Optimal Generalized Biased Estimator in Linear Regression Model

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Abstract

The paper introduces a new biased estimator namely Generalized Optimal Estimator (GOE) in a multiple linear regression when there exists multicollinearity among predictor variables. Stochastic properties of proposed estimator were derived, and the proposed estimator was compared with other existing biased estimators based on sample information in the the Scalar Mean Square Error (SMSE) criterion by using a Monte Carlo simulation study and two numerical illustrations.

Keywords

Multicollinearity, Biased Estimator, Generalized Optimal Estimator, Scalar Mean Square Error

1. Introduction

To overcome the multicollinearity problem in the linear regression model, several biased estimators were defined in the place of Ordinary Least Squares Estimator (OLSE) to estimate the regression coefficients, and the properties of these were discussed in the literature. Some of these estimators are based on only sample information such as Ridge Estimator (RE) [1], Almost Unbiased Ridge Estimator (AURE) [2], Liu Estimator (LE) [3] and Almost Unbiased Liu Estimator (AULE) [4]. However for each case, the researcher has to obtain their properties and to compare the superiority of one estimator over another estimator when selecting a suitable estimator for a practical situation. Therefore [5] proposed a Generalized Unrestricted Estimator (GURE) $\hat{\beta}_{GURE} = A\hat{\beta}$ to represent the RE, AURE, LE and AULE, where $\hat{\beta}$ is the OLSE, and A is a positive definite matrix which depends on the corresponding estimators of RE, AURE, LE and AULE.

However, the researchers are still trying to find the best estimator by changing the matrix A compared to the already proposed estimators based on sample information. Instead of changing A, in this research we introduce a more efficient new biased estimator based on optimal choice of A.

The rest of the paper is organized as follows. The model specification and estimation is given in Section 2. In Section 3, we propose a biased estimator namely Generalized Optimal Estimator (GOE), and we obtain its stochastic properties. In Section 4 we compare the proposed estimator with some biased estimators in the Scalar Mean Square Error criterion by using a real data set and a Monte Carlo simulation. Finally some conclusion remarks are given in Section 5.

2. Model Specification and Estimation

First we consider the multiple linear regression model

$$y = X\beta + \varepsilon, \ \varepsilon \sim N\left(0, \sigma^2 I\right) \tag{1}$$

where y is an $n \times 1$ observable random vector, X is an $n \times p$ known design matrix of rank p, β is a $p \times 1$ vector of unknown parameters and ε is an $n \times 1$ vector of disturbances.

The Ordinary Least Square Estimator of β and σ^2 are given by

$$\hat{\beta} = S^{-1} X' y \tag{2}$$

and

$$\hat{\sigma}^2 = \frac{\left(y - X\hat{\beta}\right)' \left(y - X\hat{\beta}\right)}{n - p} \tag{3}$$

respectively, where $S = X'X$.

The Ridge Estimator (RE), Almost Unbiased Ridge Estimator (AURE), Liu Estimator (LE) and Almost Unbiased Liu Estimator (AULE) are some of the biased estimators proposed to solve the multicollinearity problem which are based only on sample information. The estimators are given below:

RE: $\hat{\beta}_{RE}(k) = W\hat{\beta}$ where $W = \left(I + kS^{-1}\right)^{-1}$ for $k \geq 0$

LE: $\hat{\beta}_{LE}(d) = F_d\hat{\beta}$ where $F_d = \left(S + I\right)^{-1}\left(S + dI\right)$ for $0 \leq d \leq 1$

AURE: $\hat{\beta}_{AURE}(k) = A_k\hat{\beta}$ where $A_k = \left[I - k^2\left(S + kI\right)^{-2}\right]$ for $k \geq 0$

AULE: $\hat{\beta}_{AULE}(d) = T_d\hat{\beta}$ where $T_d = \left[I - (1-d)^2\left(S + I\right)^{-2}\right]$ for $0 \leq d \leq 1$

Since RE, LE, AURE and AULE are based on OLSE, [5] proposed a generalized form to represent these four estimators, the Generalized Unrestricted Estimator (GURE) which is given as

$$\hat{\beta}_{GURE} = A_{(i)}\hat{\beta} \tag{4}$$

where $A_{(i)}$ is a positive definite matrix and $A_{(i)}$ stands for W, F_d, A_k and T_d.

The bias vector, dispersion matrix and MSE matrix of $\hat{\beta}_{GURE}$ are given as

$$B\left(\hat{\beta}_{GURE}\right) = \left(A_{(i)} - I\right)\beta \tag{5}$$

$$D\left(\hat{\beta}_{GURE}\right) = \sigma^2 A_{(i)} S^{-1} A'_{(i)} \tag{6}$$

and

$$MSE\left(\hat{\beta}_{GURE}\right) = \sigma^2 A_{(i)} S^{-1} A'_{(i)} + A_{(i)}\left(I - A_{(i)}^{-1}\right)\beta\beta'\left(I - A_{(i)}^{-1}\right)' A'_{(i)} \tag{7}$$

respectively.

Instead of changing the matrix $A_{(i)}$ to introduce a new biased estimator, in this research we obtain the optimal choice of $A_{(i)}$ by minimizing the Mean Square Error Matrix (MSEM) of GURE with respect to $A_{(i)}$.

3. The Proposed Estimator

From (7) the following equation can be obtained by taking trace operator as

$$tr\left[MSE\left(\hat{\beta}_{GURE}\right)\right] = \sigma^2 tr\left(A_{(i)}S^{-1}A'_{(i)}\right) + \beta'\left(I - A_{(i)}^{-1}\right)' A'_{(i)}A_{(i)}\left(I - A_{(i)}^{-1}\right)\beta \tag{8}$$

By minimizing (8) with respect to $A_{(i)}$, the optimum $A_{(i)}$ can be obtained.

$$\frac{\partial\left\{tr\left[MSE\left(\hat{\beta}_{GURE}\right)\right]\right\}}{\partial A_{(i)}} = \sigma^2 \frac{\partial\left[tr\left(A_{(i)}S^{-1}A'_{(i)}\right)\right]}{\partial A_{(i)}} + \frac{\partial\beta'B\beta}{\partial A_{(i)}} \tag{9}$$

where $B = \left(I - A_{(i)}^{-1}\right)' A'_{(i)}A_{(i)}\left(I - A_{(i)}^{-1}\right)$

Now we can simplify the matrix B as

$$B = \left(I - A_{(i)}^{-1}\right)' A'_{(i)}A_{(i)}\left(I - A_{(i)}^{-1}\right)$$
$$= A'_{(i)}A_{(i)} - A'_{(i)} - A_{(i)} + I$$

Therefore

$$\beta'B\beta = \beta'A'_{(i)}A_{(i)}\beta - \beta'A'_{(i)}\beta - \beta'A_{(i)}\beta + \beta'\beta$$
$$= \beta'A'_{(i)}A_{(i)}\beta - 2\beta'A_{(i)}\beta + \beta'\beta \tag{10}$$

Now we will use the following three results (see [6], p. 521, 522) to obtain the $\dfrac{\partial\left[tr\left(A_{(i)}S^{-1}A'_{(i)}\right)\right]}{\partial A_{(i)}}$ and $\dfrac{\partial\beta'B\beta}{\partial A_{(i)}}$.

(a) Let M and X be any two matrixes with proper order. Then

$$\frac{\partial tr\left(XMX'\right)}{\partial X} = X\left(M + M'\right)$$

(b) If x is an n-vector, y is an m-vector, and C an $n \times m$ matrix, then

$$\frac{\partial}{\partial C}x'Cy = xy'$$

(c) Let x be a K vector, M a symmetric $T \times T$ matrix, and C a $T \times K$ matrix. Then

$$\frac{\partial x'C'MCx}{\partial C} = 2MCxx'$$

By applying (a) we can obtain

$$\frac{\partial\left[tr\left(A_{(i)}S^{-1}A'_{(i)}\right)\right]}{\partial A_{(i)}} = A_{(i)}\left(S^{-1} + S^{-1}\right) = 2A_{(i)}S^{-1} \tag{11}$$

Now we consider

$$\frac{\partial\beta'B\beta}{\partial A_{(i)}} = \frac{\partial\beta'A'_{(i)}A_{(i)}\beta}{\partial A_{(i)}} - 2\left(\frac{\partial\beta'A_{(i)}\beta}{\partial A_{(i)}}\right) \tag{12}$$

By using (b) and (c) we can obtain

$$\frac{\partial \boldsymbol{\beta}' \boldsymbol{A}_{(i)} \boldsymbol{\beta}}{\partial \boldsymbol{A}_{(i)}} = \boldsymbol{\beta} \boldsymbol{\beta}' \tag{13}$$

and

$$\frac{\partial \boldsymbol{\beta}' \boldsymbol{A}'_{(i)} \boldsymbol{A}_{(i)} \boldsymbol{\beta}}{\partial \boldsymbol{A}_{(i)}} = 2 \boldsymbol{A}_{(i)} \boldsymbol{\beta} \boldsymbol{\beta}' \tag{14}$$

respectively.

Hence

$$\frac{\partial \boldsymbol{\beta}' \boldsymbol{B} \boldsymbol{\beta}}{\partial \boldsymbol{A}_{(i)}} = 2 \boldsymbol{A}_{(i)} \boldsymbol{\beta} \boldsymbol{\beta}' - 2 \boldsymbol{\beta} \boldsymbol{\beta}' \tag{15}$$

Substituting (11) and (15) to (9), we can derive that

$$\frac{\partial \left\{ tr \left[MSE \left(\hat{\boldsymbol{\beta}}_{GURE} \right) \right] \right\}}{\partial \boldsymbol{A}_{(i)}} = 2\sigma^2 \boldsymbol{A}_{(i)} \boldsymbol{S}^{-1} + 2 \boldsymbol{A}_{(i)} \boldsymbol{\beta} \boldsymbol{\beta}' - 2 \boldsymbol{\beta} \boldsymbol{\beta}' \tag{16}$$

Equating (16) to a null matrix, we can obtain the optimal matrix $\boldsymbol{A}_{(i)}$ as

$$\tilde{\boldsymbol{A}}_{(i)} = \boldsymbol{\beta} \boldsymbol{\beta}' \left(\sigma^2 \boldsymbol{S}^{-1} + \boldsymbol{\beta} \boldsymbol{\beta}' \right)^{-1} \tag{17}$$

Note that $\left(\sigma^2 \boldsymbol{S}^{-1} + \boldsymbol{\beta} \boldsymbol{\beta}' \right)^{-1}$ exists since $1 + \sigma^{-2} \boldsymbol{\beta}' \boldsymbol{S} \boldsymbol{\beta} \neq 0$ (see [6], p. 494).

Now we are ready to propose a biased estimator namely **Generalized Optimal Estimator (GOE)** as

$$\tilde{\boldsymbol{\beta}}_{GOE} = \tilde{\boldsymbol{A}}_{(i)} \hat{\boldsymbol{\beta}} \tag{18}$$

The bias vector, dispersion matrix, mean square error matrix and scalar mean square error of $\tilde{\boldsymbol{\beta}}_{GOE}$ can be obtained as

$$\boldsymbol{B} \left(\tilde{\boldsymbol{\beta}}_{GOE} \right) = \left(\tilde{\boldsymbol{A}}_{(i)} - \boldsymbol{I} \right) \boldsymbol{\beta} \tag{19}$$

$$\boldsymbol{D} \left(\tilde{\boldsymbol{\beta}}_{GOE} \right) = \sigma^2 \tilde{\boldsymbol{A}}_{(i)} \boldsymbol{S}^{-1} \tilde{\boldsymbol{A}}'_{(i)} \tag{20}$$

$$MSE \left(\tilde{\boldsymbol{\beta}}_{GOE} \right) = \sigma^2 \tilde{\boldsymbol{A}}_{(i)} \boldsymbol{S}^{-1} \tilde{\boldsymbol{A}}'_{(i)} + \tilde{\boldsymbol{A}}_{(i)} \left(\boldsymbol{I} - \tilde{\boldsymbol{A}}_{(i)}^{-1} \right) \boldsymbol{\beta} \boldsymbol{\beta}' \left(\boldsymbol{I} - \tilde{\boldsymbol{A}}_{(i)}^{-1} \right)' \tilde{\boldsymbol{A}}'_{(i)} \tag{21}$$

and

$$SMSE \left(\tilde{\boldsymbol{\beta}}_{GOE} \right) = \sigma^2 tr \left(\tilde{\boldsymbol{A}}_{(i)} \boldsymbol{S}^{-1} \tilde{\boldsymbol{A}}_{(i)} \right) + \boldsymbol{\beta}' \left(\boldsymbol{I} - \tilde{\boldsymbol{A}}_{(i)}^{-1} \right)' \tilde{\boldsymbol{A}}'_{(i)} \tilde{\boldsymbol{A}}_{(i)} \left(\boldsymbol{I} - \tilde{\boldsymbol{A}}_{i}^{-1} \right) \boldsymbol{\beta} \tag{22}$$

respectively.

Note that since $\boldsymbol{P} = \boldsymbol{\beta} \boldsymbol{\beta}'$ is symmetric and $\boldsymbol{P}^2 = \boldsymbol{\beta} \boldsymbol{\beta}' \boldsymbol{\beta} \boldsymbol{\beta}' = \|\boldsymbol{\beta}\|^2 \boldsymbol{\beta} \boldsymbol{\beta}' = c \boldsymbol{P}$ where $c = \|\boldsymbol{\beta}\|^2 = \sum_{i=1}^{p} \boldsymbol{\beta}_i^2$ it can

be defined $\boldsymbol{P} = \boldsymbol{\beta} \boldsymbol{\beta}' = c \left(\frac{1}{c} \boldsymbol{\beta} \boldsymbol{\beta}' \right) = c \boldsymbol{R}$. Then $\boldsymbol{R}' = \boldsymbol{R}$ and $\boldsymbol{R}^2 = \boldsymbol{R}$. Therefore $\boldsymbol{R}$ is symmetric and idempotent

matrix. Now we write the optimal matrix $\boldsymbol{A}_{(i)}$ as $\tilde{\boldsymbol{A}}_{(i)} = c \boldsymbol{R} \left(\sigma^2 \boldsymbol{S}^{-1} + c \boldsymbol{R} \right)^{-1}$.

Now the bias vector, dispersion matrix, mean square error matrix and scalar mean square error of $\tilde{\boldsymbol{\beta}}_{GOE}$ can be rewritten as

$$\boldsymbol{B} \left(\tilde{\boldsymbol{\beta}}_{GOE} \right) = -\sigma^2 \left(\sigma^2 \boldsymbol{I} + c \boldsymbol{R} \boldsymbol{S} \right)^{-1} \boldsymbol{\beta}, \tag{23}$$

$$D\left(\tilde{\boldsymbol{\beta}}_{GOE}\right) = c^2\sigma^2\boldsymbol{R}\left(\sigma^2\boldsymbol{S}^{-1}+c\boldsymbol{R}\right)^{-1}\boldsymbol{S}^{-1}\left(\sigma^2\boldsymbol{S}^{-1}+c\boldsymbol{R}\right)^{-1}\boldsymbol{R}, \tag{24}$$

$$MSE\left(\tilde{\boldsymbol{\beta}}_{GOE}\right) = c^2\sigma^2\boldsymbol{R}\left(\sigma^2\boldsymbol{S}^{-1}+c\boldsymbol{R}\right)^{-1}\boldsymbol{S}^{-1}\left(\sigma^2\boldsymbol{S}^{-1}+c\boldsymbol{R}\right)^{-1}\boldsymbol{R} + c\sigma^4\left(\sigma^2\boldsymbol{I}+c\boldsymbol{R}\boldsymbol{S}\right)^{-1}\boldsymbol{R}\left(\sigma^2\boldsymbol{I}+c\boldsymbol{R}\boldsymbol{S}\right)^{-1} \tag{25}$$

and

$$SMSE\left(\tilde{\boldsymbol{\beta}}_{GOE}\right) = c^2\sigma^2 tr\left[\sigma^2\boldsymbol{R}\left(\sigma^2\boldsymbol{S}^{-1}+c\boldsymbol{R}\right)^{-1}\boldsymbol{S}^{-1}\left(\sigma^2\boldsymbol{S}^{-1}+c\boldsymbol{R}\right)^{-1}\boldsymbol{R}\right] + \sigma^4\boldsymbol{\beta}'\left(\sigma^2\boldsymbol{I}+c\boldsymbol{R}\boldsymbol{S}\right)^{-2}\boldsymbol{\beta} \tag{26}$$

respectively.

For practical situations we have to replace the unknown parameters $\boldsymbol{\beta}$ and σ^2. For an estimated value for $\boldsymbol{\beta}$ the OLSE, RE, LE, AURE or AULE can be used. For the estimated value for $\hat{\sigma}^2$ we can use either (2) or replace $\hat{\boldsymbol{\beta}}$ in (2) by RE, LE, AURE or AULE accordingly. In the next section we will discuss the superiority of estimators when replacing each of these estimated values by using a simulation study, and then we use numerical examples for further illustration.

4. Numerical Illustration

4.1. Monte Carlo Simulation

To study the behavior of our proposed estimator, we perform the Monte Carlo Simulation study by considering different levels of multicollinearity. Following [7] we generate explanatory variables as follows:

$$x_{ij} = \left(1-\gamma^2\right)^{1/2} z_{ij} + \gamma z_{i,p+1}, \ i=1,2,\cdots,n, \ j=1,2,\cdots,p,$$

where z_{ij} is an independent standard normal pseudo random number, and γ is specified so that the theoretical correlation between any two explanatory variables is given by γ^2. A dependent variable is generated by using the equation.

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \varepsilon_i, \ i=1,2,\cdots,n,$$

where ε_i is a normal pseudo random number with mean zero and variance σ_i^2. [8] have noted that if the MSE is a function of σ^2 and $\boldsymbol{\beta}$, and if the explanatory variables are fixed, then subject to the constraint $\boldsymbol{\beta}'\boldsymbol{\beta}$, the MSE is minimized when $\boldsymbol{\beta}$ is the normalized eigenvector corresponding to the largest eigenvalue of the $\boldsymbol{X}'\boldsymbol{X}$ matrix. In this study we choose the normalized eigenvector corresponding to the largest eigenvalue of $\boldsymbol{X}'\boldsymbol{X}$ as the coefficient vector $\boldsymbol{\beta}$, $n=50$, $p=4$ and $\sigma_i^2=1$. Four different sets of correlations are considered by selecting the values as $\gamma=0.8$, 0.9, 0.99 and 0.999.

Table 1 can be obtained by using estimated SMSE values obtained by using equations (7) and (21) for different shrinkage parameter d or k values selected from the interval (0, 1). The SMSE of GOE-OLSE, GOE-RE, GOE-LE, GOE-AURE and GOE-AULE are obtained by substituting OLSE, RE, LE, AURE and AULE in equation (21) respectively instead of $\boldsymbol{\beta}$ and σ^2.

According to Table 1, we can say that the GOE-OLSE has the smallest scalar mean square error values with compared to RE, LE, AURE, AULE, GOE-RE, GOE-LE, GOE-AURE and GOE-AULE when $\gamma = 0.8$, 0.9, 0.99 and 0.999.

4.2. Numerical Example

To further illustrate the behavior of our proposed estimator we consider two data sets. First we consider the data set on Portland cement originally due to [9]. This data set has since then been widely used by many researchers such as [10]-[13]. This data set came from an experimental investigation of the heat evolved during the setting and hardening of Portland cements of varied composition and the dependence of this heat on the percentages of four compounds in the clinkers from which the cement was produced. The four compounds considered by [9] are tricalcium aluminate: $3CaO \cdot Al_2O_3$, tricalcium silicate: $3CaO \cdot SiO_2$, tetracalcium aluminaferrite: $4CaO \cdot Al_2O_3 \cdot Fe_2O_3$, and beta-dicalcium silicate: $2CaO \cdot SiO_2$, which we will denote by X_1, X_2, X_3 and X_4, respectively. The dependent variable y is the heat evolved in calories per gram of cement after 180 days of curing.

Table 1. Estimated SMSE values of the RE, LE, AURE, AULE, GOE-OLSE, GOE-RE, GOE-LE, GOE-AURE and GOE-AULE when γ = 0.8, 0.9, 0.99 and 0.999.

k/d	0.1	0.2	0.5	0.7	0.85	0.95	1
			γ = 0.8 (Condition number = 3.40)				
RE	0.2432	0.4089	0.6532	0.7297	0.7684	0.7887	0.7976
LE	0.6482	0.5179	0.2409	0.1514	0.1342	0.1465	0.1598
AURE	0.0166	0.0220	0.1647	0.3583	0.5557	0.7122	0.7979
AULE	0.4257	0.2845	0.1320	0.1385	0.1532	0.1590	0.1598
GOE-OLSE	0.0065	0.0065	0.0065	0.0065	0.0065	0.0065	0.0065
GOE-RE	0.0721	0.0912	0.1786	0.2795	0.3624	0.4163	0.4423
GOE-LE	0.1407	0.0527	0.0125	0.0083	0.0069	0.0065	0.0065
GOE-AURE	0.0555	0.0737	0.0981	0.1178	0.1390	0.1566	0.1663
GOE-AULE	0.1532	0.1207	0.0328	0.0135	0.0081	0.0067	0.0065
			γ = 0.9 (Condition number = 4.95)				
RE	0.3641	0.5621	0.8688	0.9687	1.0200	1.0471	1.0590
LE	0.8613	0.6906	0.3402	0.2415	0.2383	0.2698	0.2957
AURE	0.0140	0.0268	0.2183	0.4753	0.7374	0.9453	1.0593
AULE	0.5673	0.3875	0.2215	0.2527	0.2832	0.2942	0.2957
GOE-OLSE	0.0062	0.0062	0.0062	0.0062	0.0062	0.0062	0.0062
GOE-RE	0.1768	0.2002	0.2757	0.3693	0.4546	0.5137	0.5432
GOE-LE	0.1844	0.0740	0.0154	0.0088	0.0068	0.0062	0.0062
GOE-AURE	0.1521	0.1801	0.2080	0.2246	0.2418	0.2563	0.2644
GOE-AULE	0.2464	0.2004	0.0578	0.0196	0.0090	0.0065	0.0062
			γ = 0.99 (Condition number = 15.99)				
RE	2.4054	2.5669	2.8182	2.9021	2.9456	2.9686	2.9788
LE	2.4411	2.0183	1.4392	1.6275	2.0703	2.5091	2.7716
AURE	0.0285	0.1052	0.7054	1.4226	2.1285	2.6795	2.9791
AULE	1.9373	1.5078	1.7362	2.3188	2.6500	2.7578	2.7716
GOE-OLSE	0.0167	0.0167	0.0167	0.0167	0.0167	0.0167	0.0167
GOE-RE	2.2621	2.2908	2.3495	2.4159	2.4778	2.5219	2.5442
GOE-LE	1.1276	0.5055	0.0754	0.0304	0.0195	0.0170	0.0167
GOE-AURE	2.2257	2.2685	2.3000	2.3135	2.3260	2.3361	2.3418
GOE-AULE	2.2043	1.8384	0.5393	0.1438	0.0406	0.0190	0.0167
			γ = 0.999 (Condition number = 50.55)				
RE	23.6901	23.8842	24.1905	24.2933	24.3466	24.3749	24.3874
LE	20.0298	16.7109	12.9858	15.6956	20.4542	24.9250	27.5498
AURE	0.2405	0.9577	6.0485	11.9042	17.5907	21.9986	24.3876
AULE	16.8509	13.4711	17.0094	23.0107	26.3365	27.4124	27.5498
GOE-OLSE	0.1041	0.1041	0.1041	0.1041	0.1041	0.1041	0.1041
GOE-RE	23.5189	23.5494	23.6133	23.6889	23.7612	23.8133	23.8399
GOE-LE	10.7719	4.8376	0.6625	0.2305	0.1290	0.1065	0.1041
GOE-AURE	23.4801	23.5255	23.5590	23.5735	23.5871	23.5982	23.6045
GOE-AULE	22.2149	18.5172	5.3798	1.3793	0.3411	0.1261	0.1041

We assemble our data set in the matrix form as follows:

$$X = \begin{pmatrix} 7 & 26 & 6 & 60 \\ 1 & 29 & 15 & 52 \\ 11 & 56 & 8 & 20 \\ 11 & 31 & 8 & 47 \\ 7 & 52 & 6 & 33 \\ 11 & 55 & 9 & 22 \\ 3 & 71 & 17 & 6 \\ 1 & 31 & 22 & 44 \\ 2 & 54 & 18 & 22 \\ 21 & 47 & 4 & 26 \\ 1 & 40 & 23 & 34 \\ 11 & 66 & 9 & 12 \\ 10 & 68 & 8 & 12 \end{pmatrix} \text{ and } y = \begin{pmatrix} 78.5 \\ 74.3 \\ 104.3 \\ 87.6 \\ 95.9 \\ 109.2 \\ 102.7 \\ 72.5 \\ 93.1 \\ 115.9 \\ 83.8 \\ 113.3 \\ 109.4 \end{pmatrix}.$$

For this particular data set, we obtain the following results:

a) The eigen values of $X'X$: 105, 810, 5965, 44663

b) The condition number = 20.58464

c) The OLSE of β is $\hat{\beta}$: $\hat{\beta}' = (2.1930, 1.1533, 0.7585, 0.4863)$

d) The OLSE of σ^2 : $\hat{\sigma}^2 = 5.8455$.

Table 2 can be obtained by using estimated SMSE values obtained by using Equations (7) and (21) for different shrinkage parameter d or k values selected from the interval (0, 1). The SMSE of GOE-OLSE, GOE-RE, GOE-LE, GOE-AURE and GOE-AULE are obtained by substituting OLSE, RE, LE, AURE and AULE in Equation (21) respectively instead of β and σ^2 .

From **Table 2** we can notice that the GOE-OLSE, GOE-RE, GOE-LE, GOE-AURE and GOE-AULE have the smallest scalar mean square error value with compared to RE, LE, AURE, AULE. Therefore we can suggest the GOE to estimate the regression coefficients. When k is large, GOE-OLSE has smallest SMSE than GOE-RE. When d is small, GOE-OLSE has smallest SMSE than GOE-LE.

Now we consider the second data set on Total National Research and Development Expenditures as a Percent of Gross National product originally due to [14], and later considered by [15]-[17].

Table 2. Estimated SMSE values of RE, LE, AURE, AULE, GOE-OLSE, GOE-RE, GOE-LE, GOE-AURE and GOE-AULE the data set on Portland Cement.

k/d	0.1	0.2	0.5	0.7	0.85	0.95	1
RE	0.0637	0.0636	0.0633	0.0632	0.0631	0.0631	0.0630
LE	0.0631	0.0631	0.0633	0.0635	0.0636	0.0637	0.0638
AURE	0.0638	0.0638	0.0638	0.0639	0.0639	0.0640	0.0641
AULE	0.0638	0.6368	0.0638	0.0638	0.0638	0.0638	0.0638
GOE-OLSE	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
GOE-RE	0.0003	0.0003	0.0004	0.0005	0.0006	0.0006	0.0006
GOE-LE	0.0006	0.0005	0.0004	0.0004	0.0003	0.0003	0.0003
GOE-AURE	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
GOE-AULE	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003

The data set is given below:

$$X = \begin{pmatrix} 1.9 & 2.2 & 1.9 & 3.7 \\ 1.8 & 2.2 & 2.0 & 3.8 \\ 1.8 & 2.4 & 2.1 & 3.6 \\ 1.8 & 2.4 & 2.2 & 3.8 \\ 2.0 & 2.5 & 2.3 & 3.8 \\ 2.1 & 2.6 & 2.4 & 3.7 \\ 2.1 & 2.6 & 2.6 & 3.8 \\ 2.2 & 2.6 & 2.6 & 4.0 \\ 2.3 & 2.8 & 2.8 & 3.7 \\ 2.3 & 2.7 & 2.8 & 3.8 \end{pmatrix}, \; y = \begin{pmatrix} 2.3 \\ 2.2 \\ 2.2 \\ 2.3 \\ 2.4 \\ 2.5 \\ 2.6 \\ 2.6 \\ 2.7 \\ 2.7 \end{pmatrix}.$$

The four column of the 10×4 matrix X comprise the data on x_1, x_2, x_3 and x_4 respectively, and y is the predictor variable.

For this particular data set, we obtain the following results:

a) The eigen values of $X'X$: 302.9626, 0.7283, 0.7283 and 0.00345

b) The condition number = 93.68

c) The OLSE of β is $\hat{\beta} : \hat{\beta}' = (0.6455, 00896, 0.1436, 0.1526)$

d) The OLSE of σ^2 : $\hat{\sigma}^2 = 0.0015$.

Table 3 can also be obtained by using estimated SMSE values obtained by using Equations (7) and (21) for different shrinkage parameter d or k values selected from the interval (0, 1). The SMSE of GOE-OLSE, GOE-RE, GOE-LE, GOE-AURE and GOE-AULE are obtained by substituting OLSE, RE, LE, AURE and AULE in Equation (21) respectively instead of β and σ^2.

From **Table 3**, we can say that our proposed estimator is superior to RE, LE, AURE and AULE, GOE-RE, GOE-LE, GOE-AURE and GOE-AULE.

5. Conclusions

In this paper we proposed a new biased estimator namely Generalized Optimal Estimator (GOE) in a multiple linear regression when there exists multicollinearity problem in the independent variables. The proposed estimator is superior to biased estimators which are based on sample information and takes the form $\hat{\beta}_{GURE} = A_{(i)}\hat{\beta}$. Based on **Tables 1-3**, it can be concluded that the proposed estimator has smallest scalar mean square error values compared with RE, LE, AURE and AULE. We can also suggest that GOE-OLSE is the best estimator with compared to GOE-RE, GOE-LE, GOE-AURE and GOE-AULE.

Table 3. Estimated SMSE values of RE, LE, AURE, AULE, GOE-OLSE, GOE-RE, GOE-LE, GOE-AURE and GOE-AULE the data set on total national research and development expenditures.

k/d	0.1	0.2	0.5	0.7	0.85	0.95	1
RE	0.1247	0.1664	0.2088	0.2202	0.226	0.2291	0.2304
LE	0.1881	0.1519	0.0797	0.0619	0.0645	0.0739	0.0808
AURE	0.0214	0.0155	0.0549	0.1096	0.1645	0.2077	0.2313
AULE	0.1362	0.086	0.0213	0.025	0.0453	0.0671	0.0808
GOE-OLSE	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GOE-RE	0.1169	0.163	0.2076	0.2194	0.2253	0.2284	0.2298
GOE-LE	0.186	0.1469	0.0573	0.0206	0.0052	0.0006	0.0000
GOE-AURE	0.0568	0.1093	0.1724	0.1899	0.1987	0.2033	0.2053
GOE-AULE	0.2012	0.1892	0.1155	0.0534	0.0158	0.002	0.0000

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