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A Multivariate Student's *t*-Distribution

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Abstract

A multivariate Student's *t*-distribution is derived by analogy to the derivation of a multivariate normal (Gaussian) probability density function. This multivariate Student's *t*-distribution can have different shape parameters ν_i for the marginal probability density functions of the multivariate distribution. Expressions for the probability density function, for the variances, and for the covariances of the multivariate *t*-distribution with arbitrary shape parameters for the marginals are given.

Keywords

Multivariate Student's *t*, Variance, Covariance, Arbitrary Shape Parameters

1. Introduction

An expression for a multivariate Student's ***t***-distribution is presented. This expression, which is different in form than the form that is commonly used, allows the shape parameter ν for each marginal probability density function (pdf) of the multivariate pdf to be different.

The form that is typically used is [1]

$$\frac{\Gamma\left(\left(\nu+n\right)/2\right)}{\Gamma\left(\nu/2\right)\left(\pi\nu\right)^{n/2}\left|\Sigma\right|}\left(1+\left[x\right]^{\mathrm{T}}\Sigma^{-1}\left[x\right]\right)^{-(\nu+n)/2}. \tag{1}$$

This "typical" form attempts to generalize the univariate Student's ***t***-distribution and is valid when the n marginal distributions have the same shape parameter ν. The shape of this multivariate ***t***-distribution arises from the observation that the pdf for $[\mathbf{x}]=[\mathbf{y}]/\sigma$ is given by Equation (1) when $[\mathbf{y}]$ is distributed as a multivariate normal distribution with covariance matrix $[\Sigma]$ and σ^2 is distributed as chi-squared.

The multivariate Student's ***t***-distribution put forth here is derived from a Cholesky decomposition of the scale matrix by analogy to the multivariate normal (Gaussian) pdf. The derivation of the multivariate normal pdf is

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given in Section 2 to provide background. The multivariate Student's ***t***-distribution and the variances and covariances for the multivariate ***t***-distribution are given in Section 3. Section 4 is a conclusion.

2. Background Information

2.1. Cholesky Decomposition

A method to produce a multivariate pdf with known scale matrix $[\Sigma_s]$ is presented in this section. For normally distributed variables, the covariance matrix $[\Sigma]=[\Sigma_s]$ since the scale factor for a normal distribution is the standard deviation of the distribution. An example with $n=4$ is used to provide concrete examples.

Consider the transformation $[\boldsymbol{y}]=[M][\boldsymbol{x}]$ where $[\boldsymbol{y}]$ and $[\boldsymbol{x}]$ are 4×1 column matrices, $[M]$ is 4×4 square matrix, and the elements of $[\boldsymbol{x}]$ are independent random variables. The off-diagonal elements of $[M]$ introduce correlations between the elements of $[\boldsymbol{y}]$.

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \boldsymbol{y}_3 \\ \boldsymbol{y}_4 \end{bmatrix} = \begin{bmatrix} m_{1,1} & 0 & 0 & 0 \\ m_{2,1} & m_{2,2} & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & 0 \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \end{bmatrix} \tag{2}$$

The scale matrix $[\Sigma_s]=[M][M]^{\mathrm{T}}$. The covariance matrix $[\Sigma]$ has elements $\Sigma_{i,j}=\mathrm{E}\{\boldsymbol{y_i}\,\boldsymbol{y_j}\}$ where $\mathrm{E}\{\boldsymbol{y_i}\,\boldsymbol{y_j}\}$ is the expectation of $\boldsymbol{y_i}\,\boldsymbol{y_j}$ and $\Sigma_{i,j}=\Sigma_{j,i}$. If the $[\boldsymbol{x}]$ are normally distributed, then $[\Sigma]=[\Sigma_s]=[M][M]^{\mathrm{T}}$, where the superscript T indicates a transpose of the matrix. If $[\Sigma_s]$ is known, then $[M]$ is the Cholesky decomposition of the matrix $[\Sigma_s]$ [2].

For the 4×4 example of Equation (2),

$$[\Sigma_s]=\begin{bmatrix} m_{1,1}^2 & m_{1,1}m_{2,1} & m_{1,1}m_{3,1} & m_{1,1}m_{4,1} \\ m_{1,1}m_{2,1} & m_{2,1}^2+m_{2,2}^2 & m_{2,1}m_{3,1}+m_{2,2}m_{3,2} & m_{2,1}m_{4,1}+m_{2,2}m_{4,2} \\ m_{1,1}m_{3,1} & m_{2,1}m_{3,1}+m_{2,2}m_{3,2} & m_{3,1}^2+m_{3,2}^2+m_{3,3}^2 & m_{3,1}m_{4,1}+m_{3,2}m_{4,2}+m_{3,3}m_{4,3} \\ m_{1,1}m_{4,1} & m_{2,1}m_{4,1}+m_{2,2}m_{4,2} & m_{3,1}m_{4,1}+m_{3,2}m_{4,2}+m_{3,3}m_{4,3} & m_{4,1}^2+m_{4,2}^2+m_{4,3}^2+m_{4,4}^2 \end{bmatrix}. \tag{3}$$

From linear algebra, $\det\left([M]\left[M^{\mathrm{T}}\right]\right)=\det[M]\det[M]^{\mathrm{T}}=\left(\det[M]\right)^2$. For $[M]$ as defined in Equation (2), $\det[M]=m_{1,1}m_{2,2}m_{3,3}m_{4,4}$ and $\det[\Sigma_s]=m_{1,1}^2m_{2,2}^2m_{3,3}^2m_{4,4}^2=\left(\det[M]\right)^2$ whereas $\Sigma_{i,i}=\mathrm{E}\{\boldsymbol{y_i}\boldsymbol{y_i}\}=\sigma_{y_i}^2$ is the variance of the zero-mean random variable $\boldsymbol{y_i}$ and $\Sigma_{i,j}=\mathrm{E}\{\boldsymbol{y_i}\boldsymbol{y_j}\}=\sigma_{y_iy_j}^2$ is the covariance of the zero-mean random variables $\boldsymbol{y_i}$ and $\boldsymbol{y_j}$.

2.2. Multivariate Normal Probability Density Function

To create a multivariate normal pdf, start with the joint pdf $f_N([x])$ for *n* unit normal, zero mean, independent random variables $[\boldsymbol{x}]$:

$$f_N([x])=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_i^2\right)=\frac{1}{\sqrt{(2\pi)^n}}\exp\left(-\frac{1}{2}[x]^{\mathrm{T}}[x]\right) \tag{4}$$

where $[x]$ is an *n*-row column matrix: $[x]^{\mathrm{T}}=[x_1,x_2,\cdots,x_n]$. $f_N([x])\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$ gives the probability that the random variables $[\boldsymbol{x}]$ lie in the interval $[x]<[\boldsymbol{x}]\le[x]+[\mathrm{d}x]$.

The requirement for zero mean random variables is not a restriction. If $\mathrm{E}\{\boldsymbol{x_i}\}=\mu_i$, then $\boldsymbol{x}_i'=\boldsymbol{x_i}-\mu_i$ is a zero mean random variable with the same shape and scale parameters as $\boldsymbol{x_i}$.

Use Equation (2) to transform the variables. The Jacobian determinant of the transformation relates the products of the infinitesimals of integration such that

$$\mathrm{d}x_i\,\mathrm{d}x_2\cdots\mathrm{d}x_n=\left|\frac{\partial(x_1,x_2,\cdots,x_n)}{\partial(y_1,y_2,\cdots,y_n)}\right|\mathrm{d}y_1\,\mathrm{d}y_2\cdots\mathrm{d}y_n. \tag{5}$$

The magnitude of the Jacobian determinant of the transformation $[y]=[M][x]$ is (**Appendix**)

$$\left|\frac{\partial(x_1,x_2,\cdots,x_n)}{\partial(y_1,y_2,\cdots,y_n)}\right| = \left|\frac{1}{\det[M]}\right| = \left|\frac{1}{\sqrt{\det[\Sigma_s]}}\right| \tag{6}$$

where the equality $\det[\Sigma_s]=(\det[M])^2$ has been used.

Since $[\Sigma_s]=[M][M]^{\mathrm{T}}$, $[\Sigma_s]^{-1}=([M]^{\mathrm{T}})^{-1}[M]^{-1}$, and since $[x]=[M]^{-1}[y]$, the multivariate "z-score" $[x]^{\mathrm{T}}[x]$ becomes $[y]^{\mathrm{T}}([M]^{\mathrm{T}})^{-1}[M]^{-1}[y]=[y]^{\mathrm{T}}[\Sigma_s]^{-1}[y]$, which equals $[y]^{\mathrm{T}}[\Sigma]^{-1}[y]$ since $[\Sigma_s]=[\Sigma]$ for normally distributed variables.

The result is that the unit normal, independent, multivariate pdf, Equation (4), becomes under the transformation Equation (2)

$$f_N([y]) = \frac{1}{\sqrt{(2\pi)^n|\det[\Sigma_s]|}}\exp\left(-\frac{1}{2}[y]^{\mathrm{T}}[\Sigma_s]^{-1}[y]\right) \tag{7}$$

where $[y]$ is a n-row column matrix: $[y]^{\mathrm{T}}=[y_1,y_2,\cdots,y_n]$ and $[\Sigma_s]=[\Sigma]$.

For the 4×4 example,

$$[M]^{-1} = \begin{bmatrix} \frac{1}{m_{1,1}} & 0 & 0 & 0 \\ -\frac{m_{2,1}}{m_{1,1}m_{2,2}} & \frac{1}{m_{2,2}} & 0 & 0 \\ \frac{m_{3,2}m_{2,1}-m_{3,1}m_{2,2}}{m_{1,1}m_{2,2}m_{3,3}} & -\frac{m_{3,2}}{m_{2,2}m_{3,3}} & \frac{1}{m_{3,3}} & 0 \\ m_{1,4}^{-1} & \frac{m_{4,3}m_{3,2}-m_{4,2}m_{3,3}}{m_{2,2}m_{3,3}m_{4,4}} & -\frac{m_{4,3}}{m_{3,3}m_{4,4}} & \frac{1}{m_{4,4}} \end{bmatrix}, \tag{8}$$

from which $[\Sigma_s]^{-1}=([M]^{\mathrm{T}})^{-1}[M]^{-1}$ can be calculated. In Equation (8),

$$m_{1,4}^{-1} = -\frac{m_{2,1}m_{3,2}m_{4,3}-m_{4,2}m_{2,1}m_{3,3}-m_{2,2}m_{3,1}m_{4,3}+m_{4,1}m_{2,2}m_{3,3}}{m_{1,1}m_{2,2}m_{3,3}m_{4,4}}. \tag{9}$$

The denominator in the expression for $m_{1,4}^{-1}$ is $\det[M]$.

3. Multivariate Student's *t* Probability Density Function

A similar approach can be used to create a multivariate Student's ***t*** pdf. Assume truncated or effectively truncated ***t***-distributions, so that moments exist [3] [4]. For simplicity, assume that support is $[\mu-b\beta,\mu+b\beta]$ where b is a positive, large number, β is the scale factor for the distribution, and μ is the location parameter for the distribution. If b is a large number, then a significant portion of the tails of the distribution are included. If $b=\infty$ then all of the tails are included.

Start with the joint pdf for n independent, zero-mean (location parameters $[\mu]=0$) Student's ***t*** pdfs with shape parameters $[\nu]$, and scale parameters $[\beta]=1$:

$$f_t([x];[\nu]) = \prod_{i=1}^{n}\frac{\Gamma((\nu_i+1)/2)}{\Gamma(\nu_i/2)\sqrt{\pi\nu_i}}\left(1+\frac{x_i^2}{\nu_i}\right)^{-(\nu_i+1)/2} = \prod_{i=1}^{n} g_i(x_i;\nu_i) \tag{10}$$

with $-\infty\le x_i\le+\infty$. $f_t([x];[\nu])\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n$ gives the probability that a random draw of the column matrix $[\boldsymbol{x}]$ from the joint Student's ***t***-distribution lies in the interval $[x]<[\boldsymbol{x}]\le[x]+[\mathrm{d}x]$. The pdf $g_i(x_i;\nu_i)$ is a function of only x_i and the shape parameter ν_i, and thus is independent of any other $g_j(x_j;\nu_j)$, $j\ne i$.

Use the transformation of Equation (2) to create a multivariate pdf

$$f_t([y];[\nu]) = \frac{1}{|\det[M]|}\prod_{i=1}^{n}\frac{\Gamma((\nu_i+1)/2)}{\Gamma(\nu_i/2)\sqrt{\pi\nu_i}}\left(1+\frac{\left(\sum_{j=1}^{i} m_{i,j}^{-1}y_j\right)^2}{\nu_i}\right)^{-(\nu_i+1)/2}. \tag{11}$$

The solution $x_i = \sum_{j=1}^{i} m_{i,j}^{-1} y_j$ of the transformation Equation (2) was used. The elements of the inverse matrix $[M]^{-1}$, $m_{i,j}^{-1}$, are given in terms of the $m_{i,j}$ by Equation (8) for the $n=4$ example. Note that the shape parameters ν_i of the constituent distributions need not be the same in the multivariate $\boldsymbol{t}$-distribution given by $f_t([y];[\nu])$.

$f_t([y];[\nu])\mathrm{d}y_1\mathrm{d}y_2\cdots\mathrm{d}y_n$ gives the probability that a random draw of the column matrix $[\boldsymbol{y}]$ from the multivariate Student's $\boldsymbol{t}$-distribution with shape parameters $[\nu]$ lies in the interval $[y]<[\boldsymbol{y}]\le[y]+[\mathrm{d}y]$.

From the definition of the exponential function $\mathrm{e}^x = \lim_{n\to\infty}(1+x/n)^n$ where $\mathrm{e}=2.718281828\cdots$ is Euler's number, then

$$\lim_{\nu\to\infty}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} = \exp\left(-t^2/2\right) \tag{12}$$

and

$$\begin{aligned}
\lim_{\nu\to\infty} f_t([y];[\nu]) &= \frac{1}{|\det[M]|}\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}}\exp\left(-0.5\left(\sum_{j=1}^{i} m_{i,j}^{-1}y_j\right)^2\right)\\
&= \frac{1}{|\det[M]|\sqrt{(2\pi)^n}}\exp\left(-0.5\left(\sum_{i=1}^{n}\sum_{j=1}^{i} m_{i,j}^{-1}y_j\right)^2\right)\\
&= \frac{1}{\sqrt{(2\pi)^n|\det[\Sigma_s]|}}\exp\left(-\frac{1}{2}[y]^{\mathrm{T}}[\Sigma_s]^{-1}[y]\right)\\
&= f_N([y]).
\end{aligned} \tag{13}$$

In the limit as $[\nu\to\infty]$, the multivariate Student's $\boldsymbol{t}$-distribution $f_t([y];[\nu])$, Equation (11), becomes a multivariate normal distribution.

3.1. Some $\Sigma_{i,j}$ for the $n=4$ Example

In this subsection some examples for the variances and covariances of a multivariate Student's $\boldsymbol{t}$-distribution using the $n=4$ example of Equation (2) are given.

The variance of the random variable $\boldsymbol{y}_3$ is

$$E\left\{\boldsymbol{y}_3^2\right\} = \iiiint y_3^2 f_t\left([M]^{-1}[y];[\nu]\right)\mathrm{d}y_4\mathrm{d}y_3\mathrm{d}y_2\mathrm{d}y_1 \tag{14}$$

with the limits of the integrations equal to $\mu_i - b\beta_i$ and $\mu_i + b\beta_i$, $i=1,2,3,4$.

Perform the integrations as listed. The integral over $\mathrm{d}y_4$ is unity since only x_4 depends on y_4 (c.f. Equation (2)) and $f_t([x];[\nu])$ factors into a product $g_1(x_1)g_2(x_2)g_3(x_3)g_4(x_4)$—see Equation (10). Write

$$x_4 = m_{4,4}^{-1}y_4 + m_{4,3}^{-1}y_3 + m_{4,2}^{-1}y_2 + m_{4,1}^{-1}y_1 \tag{15}$$

$$= m_{4,4}^{-1}y_4 - \mu_4 \tag{16}$$

where the $m_{i,j}^{-1}$ are the elements of the inverse of matrix $[M]$ and are as given by $[M]^{-1}$, Equation (8), and μ_4 is a constant as far as the integral over y_4 is concerned.

Repeat the procedure for the integrals for $\mathrm{d}y_3$, $\mathrm{d}y_2$, and $\mathrm{d}y_1$. These integrals are not equal to unity owing to the presence of the $\boldsymbol{y}_3^2$ term.

The variance of the random variable $\boldsymbol{y}_i$ for the multivariate Student's ***t***-distribution with support $[-\infty,\infty]$ and with $\nu_i = \nu$ for all *i* is given by

$$E\{\boldsymbol{y}_i^2\} = \sigma_{y_i}^2 = \sum_{j=1}^{i} m_{j,j}^2 \frac{\nu}{\nu-2} \tag{17}$$

The expression for $\sigma_{y_i}^2$ is valid only for $\nu > 2$. The expression would be valid for $\nu \ge 1$ if the region of support was $[\mu_i - b\beta_i, \mu_i + b\beta_i]$ rather than $[-\infty,\infty]$ where β_i is a scale factor and $b < \infty$ [3]-[5]. Note that the scale factors for the multivariate ***t***-distribution are $\beta_i = |m_{i,i}|$.

Truncation or effective truncation of the pdf keeps the moments finite [3]-[5]. For example, the second central moment for a $\nu = 1$ Student's ***t***-distribution with scale factor β and support $[\mu - b\beta, \mu + b\beta]$ is

$$\mu_2(\nu = 1, \beta, b) = \beta^2 \times \frac{(b - \arctan(b))}{\arctan(b)}, \tag{18}$$

which is finite provided that $b < \infty$.

In the interest of brevity, only variances and covariances that were calculated for support of $[-\infty,\infty]$ will be discussed. The requirement that $\nu_i > 2$ will be understood to be waived if the pdf is truncated or effectively truncated. It is also to be understood that the variances and covariances as calculated for support of $[-\infty,\infty]$ provide upper limits for variances and covariances calculated for truncation or effective truncation of the pdf.

If the ν_i are not equal, then for the $n = 4$ example of Equation (2)

$$E\{\boldsymbol{y}_3^2\} = \frac{\nu_1}{\nu_1 - 2} \times m_{3,1}^2 + \frac{\nu_2}{\nu_2 - 2} \times m_{3,2}^2 + \frac{\nu_3}{\nu_3 - 2} \times m_{3,3}^2. \tag{19}$$

The covariance $\mathrm{E}\{\boldsymbol{y}_2 \boldsymbol{y}_3\}$ for the $\nu_i = \nu$ for all *i* is given by

$$\mathrm{E}\{\boldsymbol{y}_2 \boldsymbol{y}_3\} = (m_{2,1} m_{3,1} + m_{2,2} m_{3,2}) \frac{\nu}{\nu - 2}. \tag{20}$$

If the ν_i are not equal, then the covariance $\mathrm{E}\{\boldsymbol{y}_2 \boldsymbol{y}_3\}$

$$\mathrm{E}\{\boldsymbol{y}_2 \boldsymbol{y}_3\} = \frac{\nu_1}{\nu_1 - 2} \times m_{2,1} m_{3,1} + \frac{\nu_2}{\nu_2 - 2} \times m_{2,2} m_{3,2}. \tag{21}$$

The expression for $\mathrm{E}\{\boldsymbol{y}_1 \boldsymbol{y}_3\}$, which is valid for the ν_i not equal, is

$$\mathrm{E}\{\boldsymbol{y}_1 \boldsymbol{y}_3\} = \frac{\nu_1}{\nu_1 - 2} \times m_{1,1} m_{3,1}. \tag{22}$$

The expressions for $\mathrm{E}\{\boldsymbol{y}_3 \boldsymbol{y}_3\}$, $\mathrm{E}\{\boldsymbol{y}_2 \boldsymbol{y}_3\}$, and $\mathrm{E}\{\boldsymbol{y}_1 \boldsymbol{y}_3\}$ show a simple pattern for the relationship between the covariance matrix Σ, the scale matrix $[\Sigma_s]$ Equation (3), and the matrix $[M]$ Equation (2).

3.2. General Expressions for $\Sigma_{i,j}$

Given a matrix $[M]$ that is an $n \times n$ square matrix with elements $m_{i,j}$, an expression for the variance (assuming support $[-\infty,\infty]$, $\nu_i > 2$ for all *i*, and $i \le n$) for the multivariate Student's ***t***-distribution $f_t([y];[\nu])$ is

$$\mathrm{E}\{\boldsymbol{y}_i^2\} = \sum_{j=1}^{i} \frac{\nu_j}{\nu_j - 2} \times m_{j,j}^2. \tag{23}$$

A general expression for the covariance (assuming support $[-\infty,\infty]$, $\nu_i > 2$ for all *i*, and $i \le n, j \le n$) for the multivariate Student's ***t***-distribution $f_t([y];[\nu])$ is

$$\mathrm{E}\{\boldsymbol{y}_i \boldsymbol{y}_j\} = \sum_{k=1}^{\min(i,j)} \frac{\nu_k}{\nu_k - 2} \times m_{i,k} m_{j,k}. \tag{24}$$

If support is $[\mu - b\beta, \mu + b\beta]$, then the general expressions need to be multiplied by functions that depend on *b* and ν. Truncation or effective truncation keeps the moments finite and defined for all $\nu \ge 1$ [3]-[5]. The general expressions for the covariance, Equation (24), yields, when $i = j$, the general expression for the variance, Equation (23). The general expression for the variance, Equation (23), is given to emphasize the $m_{j,j}^2$

nature of the variance.

Unlike normally distributed random variables, the correlation matrix $[\Sigma]$ for random variables that are distributed as Student's ***t*** is not equal to $[M][M]^{\mathrm{T}}$. For normally distributed variables, the scale parameter β equals the standard deviation σ. For Student's ***t*** distributed variables, the standard deviation σ does not equal the scale parameter β. For a Student's ***t*** distribution with shape parameter ν, scale parameter β, and support $[-\infty,\infty]$, $\sigma^2 = \beta^2 \times \nu/(\nu-2)$. If the region of support for the Student's ***t*** distribution is truncated to $[\mu - b\beta, \mu + b\beta]$ then the variance $\sigma^2 < \beta^2 \times \nu/(\nu-2)$ for all $\nu \geq 2$ and is finite for all $\nu \geq 1$ [3]-[5].

Given a matrix of the variances and the covariances, $[\Sigma]$, and a column matrix of the shape parameters $[\nu]$ associated with each variable, the scale matrix $[\Sigma_s] = [M][M]^{\mathrm{T}}$ would in principle be determined sequentially, starting with $m_{1,1}$ and $m_{1,2}$. The shape parameters $[\nu]$ would be obtained from the marginal distributions or from other knowledge.

4. Conclusion

A multivariate Student's ***t***-distribution is derived by analogy to the derivation for a multivariate normal (or Gaussian) pdf. The variances and covariances for the multivariate ***t***-distribution are given. It is noteworthy that the shape parameters $[\nu]$ of the constituent Student's ***t***-distributions of the *multivariate **t**-distribution*, Equation (11), need not be the same.

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References

[1] Kotz, S. and Nadarajah, S. (2004) Multivariate t-Distributions and Their Applications. Cambridge University Press, Cambridge.

[2] Press, W.H., Teukolsky, S.A., Vetterling, W.T. and Flannery, B.P. (1992) Numerical Recipes in FORTRAN: The Art of Scientific Computing. 2nd Edition, Cambridge University Press, Cambridge, 89.

[3] Cassidy, D.T. (2011) Describing *n*-Day Returns with Student's ***t***-Distributions. *Physica A*, **390**, 2794-2802. http://dx.doi.org/10.1016/j.physa.2011.03.019

[4] Cassidy, D.T. (2012) Effective Truncation of a Student's ***t***-Distribution by Truncation of the Chi Distribution in a Mixing Integral. *Open Journal of Statistics*, **2**, 519-525. http://dx.doi.org/10.4236/ojs.2012.25067

[5] Cassidy, D.T. (2016) Student's ***t*** Increments. *Open Journal of Statistics*, **6**, 156-171. http://dx.doi.org/10.4236/ojs.2016.61014

Appendix: The Jacobian

The Jacobian determinant is used in physics, mathematics, and statistics. Many of these uses can be traced to the Jacobian determinate as a measure of the volume of an infinitesimially small, n-dimensional parallelepiped.

1. Volume of a Parallelepiped

The volume of an n-dimensional parallelepiped is given by the absolute value of the determinant of the components of the edge vectors that form the parallelepiped.

The area of a parallelogram with edge vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $|\boldsymbol{a}\times\boldsymbol{b}|$.

The volume of a parallelepiped with edge vectors $\boldsymbol{a}=(a_1,a_2,a_3)$, $\boldsymbol{b}=(b_1,b_2,b_3)$, and $\boldsymbol{c}=(c_1,c_2,c_3)$ is given by the determinant

$$\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})=\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{25}$$

2. Inversion Exists

Assume that there are n functions $x_i = f_i(q_1,q_2,\cdots,q_n)$. The necessary and sufficient condition that the functions can be inverted to find $q_i = f_i^{-1}(x_1,x_2,\cdots,x_n)$ is that the Jacobian determinant is nonzero, *i.e.*,

$$\frac{\partial(x_1,x_2,\cdots,x_n)}{\partial(q_1,q_2,\cdots,q_n)}\neq 0 \tag{26}$$

where

$$\frac{\partial(x_1,x_2,\cdots,x_n)}{\partial(q_1,q_2,\cdots,q_n)}\equiv\begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & . & . & . & \frac{\partial x_1}{\partial q_n} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & . & . & . & \frac{\partial x_2}{\partial q_n} \\ . & . & . & . & . & . \end{vmatrix}. \tag{27}$$

To simplify the notation, assume that $n=3$ so that $x_i = f_i(q_1,q_2,q_3)$, $i=1,\cdots,3$. The total differential is

$$\mathrm{d}x_i=\frac{\partial f_i}{\partial q_1}\mathrm{d}q_1+\frac{\partial f_i}{\partial q_2}\mathrm{d}q_2+\frac{\partial f_i}{\partial q_3}\mathrm{d}q_3. \tag{28}$$

These equations can be put in matrix form

$$\begin{bmatrix}\mathrm{d}x_1\\ \mathrm{d}x_2\\ \mathrm{d}x_3\end{bmatrix}=\begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \frac{\partial f_1}{\partial q_3} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \frac{\partial f_2}{\partial q_3} \\ \frac{\partial f_3}{\partial q_1} & \frac{\partial f_3}{\partial q_2} & \frac{\partial f_3}{\partial q_3} \end{bmatrix}\begin{bmatrix}\mathrm{d}q_1\\ \mathrm{d}q_2\\ \mathrm{d}q_3\end{bmatrix}. \tag{29}$$

These three equations can be solved for the $\mathrm{d}q_i$ if the determinant of the 3×3 matrix is non-zero. This is a standard result from linear algebra. The determinant of the 3×3 matrix is called the Jacobian determinant of the transformation.

3. Change of Variables

The Jacobian determinant of the transformation is used in change of variables in integration:

$$\iiint \mathrm{d}V = \iiint \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = \iiint \left|\frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)}\right| \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3. \tag{30}$$

The absolute value sign is required since the determinant could be negative (*i.e.*, the volume could decrease).

The Jacobian determinant for the inverse transformation (to obtain $[x]$ as functions of $[y]$) given by Equation (8) is

$$\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & 0 & 0 & 0 \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & 0 & 0 \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} & 0 \\ \frac{\partial x_4}{\partial y_1} & \frac{\partial x_4}{\partial y_2} & \frac{\partial x_4}{\partial y_3} & \frac{\partial x_4}{\partial y_4} \end{vmatrix} = \begin{vmatrix} \frac{1}{m_{1,1}} & 0 & 0 & 0 \\ -\frac{m_{2,1}}{m_{1,1}m_{2,2}} & \frac{1}{m_{2,2}} & 0 & 0 \\ \frac{m_{3,2}m_{2,1} - m_{3,1}m_{2,2}}{m_{1,1}m_{2,2}m_{3,3}} & -\frac{m_{3,2}}{m_{2,2}m_{3,3}} & \frac{1}{m_{3,3}} & 0 \\ m_{1,4}^{-1} & \frac{m_{4,3}m_{3,2} - m_{4,2}m_{3,3}}{m_{2,2}m_{3,3}m_{4,4}} & -\frac{m_{4,3}}{m_{3,3}m_{4,4}} & \frac{1}{m_{4,4}} \end{vmatrix}, \tag{31}$$

which equals

$$\frac{1}{m_{1,1}} \times \frac{1}{m_{2,2}} \times \frac{1}{m_{3,3}} \times \frac{1}{m_{4,4}} = \frac{1}{\det[M]}. \tag{32}$$