

Asymmetry Index on Marginal Homogeneity for Square Contingency Tables with Ordered Categories

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ABSTRACT

For square contingency tables with ordered categories, the present paper considers two kinds of weak marginal homogeneity and gives measures to represent the degree of departure from weak marginal homogeneity. The proposed measures lie between -1 to 1 . When the marginal cumulative logistic model or the extended marginal homogeneity model holds, the proposed measures represent the degree of departure from marginal homogeneity. Using these measures, three kinds of unaided distance vision data are analyzed.

Keywords: Marginal Homogeneity; Marginal Cumulative Logistic Model; Measure; Square Contingency Table

1. Introduction

Consider an $R \times R$ square contingency table with ordered categories. Let p_{ij} denote the probability that an observation will fall in the i th row and j th column of the table ($i = 1, \dots, R$; $j = 1, \dots, R$). Also let X and Y denote the row and column variables, respectively. The marginal homogeneity (MH) model ([1]) is defined by

$$F_i^X = F_i^Y \quad \text{for } i = 1, \dots, R-1$$

where

$$F_i^X = \sum_{k=1}^i p_{k\cdot}, \quad F_i^Y = \sum_{k=1}^i p_{\cdot k}$$

with

$$p_{k\cdot} = \sum_{t=1}^R p_{kt}, \quad p_{\cdot k} = \sum_{s=1}^R p_{sk}$$

When the MH model does not hold, we are interested in applying the model that has weaker restriction than the MH model. As such a model, for example, there are the marginal cumulative logistic (ML) model ([2]) and the extended marginal homogeneity (EMH) model ([3-5]). We are also interested in considering the other structure of weak MH. The measures to represent the degree of departure from MH are given by, for example, [6,7]. When the structure of weak MH does not hold, we are interested in measuring what degree the departure from weak MH is.

The present paper considers two kinds of structures of weak MH and proposes the measures to represent the

degree of departure from weak MH.

2. Weak Marginal Homogeneity I and Measure

2.1. Submeasure I

Let

$$\Delta_1 = \sum_{i=1}^{R-1} (F_i^X + F_i^Y),$$

and

$$F_{1(i)}^* = \frac{F_i^X}{\Delta_1}, \quad F_{2(i)}^* = \frac{F_i^Y}{\Delta_1}, \quad \text{for } i = 1, \dots, R-1.$$

Note that $\sum_{i=1}^{R-1} (F_{1(i)}^* + F_{2(i)}^*) = 1$. Assuming that $\{F_i^X + F_i^Y \neq 0\}$, consider the submeasure defined by

$$\Psi_1 = \frac{4}{\pi} \sum_{i=1}^{R-1} (F_{1(i)}^* + F_{2(i)}^*) \left(\theta_i^{(1)} - \frac{\pi}{4} \right),$$

where

$$\theta_i^{(1)} = \sin^{-1} \left(\frac{F_i^Y}{\sqrt{(F_i^X)^2 + (F_i^Y)^2}} \right).$$

Noting that $0 \leq \theta_i^{(1)} \leq \pi/2$, we see that 1) $-1 \leq \Psi_1 \leq 1$, 2) $\Psi_1 = -1$ if and only if $F_i^Y = 0$ and $F_i^X > 0$ ($i = 1, \dots, R-1$), and 3) $\Psi_1 = 1$ if and only if $F_i^X = 0$ and $F_i^Y > 0$ ($i = 1, \dots, R-1$). When the MH model holds, Ψ_1 equals zero.

2.2. Submeasure II

Let

$$S_i^X = 1 - F_i^X, \quad S_i^Y = 1 - F_i^Y \quad \text{for } i = 1, \dots, R-1.$$

The MH model may be expressed as

$$S_i^X = S_i^Y \quad \text{for } i = 1, \dots, R-1.$$

Let

$$\Delta_2 = \sum_{i=1}^{R-1} (S_i^X + S_i^Y),$$

and for $i = 1, \dots, R-1$,

$$\theta_i^{(2)} = \sin^{-1} \left(\frac{S_i^X}{\sqrt{(S_i^X)^2 + (S_i^Y)^2}} \right),$$

$$S_{1(i)}^* = \frac{S_i^X}{\Delta_2}, \quad S_{2(i)}^* = \frac{S_i^Y}{\Delta_2}.$$

Note that $\sum_{i=1}^{R-1} (S_{1(i)}^* + S_{2(i)}^*) = 1$. Assuming that

$\{S_i^X + S_i^Y \neq 0\}$, we shall define the submeasure Ψ_2 as follows;

$$\Psi_2 = \frac{4}{\pi} \sum_{i=1}^{R-1} (S_{1(i)}^* + S_{2(i)}^*) \left(\theta_i^{(2)} - \frac{\pi}{4} \right).$$

Noting that $0 \leq \theta_i^{(2)} \leq \pi/2$, we see that 1) $-1 \leq \Psi_2 \leq 1$; 2) $\Psi_2 = -1$ if and only if $S_i^X = 0$ and $S_i^Y > 0$ ($i = 1, \dots, R-1$); and 3) $\Psi_2 = 1$ if and only if $S_i^Y = 0$ and $S_i^X > 0$ ($i = 1, \dots, R-1$). When the MH model holds, Ψ_2 equals zero.

2.3. Complete Measure

Assume that $\{F_i^X + F_i^Y \neq 0\}$ and $\{S_i^X + S_i^Y \neq 0\}$. Consider a measure defined by

$$\Psi = \frac{1}{2} (\Psi_1 + \Psi_2).$$

We see that 1) $-1 \leq \Psi \leq 1$, 2) $\Psi = -1$ if and only if $F_i^X = 1$ (then $S_i^X = 0$) and $F_i^Y = 0$ (then $S_i^Y = 1$) for all $i = 1, \dots, R-1$, and 3) $\Psi = 1$ if and only if $F_i^X = 0$ (then $S_i^X = 1$) and $F_i^Y = 1$ (then $S_i^Y = 0$) for all $i = 1, \dots, R-1$. Thus, $\Psi = -1$ indicates that $p_{1R} = 1$ and the other cell probabilities are zero (say, upper-right-marginal inhomogeneity), and $\Psi = 1$ indicates that $p_{R1} = 1$ and the other cell probabilities are zero (say, lower-left-marginal inhomogeneity). When $\Psi = 0$, we shall refer to this structure as the weak marginal homogeneity I (WMH-I). We note that if the MH model holds then the structure of WMH-I holds, but the converse does not hold.

Therefore, using the measure Ψ , we can see whether

the structure of WMH-I departs toward the upper-right-marginal inhomogeneity or toward the lower-left-marginal inhomogeneity. As the measure Ψ approaches -1 , the departure from WMH-I becomes greater toward the upper-right-marginal inhomogeneity. While as the Ψ approaches 1, it becomes greater toward the lower-left-marginal inhomogeneity.

3. Weak Marginal Homogeneity II and Measure

Let

$$T_i^X = \Pr(X \leq i | X \neq Y) = \sum_{k=1}^i p_{k.}^c,$$

$$T_i^Y = \Pr(Y \leq i | X \neq Y) = \sum_{k=1}^i p_{.k}^c,$$

for $i = 1, \dots, R-1$, where

$$p_{k.}^c = \frac{1}{\delta} (p_{k.} - p_{kk}), \quad p_{.k}^c = \frac{1}{\delta} (p_{.k} - p_{kk}), \quad \delta = \sum_{s \neq t} p_{st}.$$

The MH model may be expressed by

$$T_i^X = T_i^Y \quad \text{for } i = 1, \dots, R-1.$$

We shall consider the submeasure Υ_1 which is defined by the submeasure Ψ_1 replaced $\{F_i^X\}$ and $\{F_i^Y\}$ by $\{T_i^X\}$ and $\{T_i^Y\}$, respectively.

Let

$$U_i^X = 1 - T_i^X, \quad U_i^Y = 1 - T_i^Y \quad \text{for } i = 1, \dots, R-1.$$

The MH model may be expressed by

$$U_i^X = U_i^Y \quad \text{for } i = 1, \dots, R-1.$$

We shall consider the submeasure Υ_2 which is defined by the submeasure Ψ_2 replaced $\{S_i^X\}$ and $\{S_i^Y\}$ by $\{U_i^X\}$ and $\{U_i^Y\}$, respectively.

Assume that $\{T_i^X + T_i^Y \neq 0\}$ and $\{U_i^X + U_i^Y \neq 0\}$. Consider a measure defined by

$$\Upsilon = \frac{1}{2} (\Upsilon_1 + \Upsilon_2).$$

We see that $-1 \leq \Upsilon \leq 1$. Let $p_{ij}^c = p_{ij} / \delta$ ($i \neq j$). In a similar way to Ψ , $\Upsilon = -1$ indicates that $p_{1R}^c = 1$ and the other p_{ij}^c are zero ($i \neq j$) (say, conditional upper-right-marginal inhomogeneity), and $\Upsilon = 1$ indicates that $p_{R1}^c = 1$ and the other p_{ij}^c are zero ($i \neq j$) (say, conditional lower-left-marginal inhomogeneity). When $\Upsilon = 0$, we shall refer to this structure as the weak marginal homogeneity II (WMH-II). We note that if the MH model holds then the structure of WMH-II holds, but the converse does not hold.

4. Relationships between Measures and Models

We shall consider the relationship between the measure Ψ (or Υ) and the ML model. The ML model is given by

$$L_i^X - L_i^Y = \Delta \quad \text{for } i=1, \dots, R-1,$$

where

$$L_i^X = \log\left(\frac{F_i^X}{1-F_i^X}\right), \quad L_i^Y = \log\left(\frac{F_i^Y}{1-F_i^Y}\right).$$

A special case of this model obtained by putting $\Delta = 0$ is the MH model. The ML model may also be expressed as

$$F_i^X = \frac{\exp(\theta_i)}{1 + \exp(\theta_i)}, \quad F_i^Y = \frac{\exp(\theta_i - \Delta)}{1 + \exp(\theta_i - \Delta)},$$

for $i=1, \dots, R-1$. Therefore, when the ML model holds, 1) $\Delta > 0$ if and only if $\{F_i^X > F_i^Y\}$, 2) $\Delta < 0$ if and only if $\{F_i^X < F_i^Y\}$, and 3) $\Delta = 0$ if and only if $\{F_i^X = F_i^Y\}$. We obtain the following theorem.

Theorem 1. When the ML model holds,

- 1) $\Delta > 0$ if and only if $\Psi < 0$ ($\Upsilon < 0$),
- 2) $\Delta < 0$ if and only if $\Psi > 0$ ($\Upsilon > 0$),
- 3) $\Delta = 0$ (i.e., the MH model holds) if and only if $\Psi = 0$ ($\Upsilon = 0$).

Next, we shall consider the relationship between the measure Ψ (or Υ) and the EMH model, defined by

$$G_{1(i)} = \tau G_{2(i)} \quad \text{for } i=1, \dots, R-1,$$

with
$$a_{ij} = \frac{4}{\pi \Delta_1} \sum_{k=1}^{R-1} \left[\{I(i \leq k) + I(j \leq k)\} \theta_k^{(1)} + \frac{F_k^X + F_k^Y}{(F_k^X)^2 + (F_k^Y)^2} \{-I(i \leq k)F_k^Y + I(j \leq k)F_k^X\} \right] - \frac{\{2R - (i + j)\}(\Psi_1 + 1)}{\Delta_1},$$

$$b_{ij} = \frac{4}{\pi \Delta_2} \sum_{k=1}^{R-1} \left[\{I(i > k) + I(j > k)\} \theta_k^{(2)} + \frac{S_k^X + S_k^Y}{(S_k^X)^2 + (S_k^Y)^2} \{I(i > k)S_k^Y - I(j > k)S_k^X\} \right] - \frac{\{(i + j) - 2\}(\Psi_2 + 1)}{\Delta_2},$$

and $I(\cdot)$ is the indicator function, $I(\cdot) = 1$ if true, 0 if not.

Also, the sample version of Υ , i.e., $\hat{\Upsilon}$, is given by Υ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. We obtain the following theorem.

Theorem 4. $\sqrt{n}(\hat{\Upsilon} - \Upsilon)$ has asymptotically (as

with

$$c_{ij} = \frac{4}{\delta \pi \Delta_3} \sum_{k=1}^{R-1} \left[\{(I(i \leq k) - T_k^X) + (I(j \leq k) - T_k^Y)\} \left(\theta_k^{(3)} - \frac{\pi}{4} \right) + \frac{T_k^X + T_k^Y}{(T_k^X)^2 + (T_k^Y)^2} \{- (I(i \leq k) - T_k^X)T_k^Y + (I(j \leq k) - T_k^Y)T_k^X\} \right] - \frac{1}{\delta \Delta_3} \sum_{k=1}^{R-1} \left[(I(i \leq k) - T_k^X) + (I(j \leq k) - T_k^Y) \right] \Upsilon_1,$$

where $G_{1(i)} = \sum_{s=1}^i \sum_{t=i+1}^R p_{st}$, $G_{2(i)} = \sum_{s=i+1}^R \sum_{t=1}^i p_{st}$.

A special case of this model obtained by putting $\tau = 1$ is the MH model. Noting that $G_{1(i)} - G_{2(i)} = F_i^X - F_i^Y$

($i=1, \dots, R-1$), we obtain the following theorem.

Theorem 2. When the EMH model holds,

- 1) $\tau > 1$ if and only if $\Psi < 0$ ($\Upsilon < 0$),
- 2) $\tau < 1$ if and only if $\Psi > 0$ ($\Upsilon > 0$),
- 3) $\tau = 1$ (i.e., the MH model holds) if and only if $\Psi = 0$ ($\Upsilon = 0$).

Thus, when the ML (EMH) model holds, the measures Ψ and Υ are adequate to represent the degree of departure from MH.

5. Approximate Confidence Interval for Measures

Let n_{ij} denote the observed frequency in the i th row and j th column of the table ($i=1, \dots, R$; $j=1, \dots, R$). Assuming that a multinomial distribution applies to the $R \times R$ table, we shall consider an approximate standard error and large-sample confidence interval for the measure Ψ , using the delta method, as described by [8]. The sample version of Ψ , i.e., $\hat{\Psi}$, is given by Ψ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/n$ and $n = \sum_i \sum_j n_{ij}$. Using the delta method, we obtain the following theorem.

Theorem 3. $\sqrt{n}(\hat{\Psi} - \Psi)$ has asymptotically (as $n \rightarrow \infty$) a normal distribution with mean zero and variance $\sigma^2[\hat{\Psi}]$, where

$$\sigma^2[\hat{\Psi}] = \frac{1}{4} \sum_{i=1}^R \sum_{j=1}^R (a_{ij} + b_{ij})^2 p_{ij},$$

$n \rightarrow \infty$) a normal distribution with mean zero and variance $\sigma^2[\hat{\Upsilon}]$, where

$$\sigma^2[\hat{\Upsilon}] = \frac{1}{4} \sum_{i=1}^R \sum_{j=1}^R (c_{ij} + d_{ij})^2 p_{ij},$$

$$d_{ij} = \frac{4}{\delta\pi\Delta_4} \sum_{k=1}^{R-1} \left[\left\{ (I(i > k) - U_k^X) + (I(j > k) - U_k^Y) \right\} \left(\theta_k^{(4)} - \frac{\pi}{4} \right) + \frac{U_k^X + U_k^Y}{(U_k^X)^2 + (U_k^Y)^2} \left\{ (I(i > k) - U_k^X) U_k^Y - (I(j > k) - U_k^Y) U_k^X \right\} \right] - \frac{1}{\delta\Delta_4} \sum_{k=1}^{R-1} \left[(I(i > k) - U_k^X) + (I(j > k) - U_k^Y) \right] \Upsilon_2,$$

$$\Delta_3 = \sum_{i=1}^{R-1} (T_i^X + T_i^Y), \quad \Delta_4 = \sum_{i=1}^{R-1} (U_i^X + U_i^Y),$$

$$\theta_i^{(3)} = \sin^{-1} \left(\frac{T_i^Y}{\sqrt{(T_i^X)^2 + (T_i^Y)^2}} \right),$$

$$\theta_i^{(4)} = \sin^{-1} \left(\frac{U_i^X}{\sqrt{(U_i^X)^2 + (U_i^Y)^2}} \right).$$

Let $\hat{\sigma}^2[\hat{\Psi}]$ denote $\sigma^2[\hat{\Psi}]$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Then $\hat{\sigma}[\hat{\Psi}]/\sqrt{n}$ is estimated approximate standard error for $\hat{\Psi}$, and $\hat{\Psi} \pm z_{p/2} \hat{\sigma}[\hat{\Psi}]/\sqrt{n}$ is approximate 100(1-p) percent confidence interval for Ψ , where $z_{p/2}$ is the percentage point from the standard normal distribution corresponding to a two-tail probability equal to p . We also obtain the similar result for measure Υ .

6. Examples

Example 1: Consider the unaided distance vision data in **Table 1(a)** taken from [1]. There are data on unaided distance vision of 7477 women aged 30 to 39 employed in Royal Ordnance factories in Britain from 1943 to 1946. We see from **Table 2** that for the data in **Table 1(a)**, the estimated value of the measure Ψ is -0.0130 and all values in confidence interval for Ψ are negative. Therefore, the structure of WMH-I for a woman’s right and left eyes departs toward the upper-right-marginal inhomogeneity. Also we see from **Table 3** that for the data in **Table 1(a)**, the estimated value of the measure Υ is -0.0436 and all values in confidence interval for Υ are negative. Therefore, the structure of WMH-II for a woman’s right and left eyes departs toward the conditional upper-right-marginal inhomogeneity.

Table 4 gives the values of likelihood ratio chi-squared statistic for testing goodness-of-fit of each of MH, ML, and EMH models. We see from **Table 4** that each of ML and EMH models fits these data well. Thus the measures Ψ and Υ would indicate the degree of departure from MH. We can see from these measures that the degree of departure from MH for the vision data in **Table 1(a)** is estimated to be 1.30 (4.36) percent of the maximum departure toward the (conditional) upper-right-marginal

inhomogeneity which indicates that the grade of right eye for arbitrary woman is “Best” and the grade of her left eye is “Worst”.

Example 2: Consider the unaided vision data in **Table 1(b)**, taken from [9]. We see from **Table 2** that for the

Table 1. Unaided distance vision data of (a) 7477 women in Britain from [1]; (b) 3242 men in Britain from [9] and (c) 4746 students in Japan from [3].

(a) Women in Britain					
Right eye grade	Left eye grade				Total
	(1)	(2)	(3)	(4)	
Best (1)	1520	266	124	66	1976
Second (2)	234	1512	432	78	2256
Third (3)	117	362	1772	205	2456
Worst (4)	36	82	179	492	789
Total	1907	2222	2507	841	7477

(b) Men in Britain					
Right eye grade	Left eye grade				Total
	(1)	(2)	(3)	(4)	
Best (1)	821	112	85	35	1053
Second (2)	116	494	145	27	782
Third (3)	72	151	583	87	893
Worst (4)	43	34	106	331	514
Total	1052	791	919	480	3242

(c) Students in Japan					
Right eye grade	Left eye grade				Total
	(1)	(2)	(3)	(4)	
Best (1)	1291	130	40	22	1483
Second (2)	149	221	114	23	507
Third (3)	64	124	660	185	1033
Worst (4)	20	25	249	1429	1723
Total	1524	500	1063	1659	4746

Table 2. Estimates of Ψ , estimated approximate standard errors for $\hat{\Psi}$, and approximate 95% confidence intervals for Ψ , applied to Table 1.

Table	$\hat{\Psi}$	S. E.	C. I.
1(a)	-0.0130	0.0037	(-0.0203, -0.0056)
1(b)	0.0055	0.0064	(-0.0071, 0.0181)
1(c)	0.0125	0.0040	(0.0048, 0.0203)

Table 3. Estimates of Ψ , estimated approximate standard errors for $\hat{\Psi}$, and approximate 95% confidence intervals for Ψ , applied to Table 1.

Table	$\hat{\Psi}$	S. E.	C. I.
1(a)	-0.0436	0.0126	(-0.0683, -0.0190)
1(b)	0.0172	0.0201	(-0.0222, 0.0566)
1(c)	0.0517	0.0163	(0.0198, 0.0836)

Table 4. Values of likelihood ratio chi-squared statistic for the MH, ML, and EMH models applied to Table 1.

Applied models	Degrees of freedom	Table		
		1(a)	1(b)	1(c)
MH	3	11.99*	3.68	11.18*
ML	2	0.39	3.16	1.41
EMH	2	0.005	2.94	0.56

*Means significant at the 0.05 level.

data in **Table 1(b)** the estimated value of measure Ψ is 0.0055 and the confidence interval for Ψ includes zero. So this may indicate that there is a structure of WMH-I in the data in **Table 1(b)**. Also we see from **Table 3** that for the data in **Table 1(b)**, the estimated value of measure Ψ is 0.0172 and the confidence interval for Ψ includes zero. So this may indicate that there is a structure of WMH-II in the data in **Table 1(b)**.

Example 3: Consider the data in **Table 1(c)** taken from [3,10]. We see from **Table 2** that for the data in **Table 1(c)**, the estimated value of the measure Ψ is 0.0125 and all values in confidence interval for Ψ are positive. Therefore, the structure of WMH-I for a student's right and left eyes departs toward the lower-left-marginal inhomogeneity. Also we see from **Table 3** that for the data in **Table 1(c)**, the estimated value of the measure Ψ is 0.0517 and all values in confidence interval for Ψ are positive. Therefore, the structure of WMH-II for a student's right and left eyes departs toward the conditional lower-left-marginal inhomogeneity.

We see from **Table 4** that each of ML and EMH models fits these data well. Thus the measures Ψ and Ψ would indicate the degree of departure from MH. We can see from these measures that the degree of departure from MH for the vision data in **Table 1(c)** is estimated to be 1.25 (5.17) percent of the maximum departure toward the (conditional) lower-left-marginal inhomogeneity which indicates that the grade of right eye for arbitrary student is "Worst" and the grade of his/her left eye is "Best".

7. Concluding Remarks

For the analysis of square contingency tables with ordered categories, when the ML model, or the EMH model, or other asymmetry models, for example, [11]'s

conditional symmetry model (defined by $p_{ij}/p_{ji} = \theta$ for $i < j$) holds, the proposed measures Ψ and Ψ are adequate to represent the degree of departure from the MH model toward two maximum departures, *i.e.*, toward the (conditional) lower-left-marginal inhomogeneity or toward the (conditional) upper-right-marginal inhomogeneity.

8. Discussion

[6,7] considered the measures to represent the degree of departure from MH. The present paper has considered two types of maximum marginal inhomogeneity (*i.e.*, the lower-left-marginal inhomogeneity and the upper-right-marginal inhomogeneity). The measures in [6,7] take the value 1 in two types of maximum marginal inhomogeneity. The measures Ψ and Ψ in the present paper can distinguish these two kinds of maximum marginal inhomogeneity by the values -1 or 1 although the measures in [6,7] cannot distinguish them. Also the proposed measures can represent the degree of departure from MH when the ML or the EMH models, or the other asymmetry models hold. Therefore for the ordinal data, the proposed measures rather than those in [6,7] may be useful to represent the degree of departure from MH.

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