(1-\sigma^2\beta^2\right)^2}\right] + \sigma^2\left(\frac{\sigma^2}{1-\sigma^2\beta^2}\right) = \frac{\sigma^4\left(1+2\sigma^2\beta^2\right)}{\left(1-\sigma^2\beta^2\right)^2} \tag{2.10}$$

Hence,

$$R_y(1) = E(Y_tY_{t-1}) = E^2(Y_t) = \frac{\sigma^4(1+2\sigma^2\beta^2)}{(1-\sigma^2\beta^2)^2} - \left(\frac{\sigma^2}{1-\sigma^2\beta^2}\right)^2 = \frac{2\sigma^6\beta^2}{(1-\sigma^2\beta^2)^2} \quad (2.11)$$

$$Y_tY_{t-2} = X_t^2X_{t-2}^2 = (\beta^2X_{t-2}^2e_{t-1}^2 + 2\beta X_{t-2}e_{t-1}e_t + e_t^2)X_{t-2}^2$$

$$Y_tY_{t-2} = \beta^2X_{t-2}^4e_{t-1}^2 + 2\beta X_{t-2}^3e_{t-1}e_t + X_{t-2}^2e_t^2$$

$$E[Y_tY_{t-2}] = \sigma^2\beta^2E(X_{t-2}^4) + \sigma^2E(X_{t-2}^2)$$

$$E[Y_tY_{t-2}] = \sigma^2\beta^2E(Y_{t-2}^2) + \sigma^2E(Y_t)$$

$$\Rightarrow E[Y_tY_{t-2}] = \sigma^2\beta^2E(Y_t^2) + \sigma^2\mu_y$$

$$R_y(2) + \mu_y^2 = \sigma^2\beta^2[R_y(0) + \mu_y^2] + \sigma^2\mu_y \quad (2.12)$$

$$\begin{aligned} R_y(2) &= \sigma^2\beta^2R_y(0) + \sigma^2\beta^2\mu_y^2 + \sigma^2\mu_y - \mu_y^2 \\ &= \sigma^2\beta^2R_y(0) + \sigma^2\mu_y - \mu_y^2(1-\sigma^2\beta^2) \end{aligned}$$

Note that

$$\mu_Y = E(Y_t) = E(X_t^2) = \frac{\sigma^2}{1-\sigma^2\beta^2}$$

$$\Rightarrow (1-\sigma^2\beta^2)\mu_Y = \sigma^2$$

$$1-\sigma^2\beta^2 = \frac{\sigma^2}{\mu_Y} \quad (2.13)$$

Hence

$$\begin{aligned} R_Y(2) &= \sigma^2\beta^2R_y(0) + \sigma^2\mu_y - \mu_y^2\left(\frac{\sigma^2}{\mu_y}\right) \\ &= \sigma^2\beta^2R_y(0) + \sigma^2\mu_y - \sigma^2\mu_y = \sigma^2\beta^2R_y(0) \end{aligned} \quad (2.14)$$

We have shown that

$$\sigma^2\beta^2\mu_y^2 + \sigma^2\mu_y - \mu_y^2 = 0 \quad (2.15)$$

Similarly;

$$Y_tY_{t-3} = X_t^2X_{t-3}^2 = (\beta^2X_{t-2}^2e_{t-1}^2 + 2\beta X_{t-2}e_{t-1}e_t + e_t^2)X_{t-3}^2$$

$$Y_tY_{t-3} = \beta^2X_{t-3}^2X_{t-2}^2e_{t-1}^2 + 2\beta X_{t-3}^2X_{t-2}e_{t-1}e_t + X_{t-3}^2e_t^2$$

$$E[Y_tY_{t-3}] = \sigma^2\beta^2E[X_{t-2}^2X_{t-1}^2] + \sigma^2E(X_t^2) = \sigma^2\beta^2E[Y_tY_{t-1}] + \sigma^2E(Y_t)$$

$$\begin{aligned} \Rightarrow R_y(3) + \mu_y^2 &= \sigma^2\beta^2[R_y(1) + \mu_y^2] + \mu_y^2 \\ &= \sigma^2\beta^2R_y(1) + \sigma^2\beta^2\mu_y^2 + \sigma^2\mu_y - \mu_y^2 \\ &= \sigma^2\beta^2R_y(1) \end{aligned} \quad (2.16)$$

Generally;

$$R_Y(k) = \sigma^2\beta^2R_Y(k-2),\ k = 2,3,\cdots \quad (2.17)$$

Hence,

$$R_Y(k)=\begin{cases}\dfrac{2\sigma^4}{(1-\sigma^2\beta^2)^2(1-3\sigma^4\beta^4)}, \sigma^2\beta^2<\dfrac{1}{\sqrt{3}}, k=0\\ \dfrac{2\sigma^6\beta^2}{(1-\sigma^2\beta^2)^2}, \ \sigma^2\beta^2<1,\ k=1\\ \sigma^2\beta^2R_Y(k-2),\ k=2,3,\cdots\end{cases} \tag{2.18}$$

and

$$\rho_Y(k)=\begin{cases}1, & k=0\\ \sigma^2\beta^2(1-3\sigma^4\beta^4), & k=1\\ \sigma^2\beta^2\rho_Y(k-2), & k=2,3,\cdots\end{cases} \tag{2.19}$$

With this result, it is clear that when $X_t, t\in Z$ is defined by (1.20), $Y_t=X_t^2$ has the same covariance structure as the linear ARMA(2, 1) process. Its linear equivalence is

$$Y_t=\lambda+\phi_1X_{t-1}+\phi_2Y_{t-2}+\theta_1a_{t-1}+a_t,\ \phi_1=0 \tag{2.20}$$

where a_t is the purely random process with $E(a_t)=0$ and $Var(a_t)=\sigma_1^2<\infty$. Table 2 compares $Y_t=X_t^2$ with its linear ARMA(2, 1) equivalence.

Theorem 2.2.:

Let $e_t, t\in Z$ be the linear Gaussian white noise process with $E(e_t)=0$ and $E(e_t^2)=\sigma^2<\infty$. Suppose there exists a stationary and invertible process $X_t, t\in Z$ satisfying $X_t=\beta X_{t-2}e_{t-1}+e_t$ for every $t\in Z$ and some constant β, then the mean and variance of $Y_t=X_t^3, t\in Z$ are

$$E(Y_t)=\mu_Y=0 \tag{2.21}$$

Table 2. Covariance analysis of $Y_t=X_t^2$ when $X_t=\beta X_{t-2}e_{t-1}+e_t$, $e_t\sim N(0,\sigma^2)$ and its linear ARMA(2, 1) equivalence.

Properties	Process: Bilinear	Process: Linear ARMA(2, 1)
Structure	$X_t=\beta X_{t-2}e_{t-1}+e_t, e_t\sim N(0,\sigma^2)$, $Y_t=X_t^2\sim \text{ARMA}(2,1)$ with $\phi_1=0$	$Y_t=\lambda+\phi_2Y_{t-2}+\theta_1a_{t-1}+a_t$, $E(a_t)=0$, $Var(a_t)=\sigma_1^2$
Mean	$\mu_Y=E(Y_t)=E(X_t^2)=\dfrac{\sigma^2}{1-\sigma^2\beta^2};\sigma^2\beta^2<1$	$\mu_Y=E(Y_t)=\dfrac{\lambda}{1-\phi_2}$, $[\lambda=(1-\phi_2)\mu_X]$
Autocovariance	$R_Y(k)=\begin{cases}\dfrac{2\sigma^4}{(1-\sigma^2\beta^2)^2(1-3\sigma^4\beta^4)}, \sigma^2\beta^2<\dfrac{1}{\sqrt{3}}, k=0\\ \dfrac{2\sigma^6\beta^2}{(1-\sigma^2\beta^2)^2}, \sigma^2\beta^2<1, k=1\\ \sigma^2\beta^2R_Y(k-2), k=2,3,\cdots\end{cases}$	$R_Y(k)=\begin{cases}\dfrac{\sigma_1^2(1+\theta_1^2)}{1-\phi_2^2}, \lvert\phi_2\rvert<1, k=0\\ \dfrac{\sigma_1^2\theta_1}{1-\phi_2}, \phi_2\neq1, k=1\\ \phi_2R_Y(k-2), k=2,3,\cdots\end{cases}$
Autocorrelation	$\rho_Y(k)=\begin{cases}1, & k=0\\ \sigma^2\beta^2(1-3\sigma^4\beta^4), & k=1\\ \sigma^2\beta^2\rho_Y(k-2), & k=2,3,\cdots\end{cases}$	$\rho_Y(k)=\begin{cases}1, & k=0\\ \dfrac{\theta_1(1+\phi_2)}{1+\theta_1^2}, & k=1\\ \phi_2\rho_Y(k-2), & k=2,3,\cdots\end{cases}$

$$R_Y(k)=\begin{cases}\dfrac{15\sigma^6\left(1+2\sigma^2\beta^2+6\sigma^4\beta^4+3\sigma^6\beta^6\right)}{\left(1-\sigma^2\beta^2\right)\left(1-3\sigma^4\beta^4\right)\left(1-15\sigma^6\beta^6\right)}, & \sigma^2\beta^2<\dfrac{1}{\sqrt[3]{15}},\ k=0\\ 0,\ k\neq 0\end{cases} \tag{2.22}$$

$$\rho_k(k)=\begin{cases}1, & k=0\\ 0, & k\neq 0\end{cases} \tag{2.23}$$

The covariance structure of $Y_t=X_t^3, t\in Z$ is that of the linear white noise process.

Proof:

Let $Y_t=X_t^3=\left(\beta X_{t-2}e_{t-1}+e_t\right)^3=\beta^3X_{t-2}^3e_{t-1}^3+3\beta^2X_{t-2}^2e_{t-1}^2e_t+3\beta X_{t-2}e_{t-1}e_t^2+e_t^3$ (2.24)

$$\begin{aligned}E(Y_t)=E\left(X_t^3\right)=\mu_y&=\beta^3E\left(X_{t-2}^3e_{t-1}^3\right)+3\sigma^2\beta^2E\left(X_{t-2}e_{t-1}\right)\\&=\beta^3E\left(X_{t-1}^3e_t^3\right)+3\sigma^2\beta^2E\left(X_{t-1}e_t\right)=0\end{aligned} \tag{2.25}$$

$$\begin{aligned}Y_t^2=X_t^6&=\left(\beta X_{t-2}e_{t-1}+e_t\right)^6\\&=\beta^6X_{t-2}^6e_{t-1}^6+6\beta^5X_{t-2}^5e_{t-1}^5e_t+15\beta^4X_{t-2}^4e_{t-1}^4e_t^2+20\beta^3X_{t-2}^3e_{t-1}^3e_t^3\\&\quad+15\beta^2X_{t-2}^2e_{t-1}^2e_t^4+6\beta X_{t-2}e_{t-1}e_t^5+e_t^6\end{aligned} \tag{2.26}$$

$$\begin{aligned}E\left(Y_t^2\right)&=\beta^6E\left(X_{t-2}^6e_{t-1}^6\right)+6\beta^5E\left(X_{t-2}^5e_{t-1}^5e_t\right)+15\beta^4E\left(X_{t-2}^4e_{t-1}^4e_t^2\right)\\&\quad+20\beta^3E\left(X_{t-2}^3e_{t-1}^3e_t^3\right)+15\beta^2E\left(X_{t-2}^2e_{t-1}^2e_t^4\right)+6\beta E\left(X_{t-2}e_{t-1}e_t^5\right)+E\left(e_t^6\right)\\&=\beta^6E\left(X_{t-2}^6e_{t-1}^6\right)+15\sigma^2\beta^4E\left(X_{t-2}^4e_{t-1}^4\right)+45\sigma^4\beta^2E\left(X_{t-2}^2e_{t-1}^2\right)+15\sigma^6\\&=15\sigma^6\beta^6E\left(X_t^6\right)+45\sigma^6\beta^4E\left(X_t^4\right)+45\sigma^6\beta^2E\left(X_t^2\right)+15\sigma^6\\&=15\sigma^6\beta^6E\left(Y_t^2\right)+45\sigma^6\beta^4\left[\frac{3\sigma^4\left(1+\sigma^2\beta^2\right)}{\left(1-\sigma^2\beta^2\right)\left(1-3\sigma^4\beta^4\right)}\right]\\&\quad+45\sigma^6\beta^2\left(\frac{\sigma^2}{1-\sigma^2\beta^2}\right)+15\sigma^6\end{aligned}$$

$$\begin{aligned}&\left(1-15\sigma^6\beta^6\right)E\left(Y_t^2\right)\\&=45\sigma^6\beta^4\left[\frac{3\sigma^4\left(1+\sigma^2\beta^2\right)}{\left(1-\sigma^2\beta^2\right)\left(1-3\sigma^4\beta^4\right)}\right]+45\sigma^6\beta^2\left(\frac{\sigma^2}{1-\sigma^2\beta^2}\right)+15\sigma^6\\&=\frac{1}{\left(1-\sigma^2\beta^2\right)\left(1-3\sigma^4\beta^4\right)}\Big[45\sigma^6\beta^4\left[3\sigma^4\left(1+\sigma^2\beta^2\right)\right]\\&\quad+45\sigma^6\beta^2\left[\sigma^2\left(1-3\sigma^4\beta^4\right)\right]+15\sigma^6\left(1-\sigma^2\beta^2\right)\left(1-3\sigma^4\beta^4\right)\Big]\\&=\frac{1}{\left(1-\sigma^2\beta^2\right)\left(1-3\sigma^4\beta^4\right)}\Big[135\sigma^{10}\beta^4+135\sigma^{12}\beta^6+45\sigma^8\beta^2\\&\quad-135\sigma^{12}\beta^6+15\sigma^6-45\sigma^{10}\beta^4-15\sigma^8\beta^2+45\sigma^{12}\beta^6\Big]\\&=\frac{1}{\left(1-\sigma^2\beta^2\right)\left(1-3\sigma^4\beta^4\right)}\left[90\sigma^{10}\beta^4+30\sigma^8\beta^2+15\sigma^6+45\sigma^{12}\beta^6\right]\\&=\frac{15\sigma^6\left(1+2\sigma^2\beta^2+6\sigma^4\beta^4+3\sigma^6\beta^6\right)}{\left(1-\sigma^2\beta^2\right)\left(1-3\sigma^4\beta^4\right)},\quad\sigma^2\beta^2<\frac{1}{\sqrt[3]{15}}\end{aligned} \tag{2.27}$$

$$\therefore E\left(Y_t^2\right) = R_y\left(0\right) + \mu_y^2 \tag{2.28}$$

$$\Rightarrow Var\left(Y_t\right) = Var\left(X_t^3\right) = R_y\left(0\right) = E\left(Y_t^2\right) - \mu_y^2$$

$$= \frac{15\sigma^6\left(1+2\sigma^2\beta^2+6\sigma^4\beta^4+3\sigma^6\beta^6\right)}{\left(1-\sigma^2\beta^2\right)\left(1-3\sigma^4\beta^4\right)\left(1-15\sigma^6\beta^6\right)}, \ \sigma^2\beta^2 < \frac{1}{\sqrt[3]{15}} \tag{2.29}$$

Some Results

$$E\left(X_{t-1}X_te_t\right) = \sigma^2 E\left(X_t\right) = 0$$

Proof:

$$X_{t-1}X_te_t = X_{t-1}\left[\beta X_{t-2}e_{t-1} + e_t\right]e_t = \beta X_{t-2}X_{t-1}e_{t-1}e_t + X_{t-1}e_t^2$$

$$E\left(X_{t-1}X_te_t\right) = \sigma^2 E\left(X_{t-1}\right) = \sigma^2 E\left(X_t\right) = 0$$

$$E\left(X_{t-1}X_t^2e_t\right) = 2\sigma^2\beta E\left(X_{t-1}X_te_t\right) = 0$$

Proof:

$$X_{t-1}X_t^2e_t = X_{t-1}\left[\beta^2X_{t-2}^2e_{t-1}^2 + 2\beta X_{t-2}e_{t-1}e_t + e_t^2\right]e_t$$

$$= \beta^2X_{t-2}^2X_{t-1}e_{t-1}^2e_t + 2\beta X_{t-2}X_{t-1}e_{t-1}e_t^2 + e_t^3$$

$$E\left(X_{t-1}X_t^2e_t\right) = 2\beta\sigma^2E\left(X_{t-2}X_{t-1}e_{t-1}\right) = 2\beta\sigma^2E\left(X_{t-1}X_te_t\right) = 0$$

$$E\left(X_{t-1}^2X_te_t^2\right) = \sigma^2\beta E\left(X_{t-1}X_t^2e_t\right) = 0$$

Proof:

$$X_{t-1}^2X_te_t^2 = X_{t-1}^2\left[\beta X_{t-2}e_{t-1} + e_t\right]e_t^2 = \beta X_{t-2}X_{t-1}^2e_{t-1} + X_{t-1}^2e_t^3$$

$$E\left(X_{t-1}^2X_te_t^2\right) = \sigma^2\beta E\left(X_{t-2}X_{t-1}^2e_{t-1}\right) = \sigma^2\beta E\left(X_{t-1}X_t^2e_t\right) = 0$$

$$E\left(X_{t-1}X_t^3e_t\right) = 3\sigma^2\beta^2E\left(X_{t-1}^2X_te_t^2\right) = 0$$

Proof:

$$X_{t-1}X_t^3e_t = X_{t-1}\left[\beta^3X_{t-2}^3e_{t-1}^3 + 3\beta^2X_{t-2}^2e_{t-1}^2e_t + 3\beta X_{t-2}e_{t-1}e_t^2 + e_t^3\right]e_t$$

$$= \beta^3X_{t-2}^3X_{t-1}e_{t-1}^3e_t + 3\beta^2X_{t-2}^2X_{t-1}e_{t-1}^2e_t^2 + 3\beta X_{t-2}X_{t-1}e_{t-1}e_t^3 + X_{t-1}e_t^4$$

$$E\left(X_{t-1}X_t^3e_t\right) = 3\sigma^2\beta^2E\left(X_{t-2}^2X_{t-1}e_{t-1}^2\right) = 3\sigma^2\beta^2E\left(X_{t-1}^2X_te_t^2\right) = 0$$

⇒Now

$$Y_tY_{t-1} = X_t^3X_{t-1}^3 = \left[\beta^3X_{t-2}^3e_{t-1}^3 + 3\beta^2X_{t-2}^2e_{t-1}^2e_t + 3\beta X_{t-2}e_{t-1}e_t^2 + e_t^3\right]X_{t-1}^3$$

$$= \beta^3X_{t-2}^3X_{t-1}^3e_{t-1}^3 + 3\beta^2X_{t-2}^2X_{t-1}^3e_{t-1}^2e_t + 3\beta X_{t-2}X_{t-1}^3e_{t-1}e_t^2 + X_{t-1}^3e_t^3$$

$$E\left(Y_tY_{t-1}\right) = \beta^3E\left(X_{t-2}^3X_{t-1}^3e_{t-1}^3\right) + 3\sigma^2\beta E\left(X_{t-2}X_{t-1}^3e_{t-1}\right)$$

$$= \beta^3E\left(X_{t-1}^3X_t^3e_t^3\right) + 3\sigma^2\beta E\left(X_{t-1}X_t^3e_t\right)$$

$$= \beta^3E\left(X_{t-1}^3X_t^3e_t^3\right)$$

$$X_{t-1}^3X_t^3e_t^3 = X_{t-1}^3\left[\beta^3X_{t-2}^3e_{t-1}^3 + 3\beta^2X_{t-2}^2e_{t-1}^2e_t + 3\beta X_{t-2}e_{t-1}e_t^2 + e_t^3\right]e_t^3$$

$$= \beta^3X_{t-2}^3X_{t-1}^3e_{t-1}^3e_t^3 + 3\beta^2X_{t-2}^2X_{t-1}^3e_{t-1}^2e_t^4 + 3\beta X_{t-2}X_{t-1}^3e_{t-1}e_t^5 + X_{t-1}^3e_t^6$$

$$\begin{aligned}E\left(X_{t-1}^3X_t^3e_t^3\right)&=3\beta^2\left(3\sigma^4\right)E\left(X_{t-2}^2X_{t-1}^3e_{t-1}^2\right)\\&=9\sigma^4\beta^2E\left(X_{t-2}^2X_{t-1}^3e_{t-1}^2\right)\\&=9\sigma^4\beta^2E\left(X_{t-1}^2X_t^3e_t^2\right)\end{aligned}$$

Hence,

$$E\left(Y_tY_{t-1}\right)=\beta^3\left[9\sigma^4\beta^2E\left(X_{t-1}^2X_t^3e_t^2\right)\right]=9\sigma^4\beta^5E\left(X_{t-1}^2X_t^3e_t^2\right)$$

Now

$$\begin{aligned}X_{t-1}^2X_t^3e_t^2&=X_{t-1}^2\left[\beta^3X_{t-2}^3e_{t-1}^3+3\beta^2X_{t-2}^2e_{t-1}^2e_t+3\beta X_{t-2}e_{t-1}e_t^2+e_t^3\right]e_t^2\\&=\beta^3X_{t-2}^3X_{t-1}^2e_{t-1}^3e_t^2+3\beta^2X_{t-2}^2X_{t-1}^2e_{t-1}^2e_t^3+3\beta X_{t-2}X_{t-1}^2e_{t-1}e_t^4+X_{t-1}^2e_t^5\end{aligned}$$

$$\begin{aligned}E\left(X_{t-1}^2X_t^3e_t^2\right)&=\sigma^2\beta^3E\left(X_{t-2}^3X_{t-1}^2e_{t-1}^3\right)+3\beta\left(3\sigma^4\right)E\left(X_{t-2}X_{t-1}^2e_{t-1}\right)\\&=\sigma^2\beta^3E\left(X_{t-1}^3X_t^2e_t^3\right)+9\sigma^4\beta E\left(X_{t-1}X_t^2e_t\right)\\&=\sigma^2\beta^3E\left(X_{t-1}^3X_t^2e_t^3\right)\end{aligned}$$

But,

$$\begin{aligned}E\left(Y_tY_{t-1}\right)&=9\sigma^4\beta^5E\left(X_{t-1}^2X_t^3e_t^2\right)=9\sigma^4\beta^5\left(\sigma^2\beta^3E\left(X_{t-1}^3X_t^2e_t^3\right)\right)\\&=9\sigma^6\beta^8E\left(X_{t-1}^3X_t^2e_t^3\right)\end{aligned}$$

Now,

$$\begin{aligned}X_{t-1}^3X_t^2e_t^3&=X_{t-1}^3\left[\beta^2X_{t-2}^2e_{t-1}^2+2\beta X_{t-2}e_{t-1}e_t+e_t^2\right]e_t^3\\&=\beta^2X_{t-2}^2X_{t-1}^3e_{t-1}^2e_t^3+2\beta X_{t-2}X_{t-1}^3e_{t-1}e_t^4+X_{t-1}^3e_t^5\end{aligned}$$

$$E\left(X_{t-1}^3X_t^2e_t^3\right)=2\beta\left(3\sigma^4\right)E\left(X_{t-2}X_{t-1}^3e_{t-1}\right)=6\sigma^4\beta E\left(X_{t-1}X_t^3e_t\right)=0$$

Hence,

$$E\left(Y_tY_{t-1}\right)=9\sigma^6\beta^8\left[6\sigma^4\beta E\left(X_{t-1}X_t^3e_t\right)\right]=54\sigma^{10}\beta^9E\left(X_{t-1}X_t^3e_t\right)=0$$

$\Rightarrow R_Y(1)=0$, when $Y=X_t^3$.

$$\begin{aligned}Y_tY_{t-2}=X_t^3X_{t-2}^3&=\left[\beta^3X_{t-2}^3e_{t-1}^3+3\beta^2X_{t-2}^2e_{t-1}^2e_t+3\beta X_{t-2}e_{t-1}e_t^2+e_t^3\right]X_{t-2}^3\\&=\beta^3X_{t-2}^6e_{t-1}^3+3\beta^2X_{t-2}^5e_{t-1}^2e_t+3\beta X_{t-2}^4e_{t-1}e_t^2+X_{t-2}^3e_t^3\end{aligned}$$

$$E\left(Y_tY_{t-2}\right)=0$$

$\Rightarrow R_Y(2)=0$, when $Y=X_t^3$.

Generally, $R_Y(k)=0\ \forall k\neq0$, when $Y=X_t^3$.

Therefore, given $X_t=\beta X_{t-2}e_{t-1}+e_t$, $e_t\sim N\left(0,\sigma^2\right)$ and $Y_t=X_t^3$, the following are true $E\left(Y_t\right)=E\left(X_t^3\right)=0$.

$$R_Y(k)=\begin{cases}\dfrac{15\sigma^6\left(1+2\sigma^2\beta^2+6\sigma^4\beta^4+3\sigma^6\beta^6\right)}{\left(1-\sigma^2\beta^2\right)\left(1-3\sigma^4\beta^4\right)\left(1-15\sigma^6\beta^6\right)}, & \sigma^2\beta^2<\dfrac{1}{\sqrt[3]{15}},\ k=0\\0,\ k\neq0\end{cases}$$

$$\rho_k(k)=\begin{cases}1, & k=0\\0, & k\neq0\end{cases}$$

The covariance structure of $Y_t = X_t^3, t \in Z$ identifies the process as linear white noise.

3. Methodology

3.1. Normality Checking

The Jarque-Bera (JB) test [21] [22] [23] will be used to determine for which values of β a simple bilinear model (1.20) is normally distributed or not. The JB test statistic is

$$\text{JB} = n\left(\frac{\hat{\gamma}_1^2}{6} + \frac{(\hat{\gamma}_2 - 3)^2}{24}\right) \tag{3.1}$$

where

$$\hat{\gamma}_1 = \frac{\frac{1}{n}\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^3}{\left(\frac{1}{n}\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^2\right)^{3/2}} \tag{3.2}$$

$$\hat{\gamma}_2 = \frac{\frac{1}{n}\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^4}{\left(\frac{1}{n}\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^2\right)^{2}} \tag{3.3}$$

n is the sample size while, $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are the sample skewness and kurtosis coefficients. The asymptotic null distribution of JB is χ^2 with 2 degrees of freedom.

3.2. White Noise Test

The modified Ljung-Box test statistic [11] given by

$$Q^*(m) = n(n+2)\sum_{k=1}^{m}\left(\frac{\left[\hat{\rho}_{X^d}(k)\right]^2}{n-k}\right) \tag{3.4}$$

is used to test the *iid* hypothesis for $X_t^d, d = 1,2,3$ for the simple bilinear model (1.20). It is important to note from Theorem 2.1 that X_t^2 has ARMA(2, 1) structure while from Theorem 2.2, X_t^3 is *iid*. This test will look for β values where both X_t^2 and X_t^3 are jointly *iid*. That will help determine the values of β for which the simple bilinear model (1.20) is not distinguishable from the linear Gaussian white noise process (LGWNP). Then, the hypothesis of *iid* data is rejected at level α if the observed $Q^*(m)$ is larger than the $1-\frac{\alpha}{2}$ quartile of the $\chi^2(m)$ distribution, where $m \approx \ln(n)$ [9].

3.3. Use of Chi-Square Test for Comparison of the Simple Bilinear White Noise Process and the Linear Gaussian White Noise Process

From Theorem 2.3, the third power of the simple bilinear process is *iid*. A test is needed to confirm that the simple bilinear process (1.20) is not a linear Gaussian white noise process (LGWNP). For the LGWNP $X_t, t \in T$; $E(X_t) = \mu$, $\text{var}(X_t) = \sigma^2 < \infty$ and $\text{var}(X_t^3) = 15\sigma^6$. To show that the simple bilinear process (1.20) is not LGWNP, we need to test the hypothesis;

$$H_0 : \sigma^2_{X_t^3} = 15\sigma^6_{X_t} \tag{3.5}$$

against the alternative hypothesis

$$H_0 : \sigma^2_{X_t^3} \neq 15\sigma^6_{X_t} \tag{3.6}$$

The chi-square test [24] [25] can be used to perform the test. The chi-square test statistic is

$$\chi^2_{cal} = \frac{(n-1) S^2_{X_t^3}}{15\hat{\sigma}^6_{X_t}} \tag{3.7}$$

where $S^2_{X_t^3}$ is the sample variance of $X_t^3; X_t, t \in Z$ that follows (1.20), $\hat{\sigma}^2_{X_t}$ is an estimate of the true variance of the simple bilinear process (1.20) and n is the number of observations of the series. The null hypothesis is rejected at level α if the observed value of χ^2_{cal} is larger than $1-\frac{\alpha}{2}$ quartile of the chi-square distribution with $n-1$.degree of freedom. It should be noted that this test works well when the underlying original population $X_t, t \in Z$ is normally distributed.

4. Results and Discussion

One thousand random digits $e_t, t \in Z$ that met the condition $e_t \sim N(0,1)$ were simulated using Minitab 16 series software. Only one random digit, shown in **Appendix I**, was used for demonstration in the study because of want of space. The estimates of the descriptive statistics (mean, variance, skewness (γ_1) and kurtosis (γ_2)) and other tests (Jarque Bera (JB) test, modified Ljung Box test (Q^*) and chi-square calculated test statistic) for the powers $e_t^d, d = 1,2,3$ of the random digit are shown in **Table 3**. The results obtained using the JB, Q^* and the chi-square test indicated $e_t, t \in Z$ as a LGWNP at 5% level of significance.

The LGWNP were used to simulate the SBWNP $X_t = BX_{t-2}e_{t-1} + e_t$, $e_t \sim N(0,1)$ for $-0.60 \le \beta \le 0.60$ satisfying the existence of $E(X_t^3)$ using Fortran 77 program. The estimates of the descriptive statistic and that for the test statistic (JB, Q^* and the chi-square calculated test statistic) are shown in **Table 4**. The values of the JB statistic show that the SBWNP are normally distributed for $-0.56 \le \beta \le 0.60$. Similarly, the values of Q^* and the chi-square calculated test statistic (H_0) show that the SBWNP is *iid* and can be identified as a LGWNP for some β values. The values of β where the SBWNP will be identified as an LGWNP are summarized in **Table 5**.

5. Conclusion

We have been able to establish the covariance structure for $X_t^d, d = 1,2,3; t \in Z$

Table 3. Descriptive Statistics and estimate of the test statistic for rejecting the null hypothesis of equality of the variance of higher moment for the simulated series, $X_t = e_t, e_t \sim N(0,1)$, as linear Gaussian white noise process.

Statistic	Mean	Median	Estimated Value	Skewness	Kurtosis	JB value	Q^*	Estimate of Test Statistic	
			S^2	γ_1	γ_2			$\frac{(n-1)S^2_{X_t^2}}{2\hat{\sigma}_0^4}$	$\frac{(n-1)S^2_{X_t^3}}{15\hat{\sigma}_0^6}$
X_t	0.0000	0.1261	1.0000	−0.28	−0.04	1.87	3.36	-	-
X_t^2	0.9931	0.4763	1.9074	1.90	2.79	133.19	0.04	136.38	-
X_t^3	−0.2728	0.0020	11.5236	−0.61	6.47	259.67	−0.14	-	109.86

Table 4. Descriptive statistics and estimate of the test statistic for simulated bilinear series $X_t = \beta X_{t-2} e_{t-1} + e_t$, $e_t \sim N(0,1)$ and $-0.60 \le \beta \le 0.60$.

β	Statistic	Estimated Values				Estimate of Test Statistic		
		Mean	Variance	γ_1	γ_2	JB value	Q^*	H_0
	X_t	0.0418	1.9037	0.27	1.20	10.28	8.44	-
−0.60	X_t^2	1.8923	11.3331	3.09	11.85	1072.25	71.39	-
	X_t^3	0.9233	186.5203	3.78	26.73	4628.20	22.66	257.74
	X_t	0.0410	1.8610	0.24	1.11	8.78	8.16	.
−0.59	X_t^2	1.8490	10.5110	2.99	11.09	952.49	70.34	-
	X_t^3	0.8100	164.3700	3.61	25.83	4315.90	20.32	243.12
	X_t	0.0390	1.8200	0.21	1.02	7.30	7.89	.
−0.58	X_t^2	1.8090	9.7720	2.89	10.39	848.16	69.04	-
	X_t^3	0.7100	145.5500	3.44	24.98	4028.01	18.02	230.17
	X_t	0.0380	1.7800	0.18	0.94	6.08	7.63	.
−0.57	X_t^2	1.7700	9.1080	2.80	9.75	758.53	67.49	-
	X_t^3	0.6220	129.4920	3.29	24.13	3753.32	15.84	218.89
	X_t	0.0370	1.7430	0.15	0.87	5.12	7.39	.
−0.56	X_t^2	1.7320	8.5110	2.72	9.16	680.73	65.73	-
	X_t^3	0.5390	115.7480	3.14	23.25	3479.84	13.84	208.38
	X_t	0.0360	1.7080	0.13	0.80	4.29	7.15	.
−0.55	X_t^2	1.6970	7.9730	2.64	8.61	612.26	63.76	-
	X_t^3	0.4630	103.9380	2.99	22.32	3203.09	12.04	198.86
	X_t	0.0346	1.6739	0.10	0.74	3.58	6.93	-
−0.54	X_t^2	1.6634	7.4872	2.57	8.09	550.71	61.63	-
	X_t^3	0.3948	93.7500	2.84	21.33	2921.69	10.48	190.56
	X_t	0.0334	1.6416	0.08	0.69	3.00	6.73	-
−0.53	X_t^2	1.6313	7.0486	2.50	7.59	495.95	59.36	-
	X_t^3	0.3325	84.9253	2.69	20.27	2637.97	9.18	183.01
	X_t	0.0322	1.6108	0.06	0.64	2.52	6.54	-
−0.52	X_t^2	1.6006	6.6518	2.44	7.12	446.71	56.99	-
	X_t^3	0.2759	77.2508	2.54	19.16	2356.47	8.12	176.21
	X_t	0.0310	1.5814	0.04	0.59	2.12	6.36	-
−0.51	X_t^2	1.5714	6.2921	2.38	6.66	402.32	54.54	-
	X_t^3	0.2246	70.5498	2.39	18.01	2082.80	7.31	170.07

Continued

	X_t	0.0299	1.5534	0.02	0.55	1.79	6.20	-
−0.50	X_t^2	1.5435	5.9656	2.32	6.23	362.29	52.04	-
	X_t^3	0.1781	64.6758	2.25	16.84	1822.60	6.72	164.49
	X_t	0.0188	1.3345	−0.12	0.26	0.74	5.22	-
−0.40	X_t^2	1.3256	3.8995	1.92	3.05	144.03	29.34	-
	X_t^3	−0.1026	32.3961	1.06	7.97	408.18	7.16	129.95
	X_t	0.0096	1.1946	−0.19	0.12	0.95	4.92	-
−0.30	X_t^2	1.1864	2.9329	1.77	2.17	103.47	15.82	-
	X_t^3	−0.2082	20.8415	0.56	6.08	229.18	8.31	116.55
	X_t	0.0088	1.1836	−0.19	0.11	0.98	4.90	-
−0.29	X_t^2	1.1755	2.8659	1.77	2.16	103.03	14.92	-
	X_t^3	−0.2143	20.1484	0.53	6.07	227.89	8.32	115.84
	X_t	0.0080	1.1731	−0.20	0.10	1.01	4.88	-
−0.28	X_t^2	1.1650	2.8026	1.77	2.16	102.93	14.09	-
	X_t^3	−0.2199	19.5079	0.50	6.08	227.54	8.31	115.20
	X_t	0.0073	1.1630	−0.20	0.09	1.05	4.86	-
−0.27	X_t^2	1.1550	2.7430	1.77	2.17	103.12	13.32	-
	X_t^3	−0.2251	18.9148	0.48	6.09	227.87	8.29	114.63
	X_t	0.0065	1.1533	−0.21	0.08	1.08	4.83	-
−0.26	X_t^2	1.1453	2.6866	1.77	2.18	103.55	12.61	-
	X_t^3	−0.2298	18.3644	0.45	6.11	228.69	8.26	114.13
	X_t	0.0059	1.1440	−0.21	0.07	1.11	4.81	-
−0.25	X_t^2	1.1361	2.6333	1.77	2.20	104.19	11.95	-
	X_t^3	−0.2342	17.8527	0.42	6.13	229.83	8.22	113.68
	X_t	0.0052	1.1351	−0.22	0.07	1.15	4.78	-
−0.24	X_t^2	1.1272	2.5828	1.77	2.22	104.99	11.35	-
	X_t^3	−0.2382	17.3761	0.40	6.16	231.17	8.17	113.26
	X_t	0.0046	1.1265	−0.22	0.06	1.18	4.75	-
−0.23	X_t^2	1.1187	2.5350	1.78	2.24	105.93	10.79	-
	X_t^3	−0.2418	16.9312	0.37	6.18	232.59	8.11	112.91
	X_t	0.0040	1.1183	−0.22	0.05	1.21	4.72	-
−0.22	X_t^2	1.1105	2.4898	1.78	2.27	106.98	10.28	-
	X_t^3	−0.2452	16.5153	0.34	6.21	234.04	8.05	112.58
	X_t	0.0034	1.1103	−0.23	0.04	1.24	4.69	-
−0.21	X_t^2	1.1026	2.4468	1.79	2.29	108.11	9.81	-
	X_t^3	−0.2483	16.1256	0.31	6.23	235.45	7.98	112.32
	X_t	0.0029	1.1027	−0.23	0.04	1.28	4.65	-
−0.20	X_t^2	1.0951	2.4061	1.79	2.32	109.31	9.39	-
	X_t^3	−0.2512	15.7600	0.28	6.26	236.78	7.91	112.05
	X_t	0.0024	1.0954	−0.23	0.03	1.31	4.61	-
−0.19	X_t^2	1.0878	2.3675	1.80	2.34	110.55	9.00	-
	X_t^3	−0.2538	15.4164	0.25	6.28	238.03	7.83	111.82

Continued

	X_t	0.0020	1.0884	−0.24	0.03	1.34	4.57	-
−0.18	X_t^2	1.0808	2.3308	1.80	2.37	111.82	8.65	-
	X_t^3	−0.2561	15.0931	0.22	6.30	239.16	7.74	111.60
	X_t	0.0015	1.0816	−0.24	0.02	1.37	4.52	-
−0.17	X_t^2	1.0741	2.2959	1.81	2.40	113.10	8.33	-
	X_t^3	−0.2583	14.7883	0.18	6.32	240.20	7.66	111.42
	X_t	0.0011	1.0752	−0.24	0.01	1.40	4.48	-
−0.16	X_t^2	1.0677	2.2628	1.82	2.42	114.39	8.04	-
	X_t^3	−0.2603	14.5008	0.15	6.33	241.13	7.57	111.22
	X_t	0.0008	1.0689	−0.24	0.01	1.44	4.43	-
−0.15	X_t^2	1.0615	2.2313	1.82	2.45	115.68	7.79	-
	X_t^3	−0.2621	14.2292	0.11	6.35	241.98	7.48	111.07
	X_t	0.0005	1.0629	−0.25	0.00	1.47	4.37	-
−0.14	X_t^2	1.0555	2.2013	1.83	2.48	116.96	7.56	-
	X_t^3	−0.2638	13.9723	0.07	6.36	242.75	7.38	110.93
	X_t	0.0002	1.0571	−0.25	−0.00	1.50	4.31	-
−0.13	X_t^2	1.0498	2.1728	1.83	2.50	118.23	7.36	-
	X_t^3	−0.2652	13.7293	0.03	6.37	243.49	7.28	110.80
	X_t	−0.0001	1.0516	−0.25	−0.00	1.53	4.25	-
−0.12	X_t^2	1.0443	2.1457	1.84	2.52	119.48	7.18	-
	X_t^3	−0.2666	13.4993	−0.01	6.38	244.19	7.19	110.66
	X_t	−0.0003	1.0463	−0.25	−0.01	1.56	4.19	-
−0.11	X_t^2	1.0390	2.1199	1.85	2.55	120.71	7.03	-
	X_t^3	−0.2677	13.2813	−0.06	6.39	244.90	7.09	110.54
	X_t	−0.0004	1.0411	−0.26	−0.01	1.59	4.13	-
−0.10	X_t^2	1.0339	2.0953	1.85	2.57	121.92	6.90	-
	X_t^3	−0.2688	13.0747	−0.10	6.40	245.64	6.99	110.46
	X_t	0.0046	0.9745	−0.30	−0.04	2.18	2.51	-
0.10	X_t^2	0.9677	1.8045	1.93	2.98	142.60	6.84	-
	X_t^3	−0.2698	10.6906	−1.12	6.62	292.71	5.10	110.13
	X_t	4.19624	0.9627	−0.33	−0.01	2.61	1.77	-
0.20	X_t^2	4.19624	1.7743	1.94	3.07	146.99	7.45	-
	X_t^3	4.19624	10.4201	−1.52	6.79	331.67	4.20	111.34
	X_t	0.0149	0.9623	−0.33	−0.01	2.67	1.71	-
0.21	X_t^2	0.9558	1.7750	1.94	3.08	147.10	7.51	-
	X_t^3	−0.2654	10.4221	−1.55	6.80	334.75	4.10	111.50
	X_t	0.0161	0.9620	−0.34	−0.00	2.72	1.66	-
0.22	X_t^2	0.9556	1.7765	1.94	3.08	147.15	7.58	-
	X_t^3	−0.2651	10.4295	−1.58	6.81	337.56	4.01	111.68
	X_t	0.0174	0.9618	−0.34	0.00	2.78	1.61	-
0.23	X_t^2	0.9555	1.7786	1.94	3.08	147.18	7.65	-
	X_t^3	−0.2648	10.4424	−1.60	6.81	340.07	3.92	111.89

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0.24	X_t	0.0187	0.9618	−0.34	0.01	2.85	1.56	-
	X_t^2	0.9555	1.7813	1.94	3.08	147.18	7.72	-
	X_t^3	−0.2645	10.4608	−1.62	6.82	342.26	3.83	112.09
0.25	X_t	0.0200	0.9620	−0.35	0.01	2.91	1.52	-
	X_t^2	0.9557	1.7848	1.94	3.08	147.17	7.80	-
	X_t^3	−0.2643	10.4851	−1.64	6.82	344.13	3.75	112.28
0.26	X_t	0.0214	0.9623	−0.35	0.02	2.98	1.49	-
	X_t^2	0.9561	1.7888	1.94	3.08	147.16	7.88	-
	X_t^3	−0.2641	10.5152	−1.66	6.83	345.65	3.66	112.49
0.27	X_t	0.0229	0.9628	−0.36	0.02	3.05	1.46	-
	X_t^2	0.9566	1.7935	1.94	3.08	147.18	7.96	-
	X_t^3	−0.2638	10.5514	−1.67	6.83	346.83	3.57	112.71
0.28	X_t	0.0244	0.9634	−0.36	0.03	3.12	1.43	-
	X_t^2	0.9573	1.7989	1.94	3.08	147.23	8.05	-
	X_t^3	−0.2636	10.5939	−1.69	6.83	347.68	3.49	112.95
0.29	X_t	0.0259	0.9641	−0.36	0.03	3.19	1.41	-
	X_t^2	0.9581	1.8048	1.94	3.08	147.33	8.14	-
	X_t^3	−0.2633	10.6429	−1.69	6.82	348.21	3.41	113.22
0.30	X_t	0.0275	0.9651	−0.37	0.04	3.27	1.40	-
	X_t^2	0.9591	1.8115	1.94	3.08	147.52	8.24	-
	X_t^3	−0.2630	10.6987	−1.70	6.82	348.45	3.33	113.46
0.31	X_t	0.0291	0.9662	−0.37	0.05	3.34	1.40	-
	X_t^2	0.9603	1.8187	1.94	3.09	147.79	8.35	-
	X_t^3	−0.2626	10.7616	−1.70	6.82	348.44	3.26	113.74
0.32	X_t	0.0308	0.9675	−0.38	0.05	3.42	1.40	-
	X_t^2	0.9617	1.8266	1.95	3.09	148.18	8.46	-
	X_t^3	−0.2622	10.8318	−1.69	6.82	348.25	3.19	114.02
0.33	X_t	0.0326	0.9689	−0.38	0.06	3.50	1.41	-
	X_t^2	0.9633	1.8352	1.95	3.10	148.72	8.59	-
	X_t^3	−0.2617	10.9098	−1.68	6.83	347.93	3.12	114.35
0.34	X_t	0.0343	0.9706	−0.38	0.06	3.58	1.43	-
	X_t^2	0.9650	1.8445	1.95	3.11	149.41	8.73	-
	X_t^3	−0.2611	10.9958	−1.67	6.84	347.58	3.06	114.64
0.35	X_t	0.0362	0.9724	−0.39	0.07	3.66	1.45	-
	X_t^2	0.9670	1.8544	1.96	3.12	150.28	8.87	-
	X_t^3	−0.2603	11.0904	−1.66	6.85	347.31	3.01	114.99
0.36	X_t	0.0381	0.9744	−0.39	0.08	3.74	1.49	-
	X_t^2	0.9691	1.8651	1.96	3.14	151.36	9.03	-
	X_t^3	−0.2594	11.1938	−1.63	6.87	347.25	2.96	115.35
0.37	X_t	0.0400	0.9767	−0.40	0.08	3.81	1.53	-
	X_t^2	0.9715	1.8765	1.97	3.16	152.67	9.21	-
	X_t^3	−0.2583	11.3067	−1.61	6.90	347.55	2.91	115.69

Continued

	X_t	0.0420	0.9791	−0.40	0.09	3.88	1.59	-
0.38	X_t^2	0.9741	1.8888	1.97	3.19	154.22	9.40	-
	X_t^3	−0.2569	11.4295	−1.58	6.94	348.40	2.87	116.09
	X_t	0.0440	0.9818	−0.40	0.09	3.95	1.65	-
0.39	X_t^2	0.9769	1.9019	1.98	3.22	156.05	9.61	-
	X_t^3	−0.2553	11.5629	−1.54	6.99	349.99	2.84	116.48
	X_t	0.0461	0.9847	−0.41	0.10	4.02	1.73	-
0.40	X_t^2	0.9800	1.9159	1.99	3.26	158.16	9.84	-
	X_t^3	−0.2534	11.7074	−1.50	7.06	352.57	2.81	116.89
	X_t	0.0482	0.9879	−0.41	0.11	4.08	1.82	-
0.41	X_t^2	0.9834	1.9309	1.99	3.30	160.59	10.09	-
	X_t^3	−0.2511	11.8638	−1.45	7.14	356.40	2.79	117.31
	X_t	0.0504	0.9913	−0.41	0.11	4.13	1.92	-
0.42	X_t^2	0.9870	1.9470	2.00	3.35	163.34	10.36	-
	X_t^3	−0.2484	12.0330	−1.39	7.25	361.79	2.77	117.75
	X_t	0.0526	0.9950	−0.41	0.12	4.18	2.03	-
0.43	X_t^2	0.9909	1.9643	2.01	3.40	166.42	10.65	-
	X_t^3	−0.2452	12.2158	−1.33	7.38	369.08	2.76	118.22
	X_t	0.0549	0.9990	−0.41	0.12	4.21	2.15	-
0.44	X_t^2	0.9951	1.9829	2.02	3.46	169.86	10.97	-
	X_t^3	−0.2416	12.4134	−1.27	7.53	378.64	2.75	118.70
	X_t	0.0572	1.0033	−0.41	0.13	4.24	2.29	-
0.45	X_t^2	0.9996	2.0030	2.03	3.53	173.66	11.32	-
	X_t^3	−0.2373	12.6270	−1.19	7.71	390.89	2.75	119.20
	X_t	0.0595	1.0079	−0.42	0.14	4.25	2.44	-
0.46	X_t^2	1.0044	2.0247	2.04	3.61	177.82	11.70	-
	X_t^3	−0.2323	12.8582	−1.11	7.92	406.27	2.75	119.73
	X_t	0.0620	1.0128	−0.41	0.14	4.25	2.60	-
0.47	X_t^2	1.0096	2.0482	2.05	3.69	182.34	12.10	-
	X_t^3	−0.2267	13.1088	−1.03	8.16	425.24	2.75	120.29
	X_t	0.0644	1.0181	−0.41	0.15	4.24	2.78	-
0.48	X_t^2	1.0152	2.0737	2.06	3.78	187.25	12.54	-
	X_t^3	−0.2202	13.3809	−0.93	8.44	448.27	2.76	120.88
	X_t	0.0670	1.0238	−0.41	0.16	4.21	2.97	-
0.49	X_t^2	1.0211	2.1016	2.07	3.87	192.53	13.02	-
	X_t^3	−0.2127	13.6772	−0.83	8.75	475.84	2.78	121.52
	X_t	0.0695	1.0298	−0.41	0.16	4.16	3.18	-
0.50	X_t^2	1.0275	2.1319	2.08	3.97	198.22	13.53	-
	X_t^3	−0.2043	14.0009	−0.73	9.09	508.36	2.81	122.22
	X_t	0.0980	1.1188	−0.32	0.28	2.89	6.02	-
0.60	X_t^2	1.1207	2.6703	2.27	5.60	312.17	20.67	-
	X_t^3	−0.0402	20.1794	0.86	13.91	1178.64	5.41	137.37

Table 5. Values of β for comparison of SBWNP as a LGWNP at 0.05 and 0.10 α levels.

α% level	Values of β	
	Q^*	H_0
5	$[-0.23, 0.44]$	$[-0.18, 0.22]$
10	$[-0.19, 0.37]$	$[-0.29, 0.38]$

satisfying (1.20). We have also determined the values of β for which the simple bilinear model (1.20) is normally distributed and in which the process can be determined as a LGWNP or not. We recommend that for proper comparison of SBWNP with LGWNP, the SBWNP should be considered for normality, white noise test and test of equality of variance of its third moment being equivalent to the theoretical values of the LGWNP.

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Appendix I

Simulated Random Digits; $e_t, e_t \sim N(0,1)$ **(Read Across).**

−0.57532	−0.17491	0.35244	0.30620	−0.76520	−0.10381	−0.78604	0.19891	0.48466	−1.04050	0.25694	2.13936
0.81740	−1.61037	2.38415	0.74182	−1.83436	−0.97443	0.06649	−0.80814	−2.14835	−1.39147	−1.19600	0.16246
1.10204	−0.75625	1.43986	0.41147	0.34040	−0.27339	−0.66471	0.72426	−0.24697	−0.73065	1.22347	1.89188
−0.78388	0.99457	−0.94385	1.99912	0.00884	0.10762	−2.23041	−0.20387	1.20197	−0.12003	1.83635	−0.06882
−2.38069	0.01037	0.55983	−1.86577	0.75661	−0.83977	−0.06520	−0.25303	0.57397	−0.10694	−1.87199	−0.61338
−0.96019	−0.69799	0.41226	−0.13727	0.73620	−0.25448	0.27995	0.82692	1.07422	0.72309	0.44146	0.76731
0.72838	0.39809	0.18794	0.06831	0.45853	−0.79068	−1.97602	−1.55625	0.98349	2.09313	−1.26609	0.50341
−0.98639	0.78335	0.56394	−0.00389	−0.60469	0.68956	0.09199	−0.84437	0.28016	−0.36120	0.16969	−0.32149
−1.97702	−0.98212	−1.26901	0.93133	0.63846	−0.83151	0.68592	0.18103	−0.69071	0.35337	0.67619	0.82779
1.25023	0.50671	1.39091	−0.27367	−0.09697	1.01271	1.21921	0.67856	0.37606	1.16306	−0.11180	−2.39334
1.13787	−0.46900	−1.07178	0.09855	1.96154	−0.45406	−1.57186	0.93940	−0.00755	0.32726	0.57558	0.48859
0.45601	0.14352	−2.13818	0.23375	−1.82588	0.13979	−0.25057	1.17289	0.12739	0.35428	0.12472	−0.92299