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Characterization of Power-Function Distribution through Expectation*

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ABSTRACT

For the characterization of the power function distribution, one needs any arbitrary non constant function only in place of independence of suitable function of order statistics, linear relation of conditional expectation, recurrence relations between expectations of function of order statistics, distributional properties of exponential distribution, record valves, lower record statistics, product of order statistics and Lorenz curve, etc. available in the literature. The goal of this research is not to give a different path-breaking approach for the characterization of power function distribution through the expectation of non constant function of random variable and provide a method to characterize the power function distribution as remark. Examples are given for the illustrative purpose.

Keywords: Characterization; Power Function Distribution

1. Introduction

Several characterizations of power function distribution have been made notably by Fisz [1], Basu [2], Govindarajulu [3] and Dallas [4] using independence of suitable function of order statistics and distributional properties of transformation of exponential variable.

Other attempts were made for the characterization of exponential and related distributions assuming linear relation of conditional expectation by Beg [5], characterization based on record valves by Nagraja [6], characterization of some types of distributions using recurrence relations between expectations of function of order statistics by Alli [7], characterization results on exponential and related distributions by Tavangar [8], and characterization continuous distributions through lower record statistics by Faizan [9] included the characterization of power function distribution.

Direct characterization for power function distribution has been given in Arslan [10] who used the product of order statistics [contraction is a particular case of product of order statistics which has interesting applications such as in economic modeling and reliability see Alamatsaz [11], Kotz [12] and Alzaid [13]] where as Moothathu [14] used Lorenz curve. [Graph of fraction of total income owned by lowest p$^{\text{th}}$ fraction of the population is Lorenz curve of distribution of income [15].

This research note provides the characterization based on identity of distribution and equality of expectation of function of random variable for power-function distribution with the probability density function (p.d.f.)

$$f(x;\theta) = \begin{cases} c\theta^{-c}x^{c-1}; a < x < \theta < b; \theta = k^{-1}, k > 0, c > 0 \\ 0; \qquad \text{otherwise} \end{cases} \tag{1.1}$$

where $-\infty \le a < b \le \infty$ are known constants, x^{c-1} is positive absolutely continuous function and c/θ^c is everywhere differentiable function. Since derivative of x^c/c being positive and since range is truncated by θ from right $(a^c/c) = 0$.

The aim of the present research note is to give the new characterization through the expectation of function $\phi(x)$ for the power function distribution. Examples are given for the illustrative purpose.

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2. Characterization

Theorem 2.1 Let X be a random variable with distribution function F. Assume that F is continuous on the interval, (a,b) where $-\infty \le a < b \le \infty$. Let $\phi(x)$ and $g(X)$ be two distinct differentiable and integrable functions of X on the interval (a,b) where $-\infty \le a < b \le \infty$ and moreover $g(X)$ be non constant. Then $f(x;\theta)$ is the p.d.f. of power function distribution defined in (1.1) if and only if

$$E\left[g(x)+\frac{X}{c}\frac{\mathrm{d}}{\mathrm{d}X}g(x)\right]=g(\theta) \tag{2.1}$$

Proof Given $f(x;\theta)$ defined in (1.1), if $\phi(x)$ is such that $g(\theta)=E\phi(X)$ where $g(\theta)$ is differentiable function then

$$g(\theta)=\int_a^\theta \phi(x) f(x;\theta)\mathrm{d}x \tag{2.2}$$

Differentiating (2.2) with respect to θ on both sides and replacing X for θ and simplifying one gets

$$\phi(x)=g(x)+\frac{X}{c}\frac{\mathrm{d}}{\mathrm{d}X}g(x) \tag{2.3}$$

which establishes necessity of (2.1). Conversely given (2.1), let $k(x;\theta)$ be such that

$$g(\theta)=\int_a^\theta\left[g(x)+\frac{X}{c}\frac{\mathrm{d}}{\mathrm{d}X}g(x)\right]k(x;\theta)\mathrm{d}x \tag{2.4}$$

Since $\left(a^c/c\right)=0$ the following identity holds:

$$g(\theta)=\frac{c}{\theta^c}\int_a^\theta\left[\frac{\mathrm{d}}{\mathrm{d}x}g(x)\left(\frac{x^c}{c}\right)\mathrm{d}x\right] \tag{2.5}$$

Differentiating integrand of (2.5) and tacking $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^c/c\right)$ as one factor one gets (2.5) as

$$g(\theta)=\int_a^\theta \phi(x)\left\{c\theta^{-c}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^c}{c}\right)\right\}\mathrm{d}x \tag{2.6}$$

where $\phi(x)$ is function of X derived in (2.3). From (2.4) and (2.6) by uniqueness theorem

$$k(x;\theta)=\frac{c}{\theta^c}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^c}{c}\right) \tag{2.7}$$

Since x^c/c is decreasing function with $\left(a^c/c\right)=0$ and since $\theta=K^{-1}, K>0$, integrating (2.7) on both sides one gets

$$1=\int_a^\theta k(x;\theta)\mathrm{d}x \tag{2.8}$$

Substituting $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^c/c\right)$ in (2.7), $k(x;\theta)$ reduces to $f(x;\theta)$ defined in (1.1), which establishes sufficiency of (2.1).

Note: Author does not claim the relations between f and g in the preceding analysis.

Remark 2.1 Using $\phi(x)$ derived in (2.3), $f(x;\theta)$ given in (1.1) can be determined by

$$M(X)=\frac{\frac{\mathrm{d}}{\mathrm{d}X}g(X)}{\phi(X)-g(X)} \tag{2.9}$$

and p.d.f. is given by

$$f(x;\theta)=\frac{\frac{\mathrm{d}}{\mathrm{d}x}U(x)}{U(\theta)} \tag{2.10}$$

where $U(X)$ is increasing function in the interval (a,b) for $-\infty \le a < b \le \infty$ with $U(a)=0$ such that it satisfies

$$M(X)=\frac{\mathrm{d}}{\mathrm{d}X}\log U(X)$$

3. Illustrative Examples

Example 1 Using method described in the remark characterization of power function distribution through survival function quantile; $Q_p(\theta)=p^{\frac{1}{c}}\theta$ is illustrated.

$$g(X)=p^{1/c}X$$

$$\phi(x)=g(x)+\frac{X}{c}\frac{\mathrm{d}}{\mathrm{d}X}g(x)=\frac{c+1}{c}Xp^{1/c}$$

$$M(X)=\frac{\frac{\mathrm{d}}{\mathrm{d}X}g(X)}{\phi(X)-g(X)}=\frac{c}{X}$$

$$\frac{\mathrm{d}}{\mathrm{d}X}\log\left(\frac{X}{c}\right)=\frac{c}{X}=M(X)$$

$$U(x)=\frac{X^c}{c}$$

$$f(x;\theta)=\frac{\frac{\mathrm{d}}{\mathrm{d}x}U(x)}{U(\theta)}=\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^c}{c}\right)}{\frac{\theta^c}{c}}$$

$$=c\theta^{-c}x^{c-1}, x>\theta$$

Example 2 The p.d.f. $f(x;\theta)$ defined in (1.1) can be characterized through non constant functions of θ such as

$$g_i(\theta)=\begin{cases}\dfrac{c}{c+1}\theta;\text{mean}\\ \dfrac{c}{c+r}\theta^r;r^{th}\text{raw-moment}\\ \mathrm{e}^{\theta}\\ \mathrm{e}^{-\theta}\\ \theta p^{-1/c};p^{th}\text{quantile}\\ \left(\dfrac{t}{\theta}\right)^c;\text{distribution-function}\\ 1-\left(\dfrac{t}{\theta}\right)^c;\text{reliability-function}\\ \dfrac{\left(\dfrac{t}{\theta}\right)}{\left(\dfrac{\theta}{t}\right)^c-1};\text{hazard-function}\end{cases}$$

by using

$$\left[\phi_i(X)-g_i(X)\right]=\begin{cases}\dfrac{X}{c+1};\text{mean}\\ \dfrac{X}{c}\mathrm{e}^{X}\\ -\dfrac{X}{c}\mathrm{e}^{-X}\\ \dfrac{X}{c}p^{1/c};p^{th}\text{quantile}\\ -\left(\dfrac{t}{X}\right)^c;\text{distribution-function}\\ \left(\dfrac{t}{X}\right)^c;\text{reliability-function}\\ \dfrac{\left(\dfrac{c}{t}\right)\left(\dfrac{t}{X}\right)^c}{\left[1-\left(\dfrac{t}{X}\right)^c\right]^2};\text{hazard-function}\end{cases}$$

and defining $M(X)$ given in (2.9) and using $U(X)$ as appeared in (2.11) for (2.10).

4. Conclusion

To characterize the p.d.f. defined in (1.1), one needs any arbitrary non constant function of X which should be differentiable and integrable only.

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