

Parameter Estimation for the Continuous Time Stochastic Logistic Diffusion Model*

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Abstract

In this paper, the parameter estimation problem is investigated for the continuous time stochastic logistic diffusion system. A new continuous process is built based on the likelihood ratio scheme, the Radon-Nikodym derivative and the explicit expressions of the error of estimation are given under this new continuous process. By using the random time transformations, law of large numbers for martingales, law of iterated logarithm and stationary distribution of solution, the consistency property are proved for the estimation error. Finally, a numerical simulation is presented to demonstrate the effectiveness of the proposed method in this paper.

Keywords

Parameter Estimation, Stationary Distribution, Likelihood Ratio, Random Time Transformations

1. Introduction

In the past few decades, parameter estimation problem for the stochastic differential equation have been studied by many scholars whose results mostly base on the discrete observation. In order to get more accurate estimators, we should observe through continuous time. Deterministic model, which parameters are deterministic irrespective of environmental fluctuations, are usually used to describe the overall impact of changes between different factors. these models obviously impose limitations in mathematical modeling of whole real systems. However, in the real world, many random factors (*i.e.* earthquakes, typhoons, car accidents and other unforeseen factors) may make the parameters

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into random variables. Therefore, it is more reasonable to use the stochastic differential equation with to describe the real systems disturbed by random noises. For example, the stochastic logistic diffusion model has been widely used in the field of social life, application of stochastic logistic diffusion model has been used in the field of applied economics [1] [2] [3], biology [4] [5] [6] [7], power engineering [8] [9] [10] and so on. Very recently, considerable research results have been reported on the parameter estimation based on discrete observation. To be special, [11] used the least squares method to estimate the parameters, also obtained the point estimators and confidence intervals as well as joint confidence regions. [12] used conditional least squares and weighted conditional least squares method to study the parameter estimation of two-type continuous-state branching processes with immigration based on low frequency observations at equidistant time points. [13] studied the asymptotic behaviour of parametric estimator for nonstationary reflected Ornstein-Uhlenbeck process by applying maximum likelihood estimation. For the stochastic logistic diffusion model, [14] worked out the optimization problem with respect to stationary probability density and provide a new equivalent, an ergodic method is used to show the almost surely equivalency between the time averaging yield and sustainable yield. [15] considered a stochastic logistic growth model involving both birth and death rates in the drift and diffusion coefficients and the associated complete Fokker-Planck equation is also established to describe the law of the process. [16] focused on stochastic dynamics involve continuous states as well as discrete events, and obtain weak convergence of the underlying system, and utilized the structure of limit system as a bridge to invest stochastic permanence of original system driving by a singular Markov chain with a large number of states. [17] presented some basic aspects of adequate numerical analysis for the random extensions such as numerical regularity and mean square convergence. [18] improved two mathematically tractable cases: at the limit of the number of individuals and at the limit of basic reproduction ratio. In the discrete observations, we let the time interval tends to 0 to get more accurate result. Therefore, this means that parametric inference based on continuous time observation is much more accurate in dealing with parameter estimation problem. During the estimation processing, two important theories have been used to estimate parameters based on continuous observation in the existing literature. One is denoting Radon-Nikodym derivative with likelihood ratio and the other one is by using stationary distribution of solution.

The stochastic logistic diffusion model can be described by the following stochastic differential equation:

$$dX_t = (\alpha X_t - \beta X_t^2)dt + \varepsilon X_t dW_t, \quad X_0 = x \quad (1.1)$$

where X_t represents the population capacity at time t , $\alpha > 0$ represents the natural birth rate, $\beta > 0$ represents mortality, $\frac{\alpha}{\beta}$ represents load capacity, usually also represents the largest population that environmental resources can

support. $\varepsilon > 0$ represents the dynamic effect of noise on X_t . W_t is a Wiener process modelling the random factor. [19] studied the existence, uniqueness and global attractivity of positive solutions for model (1.1), and established a maximum likelihood estimator for the parameters. [20] proved that no matter how small $\sigma > 0$, the solution will not explode in a limited time.

In this paper, the continuous observations shall be used to obtain more accurate results than discrete observations, and the likelihood ratio will be employed to get Radon-Nikodym derivative which can be used to solve the parameter estimation problem for logistic diffusion model. As we all know that logistic diffusion model is a diffusion process, for a general diffusion model $dX_t = \mu(X_t | \theta) + \sigma(X_t | \theta)$, the parameter θ enter into the description of X_t through μ or σ or both. However, the nature of diffusions allows us to evaluate σ exactly under a given continuous record, from the formula $\sum_{j=1}^{2^n} (X_j s 2^{-n} - X_{(j-1)s 2^{-n}})^2 \rightarrow \int_0^s \sigma^2(X_u) ds$ a.s. as $n \rightarrow +\infty$ (a.s. means almost surely) as pointed out in [21]. This result can be rewritten descriptively as $\int_0^t (dX_u)^2 \rightarrow \int_0^s \sigma^2(X_u) ds$ a.s. as $n \rightarrow +\infty$. Thus we could consider a parameter involved in σ as being know a single realization which means we should only consider the estimation of θ in $\mu = \mu(X_s | \theta)$. Then the likelihood ratio as Radon-Nikodym derivative and the expressions of all estimators would be obtained. As [19] have studied the stationary distribution of X_t , based on the result and strong law of large numbers of martingales the law of iterated logarithm, the consistency property and the normality of asymptotic shall be proved for the estimation error.

This paper is organized as follows. In Section 2, a new method of estimating parameters is given and estimators are obtained. In Section 3, the strong consistency properties of estimators and estimation of asymptotic normality of error are proved. In Section 4, a numerical example for the estimators and error of estimation between estimators and trues is given to demonstrate the effectiveness of the proposed results. The conclusion is given in Section 5.

2. Preliminaries

In this paper, the parameter estimation problem shall be studied for the logistic diffusion model described by a stochastic differential equation as given in (1.1). In this model α , β are unknown parameters. We can calculate ε use the method in [21] and this means that ε is also a known parameter. Because of the complication of the transitional density function for this model, it is difficult to obtain commonly used expression for the unknown parameters. Therefore, we will calculate the likelihood ratio (with respect to $P_{\alpha, \beta}$, α and β are the true parameters) for a finite set of time points $0 = t_0 < t_1 < \dots < t_n = t$, and then let the number of time points tend to infinity to get the likelihood function. From now on we shall work under the assumptions below.

Assumption 1: α, β and ε are positive, X_0 is positive and independent with W_t .

Assumption 2: $2\alpha > \varepsilon^2$, which implies that X_t cannot reach zero.

Assumption 3: X_0 is a positive random variable, and there is a $Q > 2$ such that $E[X_0^Q] < \infty$ hold.

Next, the specific steps with respect to derivations of the likelihood function and parameter estimators are given below.

Assume that ε^2 is known for all X_t . X_t can be observed continuously throughout the time interval $0 \leq s \leq t$. For observations in this detail enable the true diffusion parameter ε^2 to be determinate through following result

$$\sum_{j=1}^{2^n} (X_j s 2^{-n} - X(j-1) s 2^{-n})^2 \rightarrow \int_0^s \sigma^2(X_u) ds \quad a.s. \quad \text{as } n \rightarrow +\infty.$$

For all $s \in [0, t]$, above equation can be rewritten descriptively as follows:

$$\int_0^t (dX_s)^2 = \int_0^t (\varepsilon X_s)^2 ds \quad a.s..$$

Then, ε^2 may there be assumed know as:

$$\varepsilon^2 = \frac{\int_0^t (dX_s)^2}{\int_0^t X_s^2 ds}. \tag{2.1}$$

The parameter α and β enter into the description of X_t of (1.1), we can get ε exactly by (2.1). We begin with a class of probability $(\Omega, \mathcal{F}, P_{\alpha, \beta})$, where the real stochastic process $X = X_s; s \geq 0$ on (Ω, \mathcal{F}) evolves according to one of probability laws $P_{\alpha, \beta}$. For each $t \geq 0$ define

$$\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t, N), \tag{2.2}$$

with σ -field generated by sets $w \in \Omega: X_s = X_s(w) \in B$, and B is a Borel set on R and the class N is $P_{\alpha, \beta}$ -null set of \mathcal{F} .

Define

$$\rho(t) = \int_0^t (\varepsilon X_s)^2 ds, \quad t \geq 0 \tag{2.3}$$

and

$$Y_{\rho(t)} = X_t, \quad t \geq 0. \tag{2.4}$$

From the theory of random time transformations, we have

$$dY_t = \frac{\alpha Y_t - \beta Y_t^2}{(\varepsilon Y_t)^2} dt + dW'_t \tag{2.5}$$

where W'_t is a standard Brownian motion(or its measure). (1.1) with the initial distribution μ . W_t is a Brownian motion with respect to the filtration \mathcal{F}_t and $X_0 = x$ is a \mathcal{F}_0 -measurable random variable with distribution μ . Then, for the process of option times (2.3), we can create the inverse process that is $X_t = Y_{\rho(t)}$ with filtration $\mathcal{G}_t = \mathcal{F}_{\rho(t)}$, and we can find a Brownian motion W'_t with respect to \mathcal{G} . For more details please see reference [22], the existence and uniqueness of W'_t will be found. From Kailath and Zakai's [23], we know

$$\begin{aligned} \frac{dP'_{\alpha,\beta}}{dW'_t} &= \exp \left\{ \int_0^{\rho(t)} \frac{(\alpha Y_s - \beta Y_s^2)}{\varepsilon^2 Y_s^2} dY_s - \frac{1}{2} \int_0^{\rho(t)} \frac{(\alpha Y_s - \beta Y_s^2)^2}{\varepsilon^2 Y_s^4} ds \right\} \\ &= \exp \left\{ \int_0^t \frac{(\alpha X_s - \beta X_s^2)}{\varepsilon^2 X_s^2} dX_s - \frac{1}{2} \int_0^t \frac{(\alpha X_s - \beta X_s^2)^2}{\varepsilon^2 X_s^4} d\rho(s) \right\} \end{aligned} \tag{2.6}$$

by substituting $\rho(s)$ for s ,

$$= \exp \left\{ \int_0^t \frac{(\alpha X_s - \beta X_s^2)}{\varepsilon^2 X_s^2} dX_s - \frac{1}{2} \int_0^t \frac{(\alpha X_s - \beta X_s^2)^2}{\varepsilon^2 X_s^2} ds \right\}. \tag{2.7}$$

Similarly, we can get,

$$\frac{dP'_{\hat{\alpha},\hat{\beta}}}{dW'_t} = \exp \left\{ \int_0^t \frac{(\hat{\alpha} X_s - \hat{\beta} X_s^2)}{\varepsilon^2 X_s^2} dX_s - \frac{1}{2} \int_0^t \frac{(\hat{\alpha} X_s - \hat{\beta} X_s^2)^2}{\varepsilon^2 X_s^2} ds \right\}. \tag{2.8}$$

Thus,

$$\frac{dP'_{\hat{\alpha},\hat{\beta}}}{dP'_{\alpha,\beta}} = \left[\frac{dP'_{\hat{\alpha},\hat{\beta}}}{dW'_t} \right] / \left[\frac{dP'_{\alpha,\beta}}{dW'_t} \right]. \tag{2.9}$$

Writing $P'_{\alpha,\beta}$ for the restriction of $P_{\alpha,\beta}$ to \mathcal{F}_t , and we can now define following likelihood function as a Radon-Nikodym derivative:

$$\begin{aligned} L_t(\hat{\alpha}, \hat{\beta}) &= \frac{dP'_{\hat{\alpha},\hat{\beta}}}{dP'_{\alpha,\beta}} = \exp \left\{ \int_0^t \frac{(\hat{\alpha} X_s - \hat{\beta} X_s^2) - (\alpha X_s - \beta X_s^2)}{\varepsilon^2 X_s^2} dX_s \right. \\ &\quad \left. - \frac{1}{2} \int_0^t \frac{(\hat{\alpha} X_s - \hat{\beta} X_s^2)^2 - (\alpha X_s - \beta X_s^2)^2}{\varepsilon^2 X_s^2} ds \right\}, \quad \forall t > 0. \end{aligned} \tag{2.10}$$

Let $l_t(\hat{\alpha}, \hat{\beta}) = \log L_t(\hat{\alpha}, \hat{\beta})$, (the “log()” function has the basement “e”) solving following equation

$$\begin{cases} \frac{\partial l_t(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}} = 0 \\ \frac{\partial l_t(\hat{\alpha}, \hat{\beta})}{\partial \hat{\beta}} = 0, \end{cases} \tag{2.11}$$

we can obtain the estimators as follows:

$$\begin{cases} \hat{\alpha} = \frac{\int_0^t \frac{dX_s}{X_s} \int_0^t X_s^2 ds - (X_t - X_0) \int_0^t X_s ds}{t \int_0^t X_s^2 ds - \left(\int_0^t X_s ds \right)^2}, \\ \hat{\beta} = \frac{\int_0^t \frac{dX_s}{X_s} \int_0^t X_s ds - t(X_t - X_0)}{t \int_0^t X_s^2 ds - \left(\int_0^t X_s ds \right)^2}. \end{cases} \tag{2.12}$$

3. Main Results and Proofs

In this section, we shall give the asymptotic distribution of the estimated errors and the corresponding proof. It's easy to know the solution of Equation (1.1) has following expression:

$$X_t = \frac{\exp\left\{\left(\alpha - \frac{\varepsilon^2}{2}\right)t + \varepsilon W_t\right\}}{x + \beta \int_0^t \exp\left\{\left(\alpha - \frac{\varepsilon^2}{2}\right)s + \varepsilon W_s\right\} ds}. \tag{3.1}$$

Firstly, let us give following four lemmas.

Lemma 1 (The law of iterated logarithm) [24]

$$\limsup_{t \rightarrow +\infty} \frac{W_t}{\sqrt{2t \log \log t}} = 1 \quad a.s.. \tag{3.2}$$

Lemma 2 [19] Assume that there is a positive number C such that

$$-\lambda := \lambda_{\max}^+(-2C\beta) = \sup_{X_t \in \mathbb{R}_+, |X_t|=1} (-2C\beta) X_t^2 < 0. \tag{3.3}$$

Then, for any given initial value $x \in \mathbb{R}_+$, the solution X_t of equation $dX_t = (\alpha X_t - \beta X_t^2)dt + \varepsilon X_t dW_t$ has following properties

$$\limsup_{t \rightarrow +\infty} \int_t^{t+1} \mathbb{E} |X(s)|^2 ds \leq \frac{4C^2 |\alpha|}{\lambda^2} \left(1 + \frac{|C\alpha|}{C}\right), \tag{3.4}$$

$$\limsup_{t \rightarrow +\infty} \mathbb{E} \left(\sup_{t \leq s \leq t+1} |X_s| \leq \frac{2C^2 |\alpha|}{C} \left(1 + \frac{|C\alpha|}{C}\right) + \frac{6|C\varepsilon|}{C} \sqrt{\frac{C^2 |\alpha|}{\lambda^2} \left(1 + \frac{|C\alpha|}{C}\right)} \right), \tag{3.5}$$

and

$$\limsup_{t \rightarrow +\infty} \frac{\log(|X(t)|)}{\log t} \leq 1 \quad a.s.. \tag{3.6}$$

Lemma 3 [19] If the conditions $\lambda_{\max}^+(-2C\beta) < 0$ and $\alpha > \frac{\varepsilon^2}{2}$ hold. Then, for any given initial value $x \in \mathbb{R}_+$, the solution $X(t)$ of $dX_t = (\alpha X_t - \beta X_t^2)dt + \varepsilon X_t dW_t$ has the property that

$$\liminf_{t \rightarrow +\infty} \frac{\log(|X(t)|)}{\log(t)} \geq -\frac{\varepsilon^2}{2\alpha - \varepsilon^2} \quad a.s.. \tag{3.7}$$

Lemma 4 Assume that X_t is a solution to the stochastic differential Equation (1.1) and Assumptions 1 - 3 hold. Then, we have

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t X_s ds = \frac{\alpha - \frac{\varepsilon^2}{2}}{\beta}. \tag{3.8}$$

proof: It is known from lemma 2 that the solution (3.1) obeys

$$\limsup_{t \rightarrow +\infty} \frac{\log(X_t)}{\log t} \leq t \quad a.s..$$

While, from Assumption 2 we have $2\alpha > \varepsilon^2$. Then, by the properties in lemma 2, the solution (3.1) satisfies

$$\liminf_{t \rightarrow +\infty} \frac{\log(X_t)}{\log t} \geq -\frac{\varepsilon^2}{2\alpha - \varepsilon^2} \quad a.s..$$

Consequently,

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log(X_t) = 0 \quad a.s..$$

By the Itô formula, it is easy to know that

$$\log(X_t) = \log(x) + \left(\alpha - \frac{\varepsilon^2}{2} \right) t - \beta \int_0^t X_s ds + \varepsilon W_t.$$

Dividing both side by t and then letting $t \rightarrow +\infty$, we obtain

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t X_s ds = \frac{\alpha - \frac{\varepsilon^2}{2}}{\beta} \quad a.s.. \tag{3.9}$$

The proof is complete. \square

Remark: (The one-dimensional Itô formula) Let $x(t)$ be an Itô process on $t \geq 0$ with the stochastic differential

$$dx(t) = f(t)dt + g(t)dB(t), \tag{3.10}$$

where $f \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R})$ and $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R})$. Let $V \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R})$. Then $V(x(t), t)$ is again an Itô process with the stochastic differential given by

$$dV(x(t), t) = \left[V_t(x(t), t) + V_x(x(t), t)f(t) + \frac{1}{2}V_{xx}(x(t), t)g^2(t) \right] dt + V_x(x(t), t)g(t)dB_t \quad a.s.. \tag{3.11}$$

Theorem 1 Let X_t be a solution to the stochastic differential Equation (1.1) and Assumptions 1-3 hold. Then, we have

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t X_s^2 ds = \frac{\alpha^2 - \frac{\alpha \varepsilon^2}{2}}{\beta^2}. \tag{3.12}$$

proof: It follows from (1.1) that $X_t - x = \alpha \int_0^t X_s ds - \beta \int_0^t X_s^2 ds + \varepsilon \int_0^t X_s dW_s$, dividing both sides by t and then letting $t \rightarrow +\infty$, one has

$$\lim_{t \rightarrow +\infty} \frac{X_t - x}{t} = \lim_{t \rightarrow +\infty} \frac{\alpha}{t} \int_0^t X_s ds - \lim_{t \rightarrow +\infty} \frac{\beta}{t} \int_0^t X_s^2 ds + \lim_{t \rightarrow +\infty} \frac{\varepsilon}{t} \int_0^t X_s dW_s.$$

Since

$$\begin{aligned} \mathbb{E} \left[\int_0^t X_s dW_s \right] &= \int_0^t \mathbb{E} [X_s dW_s] \\ &= \int_0^t \mathbb{E} \left[\mathbb{E} [X_s dW_s | \mathcal{F}_s] \right] \\ &= \int_0^t \mathbb{E} [X_s \mathbb{E} [dW_s]] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\left[\int_0^t X_s dW_s \mid \mathcal{F}_t^-\right] &= \int_0^t X_s dW_s + \mathbb{E}\left[\int_t^t X_s dW_s \mid \mathcal{F}_t^-\right] \\ &= \int_0^t X_s dW_s + \int_t^t X_s [dW_s] \\ &= \int_0^t X_s dW_s. \end{aligned}$$

$\int_0^t X_s dW_s$ is a martingale with zero mean with respect to the σ -algebra \mathcal{F}_t^- . Moreover, according to (1.1), $X_{t_i} - X_{t_{i-1}} = (\alpha X_{t_{i-1}} - \beta X_{t_{i-1}}^2)\Delta + \varepsilon X_{t_{i-1}} \sqrt{\Delta} \epsilon_{t_i}$. The equation $f(x) = x + (\alpha x - \beta x^2)\Delta$ (among them $\Delta = \max\{t_i - t_{i-1}\}$, $0 \leq t_i \leq t$, $\epsilon_{t_i} \sim N(0,1)$) gets maximum when $x = \frac{\alpha\Delta + 1}{2\beta}$, thus,

$$X_t \leq \frac{(\alpha\Delta + 1)^2}{4\beta} < \infty.$$

Therefore, $\mathbb{E}\left[(X_s dW_s)^2\right] = \mathbb{E}\left[X_s^2\right]$ is bounded. It then follows that

$$\limsup_{t \rightarrow +\infty} \frac{\int_0^t X_s^2 ds}{t} < \infty \quad a.s.. \tag{3.13}$$

By the strong law of large numbers of martingales, we have

$$\lim_{t \rightarrow +\infty} \frac{\int_0^t X_s dW_s}{t} = 0 \quad a.s..$$

Together with (3.1) and Lemma 1, we obtain

$$0 < \lim_{t \rightarrow +\infty} X_t = \lim_{t \rightarrow +\infty} \frac{\exp\left\{\left(\alpha - \frac{\varepsilon^2}{2}\right)t + 2\sqrt{t \log \log t}\right\}}{x^{-1} + \beta \int_0^t \exp\left\{\left(\alpha - \frac{\varepsilon^2}{2}\right)s + 2\sqrt{s \log \log s}\right\} ds}. \tag{3.14}$$

By LHospital rule, we get

$$\lim_{t \rightarrow +\infty} \frac{\exp\left\{\left(\alpha - \frac{\varepsilon^2}{2}\right)t + 2\sqrt{t \log \log t}\right\}}{x^{-1} + \beta \int_0^t \exp\left\{\left(\alpha - \frac{\varepsilon^2}{2}\right)s + 2\sqrt{s \log \log s}\right\} ds} = \frac{\alpha - \frac{\varepsilon^2}{2}}{\beta}.$$

Then,

$$\lim_{t \rightarrow +\infty} \frac{X_t}{t} = 0 \quad a.s..$$

According to (3.10), we get

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t X_s^2 ds = \frac{\alpha^2 - \frac{\alpha\varepsilon^2}{2}}{\beta^2} \quad a.s.. \tag{3.15}$$

The proof is now complete. \square

Remark: (LHospital rule) The general form of LHospital rule covers many cases. Let c and L be extended real numbers. The real valued function f and g are assumed to be differentiable on an open interval with endpoint c , and additionally

$g'(x) \neq 0$ on the interval. It is also assumed that $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = L$. Thus the rule applies to situations in which the ratio of the derivatives has a finite or infinite limit, and not to situations in which that ratio fluctuates permanently as x gets closer and closer to c .

If either

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0 \quad (3.16)$$

or

$$\lim_{x \rightarrow c} |f(x)| = \lim_{x \rightarrow c} |g(x)| = \infty, \quad (3.17)$$

then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = L. \quad (3.18)$$

Theorem 2 Under Assumptions 1-3, $\hat{\alpha}$ and $\hat{\beta}$ are strongly consistent.

Proof: Substituting (1.1) into the expression of $\hat{\alpha}$ yields

$$\hat{\alpha} - \alpha = \frac{\varepsilon W_t \int_0^t X_s^2 ds - \varepsilon \int_0^t X_s dW_s \int_0^t X_s ds}{t \int_0^t X_s^2 ds - \left(\int_0^t X_s ds \right)^2}. \quad (3.19)$$

Letting $t \rightarrow +\infty$, and according to Lemma 4 and Theorem 1, we have

$$\begin{aligned} \hat{\alpha} - \alpha &= \frac{\varepsilon W_t \int_0^t X_s^2 ds}{t} - \frac{\varepsilon \int_0^t X_s dW_s}{t} \frac{\int_0^t X_s ds}{t} \\ &\quad \frac{\frac{1}{t} \int_0^t X_s^2 ds - \left(\frac{1}{t} \int_0^t X_s ds \right)^2}{\frac{\varepsilon W_t}{t} \frac{\alpha^2 - \frac{\alpha \varepsilon^2}{2}}{\beta^2} - \frac{\varepsilon}{t} \frac{\alpha - \frac{\varepsilon^2}{2}}{\beta}} \\ &= \lim_{t \rightarrow +\infty} \frac{\frac{\varepsilon W_t}{t} \frac{\alpha^2 - \frac{\alpha \varepsilon^2}{2}}{\beta^2} - \frac{\varepsilon}{t} \frac{\alpha - \frac{\varepsilon^2}{2}}{\beta}}{\frac{\alpha^2 - \frac{\alpha \varepsilon^2}{2}}{\beta^2} - \left(\frac{\alpha - \frac{\varepsilon^2}{2}}{\beta} \right)^2} \\ &= \frac{2\alpha}{\varepsilon} \lim_{t \rightarrow +\infty} \frac{W_t}{t} - \frac{2\beta}{\varepsilon} \lim_{t \rightarrow +\infty} \frac{\int_0^t X_s dW_s}{t}. \end{aligned}$$

It follows from (3.12) that

$$\hat{\alpha} - \alpha \rightarrow 0 \quad a.s.. \quad (3.20)$$

Substituting (1.1) into the expression of $\hat{\beta}$ yields

$$\hat{\beta} - \beta = \frac{\varepsilon W_t \int_0^t X_s ds - t \varepsilon \int_0^t X_s dW_s}{t \int_0^t X_s^2 ds - \left(\int_0^t X_s ds \right)^2}. \quad (3.21)$$

Similarly, we have

$$\hat{\beta} - \beta = \frac{\frac{\varepsilon W_t}{t} \int_0^t X_s ds - \frac{\varepsilon}{t} \int_0^t X_s dW_s}{\frac{1}{t} \int_0^t X_s^2 ds - \left(\frac{1}{t} \int_0^t X_s ds \right)^2} = \lim_{t \rightarrow +\infty} \frac{\frac{\varepsilon W_t}{t} \frac{\alpha - \frac{\varepsilon^2}{2}}{\beta} - \frac{\varepsilon}{t} \int_0^t X_s dW_s}{\frac{\alpha^2 - \frac{\alpha \varepsilon 2}{2}}{\beta^2} - \frac{\left(\alpha - \frac{\varepsilon^2}{2} \right)^2}{\beta^2}}$$

$$= \frac{2\alpha}{\varepsilon} \lim_{t \rightarrow +\infty} \frac{W_t}{t} - \frac{4\beta^2}{2\alpha\varepsilon - \varepsilon^3} \lim_{t \rightarrow +\infty} \frac{\int_0^t X_s dW_s}{t}$$

It is easy to get from (3.12) that

$$\hat{\beta} - \beta = 0 \quad a.s.. \tag{3.22}$$

Thus, $\hat{\alpha}$ and $\hat{\beta}$ are strongly consistent. The proof is complete. \square

Theorem 3 Under Assumptions 1-3, we have

$$\sqrt{\frac{\varepsilon t}{2\alpha}} (\hat{\alpha} - \alpha) \xrightarrow{L} N(0,1),$$

$$\sqrt{\frac{\varepsilon t}{2\beta}} (\hat{\beta} - \beta) \xrightarrow{L} N(0,1). \tag{3.23}$$

Proof: It follows from (3.15) that

$$\hat{\alpha} - \alpha = \frac{\frac{\varepsilon W_t}{t} \int_0^t X_s^2 ds - \frac{X_t - x_0}{t} \int_0^t X_s ds + \alpha \left(\frac{\int_0^t X_s ds}{t} \right)^2 - \beta \frac{\int_0^t X_s^2}{t} \frac{\int_0^t X_s ds}{t}}{\frac{\int_0^t X_s^2 ds}{t} - \left(\frac{\int_0^t X_s ds}{t} \right)^2}.$$

Substituting (3.9) and (3.11) into the above expression and then letting $t \rightarrow +\infty$, we have

$$\hat{\alpha} - \alpha = \lim_{t \rightarrow +\infty} \frac{\frac{\varepsilon W_t}{t} \frac{\alpha^2 - \frac{\alpha \varepsilon^2}{2}}{\beta^2} - \frac{X_t - x}{t} \frac{\alpha - \frac{\varepsilon^2}{2}}{\beta}}{\frac{2\alpha \varepsilon^2 - \varepsilon^2}{4\beta^2}}.$$

According to (3.13), one has

$$\lim_{t \rightarrow +\infty} \sqrt{t} \frac{X_t - x}{t} \frac{\alpha - \frac{\varepsilon^2}{2}}{\beta} = 0 \quad a.s..$$

Then,

$$\lim_{t \rightarrow +\infty} \sqrt{\frac{\varepsilon t}{2\alpha}} \frac{\frac{\varepsilon W_t}{t} \frac{\alpha^2 - \frac{\alpha \varepsilon^2}{2}}{\beta^2} - \frac{X_t - x}{t} \frac{\alpha - \frac{\varepsilon^2}{2}}{\beta}}{\frac{2\alpha \varepsilon^2 - \varepsilon^2}{4\beta^2}} \xrightarrow{L} N(0,1). \tag{3.24}$$

Similarly, it follows easily from (3.17) that

$$\hat{\beta} - \beta = \frac{\frac{\varepsilon W_t \int_0^t X_s ds}{t} - X_t - x_0 + \frac{\alpha \int_0^t X_s ds}{t} - \frac{\beta \int_0^t X_s^2 ds}{t}}{\frac{\int_0^t X_s^2 ds}{t} - \left(\frac{\int_0^t X_s ds}{t} \right)^2}.$$

Substituting (3.9) and (3.11) into the above expression and then letting $t \rightarrow +\infty$ yields

$$\hat{\beta} - \beta = \lim_{t \rightarrow +\infty} \frac{\frac{\varepsilon W_t}{t} \frac{\alpha - \frac{\varepsilon^2}{2} - X_t - x}{\beta}}{\frac{2\alpha\varepsilon^2 - \varepsilon^4}{4\beta^2}} \quad (3.25)$$

Therefore,

$$\lim_{t \rightarrow +\infty} \sqrt{\frac{\varepsilon t}{2\beta}} \frac{\frac{\varepsilon W_t}{t} \frac{\alpha - \frac{\varepsilon^2}{2} - X_t - x}{\beta}}{\frac{2\alpha\varepsilon^2 - \varepsilon^4}{4\beta^2}} \xrightarrow{L} N(0,1). \quad (3.26)$$

The proof is complete. \square

4. Simulation

In this section, a numerical simulation example shall be presented to demonstrate the effectiveness of the approach results.

The simulation is based on (2.6) (2.7) and (2.12). First according to (2.6) and (2.7), for given values of α, β and t such as $\alpha = 0.3, \beta = 0.6$ and $t = 500$, we can get the sample values based on the likelihood ratio estimation and MATLAB. Then, for substituting the sample values into (2.12), the values of $(\hat{\alpha}, \hat{\beta})$ can be obtained. Subsequently, we calculate the average values of the estimators. Finally, the average errors between estimators can also be calculated. Simulation results are shown in **Table 1**. In **Table 1**, the time is represented as “ t ” and likelihood ratio estimator is shown as “ LR ”. **Table 1** lists the values of “ $\hat{\alpha} - LR$ ”, “ $\hat{\beta} - LR$ ” and the average errors of “ LR ”. **Table 1** illustrates that the average errors of α, β depended on the size of given value of α, β . But under the hypothesis of normal distribution, obvious difference can not be found between estimators and true values, estimators are good. From the **Table 1** we can see clearly that the estimators become more and more close to the true value by increasing the time t . The was of continuous time estimation is better than the discrete observation [25]. (Those data comes from web of Statistical Data and I use MATLAB to simulate those data to get the result. The confidence intervals is $[0.3 - 0.01, 0.3 + 0.01]$ for $\alpha = 0.3$, $[0.6 - 0.01, 0.6 + 0.01]$ for $\beta = 0.6$; $[0.4 - 0.01, 0.4 + 0.01]$ for $\alpha = 0.4$, $[0.7 - 0.01, 0.7 + 0.01]$ for $\beta = 0.7$; $[0.5 - 0.01, 0.5 + 0.01]$ for $\alpha = 0.5$, $[0.8 - 0.01, 0.8 + 0.01]$ for $\beta = 0.8$; $[0.6 - 0.01, 0.6 + 0.01]$ for $\alpha = 0.6$, $[0.9 - 0.01, 0.9 + 0.01]$ for $\beta = 0.9$;))

Table 1. Likelihood ratio estimator simulation results of α, β .

True value (α, β)	t	Average value		Absolute value	
		$\hat{\alpha} - LR$	$\hat{\beta} - LR$	$\hat{\alpha}$	$\hat{\beta}$
(0.3, 0.6)	500	0.2989	0.5992	0.0037	0.0013
	1000	0.2992	0.5997	0.0027	0.0005
	1500	0.2998	0.5999	0.0006	0.0001
(0.4, 0.7)	500	0.3991	0.6989	0.0023	0.0016
	1000	0.3995	0.6992	0.0013	0.0011
	1500	0.3999	0.6998	0.0003	0.0003
(0.5, 0.8)	500	0.4991	0.7992	0.0018	0.0011
	1000	0.4994	0.7995	0.0012	0.0006
	1500	0.4998	0.7998	0.0004	0.0003
(0.6, 0.9)	500	0.5991	0.8991	0.0015	0.0012
	1000	0.5995	0.8994	0.0008	0.0006
	1500	0.5999	0.8998	0.0001	0.0002

5. Conclusion

In this paper, parameter estimation problem has been studied for the continuous time stochastic logistic diffusion model by using likelihood ratio. The explicit expressions for the estimation errors have been given and the according asymptotic properties have been proved by applying the he law of iterated logarithm, random time transformations, stationary distribution of solutions of stochastic differential equations and the law of large numbers for martingales. To get more accurate results, we use continuous observation method, and the proposed estimators are closer to the true value that be demonstrated by a simulation example. In the future research, we will consider the state estimation problem for non-linear systems with incomplete observation [26] and non-linear systems with random disturbance caused by Levy jump or Poisson jump [27].

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