Open Journal of Statistics, **2011, 1, 185-198**
doi:10.4236/ojs.2011.13022 Published Online October 2011 (http://www.SciRP.org/journal/ojs)

Double Autocorrelation in Two Way Error Component Models

Jean Marcelin Bosson Brou[1], Eugene Kouassi[2], Kern O. Kymn[3]
[1]*Department of Economics*, *University of Cocody*, *Abidjan*, *Cote-d'Ivoire*
[2]*Resource Economics*, *West Virginia University*, *Morgantown*, *USA*
[3]*Division of Finance and Economics*, *West Virginia University*, *Morgantown*, *USA*
E-mail: *kern.kymn@mail.wvu.edu*
Received September 15, 2011; *revised October* 16, 2011; *revised October* 30, 2011

Abstract

In this paper, we extend the works by [1-5] accounting for autocorrelation both in the time specific effect as well as the remainder error term. Several transformations are proposed to circumvent the double autocorrelation problem in some specific cases. Estimation procedures are then derived.

Keywords: Two Way Random Effect Model, Double Autocorrelation, GLS, FGLS

1. Introduction

Following the works of [6], the regression model with error components or variance components has become a popular method for dealing with panel data. A summary of the main features of the model, together with a discussion of some applications, is available in [7-10] among others.

However, relatively little is known about the two way error component models in the presence of double autocorrelation, *i.e*, autocorrelation in the time specific effect and in the remainder error term as well.

This paper extends the works by [2-5] on the one-way random effect model in the presence of serial autocorrelation, and by [1] on the single autocorrelation two-way approach. It investigates some potential transformations to circumvent the double autocorrelation issue, along with some estimation procedures. In particular, we derive several transformations when the two disturbances follow various structures: from autoregressive and moving-average processes of order 1 to a general case of double serial correlation. We deduce several GLS estimators as well as their asymptotic properties and provide a FGLS version.

The remainder of this paper is organized as follows: Section 2 considers simple transformations on the presence of relatively manageable double autocorrelation structure. In Section 3, general transformations are considered when the double autocorrelation is more complex. GLS estimators are derived in Section 4. Asymptotic properties of the GLS estimators are considered in Section 5. Section 6 provides a FGLS counterpart approach. Finally, some concluding remarks appear in Section 7.

2. Simple Transformations

To circumvent the double autocorrelation issue, we first need to transform the model based on the variance-covariance matrix. The general regression model considered is $y_{it} = \beta_0 + x_{it}\beta + u_{it}$, $i = 1,\cdots,N$; $t = 1,\cdots,T$ where β_0 is the intercept and β is a $k\times 1$ vector of slope coefficients, x_{it} is a $1\times k$ row vector of explanatory variables which are uncorrelated with the usual two-way error components disturbances $u_{it} = \mu_i + \lambda_t + v_{it}$ (see [7]). In matrix form, we write $y = X\eta + u$.

2.1. When the Errors Follow AR(1) Structures

If the time specific term follows an AR(1) structure, $\lambda_t = \rho_\lambda \lambda_{t-1} + \varepsilon_t$, $|\rho_\lambda| < 1$, with $\varepsilon_t \sim IID(0,\sigma_\varepsilon^2)$, and the remainder error term also follow an AR(1) structure $v_{it} = \rho_v v_{i,t-1} + e_{it}$, $|\rho_v| < 1$, with $e_{it} \sim IID(0,\sigma_e^2)$, we can define two transformation matrices of dimensions $(T-2)\times(T-1)$ and $(T-1)\times T$ respectively,

$$C_\lambda = \begin{pmatrix} -\rho_\lambda & 1 & 0 & \cdots & 0 \\ 0 & -\rho_\lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\rho_\lambda & 1 \end{pmatrix} \text{ and}$$

$$C_\nu = \begin{pmatrix} -\rho_\nu & 1 & 0 & \cdots & 0 \\ 0 & -\rho_\nu & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\rho_\nu & 1 \end{pmatrix} \quad (1)$$

and since we have

$$C_\lambda C_\nu \nu_i = \begin{pmatrix} e_{i3} - \rho_\lambda e_{i2} \\ \vdots \\ e_{iT} - \rho_\lambda e_{i,T-1} \end{pmatrix}$$

and

$$C_\lambda C_\nu \lambda = \begin{pmatrix} \varepsilon_3 - \rho_\nu \varepsilon_2 \\ \vdots \\ \varepsilon_T - \rho_\nu \varepsilon_{T-1} \end{pmatrix} = \lambda^* \quad (2)$$

the transformed errors ν_i^* and λ^* follow two different MA(1) processes, of parameters ρ_λ and ρ_ν respectively. Thus, by applying the appropriate transformation matrices, the autoregressive error structure can be changed into a moving-average one. The only cost is the loss of the initial and first pseudo-differences, which has no serious consequence for a long time dimension. As a result, we focus on the MA(1) error structure.

2.2. When the Errors Follow MA(1) Structures

Here, the time specific term λ_t follows an MA(1) structure, $\lambda_t = \varepsilon_t - \rho_\lambda \varepsilon_{t-1}$, $|\rho_\lambda| < 1$ with $\varepsilon_t \sim IID(0, \sigma_\varepsilon^2)$ while the remainder error term, ν_{it} also follows an MA (1) structure, $\nu_{i,t} = e_{it} - \rho_\nu e_{i,t-1}$, $|\rho_\nu| < 1$ with $e_{it} \sim IID(0, \sigma_e^2)$. For convergence purpose and assuming normality, the initial values are defined

$$\nu_{i0} \sim N\left(0, \sigma_\nu^2 = \left(1 + \rho_\nu^2\right)\sigma_e^2\right)$$

and

$$\lambda_0 \sim N\left(0, \sigma_\lambda^2 = \left(1 + \rho_\lambda^2\right)\sigma_\varepsilon^2\right)$$

The variance-covariance matrix of the three components error term is given by,

$$\Sigma = \sigma_e^2 \left(I_N \otimes \Gamma_\nu\right) + \sigma_\mu^2 \left(I_N \otimes i_T i_T'\right) + \sigma_\varepsilon^2 \left(i_N i_N' \otimes \Gamma_\lambda\right) \quad (3)$$

where $\Gamma_\nu = \Gamma(\rho_\nu)$ and $\Gamma_\lambda = \Gamma(\rho_\lambda)$ are positive definite matrices of order T and where $\Gamma(.)$ is defined by $\Gamma(x) = \text{Toeplitz}\left(1 + x^2, -x, 0, \cdots, 0\right)$. The exact inverse of such matrices suggested by [11] and [1] does not involve the parameters ρ_ν and ρ_λ. Following [11], let P be the Pesaran orthogonal matrix whose t-th row is given by,

$$L_t = \sqrt{\frac{2}{T+1}} \times \left[\sin\left(\frac{t\pi}{T+1}\right), \sin\left(\frac{2t\pi}{T+1}\right), \cdots, \sin\left(\frac{Tt\pi}{T+1}\right)\right]$$

where

$$P\Gamma_\lambda P' = \Lambda, \quad \Lambda = \text{diag}\left(\Lambda_1, \cdots, \Lambda_T\right),$$

$$\Lambda_t = 1 + \rho_\lambda^2 - 2\rho_\lambda \cos\left(\frac{t\pi}{T+1}\right), \quad P\Gamma_\nu P' = D \quad \text{and}$$

$$D = diag\left(d_1, \cdots, d_t\right) \text{ with } d_t = 1 + \rho_\nu^2 - 2\rho_\nu \cos\left(\frac{t\pi}{T+1}\right).$$

Pre-multiplying the model by $(I_N \otimes P)$ yields the following variance-covariance matrix of $u^* = (I_N \otimes P)u$,

$$\Sigma^* = \sigma_e^2 \left(I_N \otimes D\right) + \sigma_\mu^2 \left(I_N \otimes i_T^\lambda i_T^{\lambda'}\right) + \sigma_\varepsilon^2 \left(i_N i_N' \otimes \Lambda\right) \quad (4)$$

where $i_T^\lambda = P i_T$.

3. General Transformations

We are now in the context of a general case of double autocorrelation issue and lead to a suitable error covariance matrix similar to Equation (4) and its inverse.

3.1. First Transformation

Let P_λ denote the matrix such that $P_\lambda \Gamma_\lambda P_\lambda' = I_T$. Such a matrix does exist for Γ_λ and is a positive-definite matrix. Transformation of the initial model $y = X\eta + u$ by $(I_N \otimes P_\lambda)$ yields

$$y^* = \left(I_N \otimes P_\lambda\right) y = X^* \eta + u^* \quad (5)$$

and the variance-covariance of the transformed errors is

$$\begin{aligned} \Sigma^* = E\left(u^* u^{*'}\right) &= \sigma_\nu^2 \left(I_N \otimes P_\lambda \Gamma_\nu P_\lambda'\right) \\ &+ \sigma_\mu^2 \left(I_N \otimes P_\lambda i_T i_T' P_\lambda'\right) + \sigma_\lambda^2 \left(i_N i_N' \otimes I_T\right) \end{aligned} \quad (6)$$

This transformation has removed the autocorrelation in the time-specific effect λ_t. Unfortunately, by doing so it has infected the $\nu_{it}s$ and worsened the initial correlation in the remainder disturbances. An additional “treatment” is therefore needed.

3.2. Second Transformation

We now consider an orthogonal matrix P and a diagonal matrix D such that $P\left(P_\lambda \Gamma_\nu P_\lambda'\right)P' = D$ (diagonalization of $P_\lambda \Gamma_\nu P_\lambda'$). Thus, applying a second transformation $(I_N \otimes P)$ yields,

$$y^{**} = \left(I_N \otimes P\right) y^* = X^{**}\eta + u^{**} \quad (7)$$

The underlying variance-covariance matrix of the errors is,

$$E\left(u^{**} u^{**'}\right) = \sigma_\nu^2 \left(I_N \otimes P\left(P_\lambda \Gamma_\nu P_\lambda'\right)P'\right)$$

$$+\sigma_\mu^2\left(I_N \otimes P\left(P_\lambda i_T i_T^{'} P_\lambda^{'}\right)P^{'}\right)+\sigma_\lambda^2\left(i_N i_N^{'} \otimes PP^{'}\right) \quad (8a)$$

or,

$$\Sigma^{**}=\frac{1}{\sigma^2}E\left(u^{**}u^{**'}\right)=\sigma_1^2\left(I_N\otimes D\right) +\sigma_2^2\left(I_N\otimes i_T^\lambda i_T^{\lambda'}\right)+\sigma_3^2\left(i_N i_N^{'}\otimes\Lambda\right) \quad (8b)$$

where $i_T^\lambda = PP_\lambda i_T$, $\Lambda = PP^{'}$, $P\left(P_\lambda \Gamma_v P_\lambda^{'}\right)P^{'} = D$,

$\sigma_1^2=\frac{\sigma_v^2}{\sigma^2}$, $\sigma_2^2=\frac{\sigma_\mu^2}{\sigma^2}$, and $\sigma_3^2=\frac{\sigma_\lambda^2}{\sigma^2}=1-\sigma_1^2-\sigma_2^2$,

if $\sigma^2=\sigma_v^2+\sigma_\mu^2+\sigma_\lambda^2$.

Here, because of the choice of matrices P_λ and P, we end up with $\Lambda = I_T$ since P is an orthogonal matrix. Generally speaking, Λ and D just need to have zero off-diagonal elements, *i.e.*, to be diagonal matrices. The double autocorrelation structure is thus absorbed, and one can easily accommodate with the non-spherical form of Σ^{**} by means of an accurate inversion process.

3.3. Computing the Inverse

The inverse of Σ^{**} is obtained using the procedure developed by [1]. After a bit of algebra, one gets

$$\left(\Sigma^{**}\right)^{-1}=\frac{1}{\sigma_1^2}\left(E_N\otimes K_T\right)+\frac{1}{d}\left(E_N\otimes L_T\right)+\frac{1}{N^2}\left(J_N\otimes S_T\right) \quad (9)$$

where

$$d=\left(i_T^{\lambda'}D^{-1}i_T^\lambda\right)\sigma_2^2+\sigma_1^2,\quad J_N=i_N i_N^{'},$$

$$E_N=I_N-\frac{J_N}{N},\quad K_T=D^{-1}-L_T,$$

$$L_T=\frac{1}{\left(i_T^{\lambda'}D^{-1}i_T^\lambda\right)}D^{-1}i_T^\lambda i_T^{\lambda'}D^{-1},$$

$$S_T=\left(\frac{1}{\sigma_1^2}S-\frac{\sigma_2^2}{\sigma_1^4 N+\sigma_1^2\sigma_2^2\left(i_T^{\lambda'}Si_T^\lambda\right)}Si_T^\lambda i_T^{\lambda'}S\right)$$

and

$$S=\mathrm{diag}\left(s_1,\cdots,s_T\right)\ \text{ with }\ s_T=\frac{N\sigma_1^2}{d_t\sigma_1^2+N\Lambda_t\sigma_3^2}.$$

Proof: (see the Appendix)

4. GLS Estimation

We begin with the definition of the estimator followed by its interpretation and weighted average property.

4.1. The GLS Estimator

Proposition 1:

The GLS estimator is,

$$\eta_{GLS}=\left(X^{**'}\left(\Sigma^{**}\right)^{-1}X^{**}\right)^{-1}X^{**'}\left(\Sigma^{**}\right)^{-1}y^{**} \quad (10)$$

Proof: (Straightforward)

4.2. Interpretation

In classical two-way regression models, [12,13] provide an interpretation of the GLS estimator, which is appealing in view of the sources of variation in sample data. In the straight line of their work, the GLS estimator may be viewed as obtained by pooling three uncorrelated estimators: the covariance estimator (or within estimator), the between-individual estimator and the within-individual estimator. They are the same as those suggested by [1] except for the last one which was labeled between-time estimator. We have,

1) The covariance estimator,

$$\eta_C=\left(X^{**'}A_1X^{**}\right)^{-1}X^{**'}A_1y^{**},$$

where $A_1=\left(E_N\otimes K_T\right)$;

2) The between-individual estimator,

$$\eta_B=\left(X^{**'}A_2X^{**}\right)^{-1}X^{**'}A_2y^{**},$$

where $A_2=\frac{1}{N^2}\left(J_N\otimes S_T\right)$ and,

3) The within-individual estimator,

$$\eta_T=\left(X^{**'}A_3X^{**}\right)^{-1}X^{**'}A_3y^{**},$$

where $A_3=\left(E_N\otimes L_T\right)$.

It is important to note that these estimators are obtained from some transformations of the regression Equation (7), *i.e.*,

$$y^{**}=\left(I_N\otimes P^{'}\right)y^*=X^{**}\eta+u^{**}.$$

The covariance estimator, η_C is obtained when Equation (7) is pre-multiplied by $M_1=\left(E_N\otimes K_T\right)=A_1$; the transformation annihilates the individual- and time-effects as well as the column of ones in the matrix of explanatory variables. It is equivalent to the within estimator in the classical two-way error component model (see [1-7]).

The between-individual estimator η_B comes from the transformation of Equation (7) by the matrix

$M_2 = \frac{1}{N}\left(i_N^{'} \otimes I_T\right)$. This is equivalent to averaging individual equations for each time period.

The within-individual estimator η_T is derived when Equation (7) is transformed by $M_3 = \left(E_N \otimes L_T\right) = A_3$. The presence of the idempotent matrix E_N indicates that this transformation wipes out the constant term as well as the time specific error term λ_t. However, the individual effect μ_i remains.

4.3. GLS as a Weighted Average Estimator

As in [14], the GLS estimator is a weighted average of the three estimators defined above.

Proposition 2:

$$\eta_{GLS} = F_C\eta_C + F_B\eta_B + F_T\eta_T \quad (11)$$

with,

$$F_C = \frac{1}{\sigma_1^2}\left(X^{**'}\left(\Sigma^{**}\right)^{-1}X^{**}\right)^{-1}X^{**'}A_1X^{**},$$

$$F_B = \left(X^{**'}\left(\Sigma^{**}\right)^{-1}X^{**}\right)^{-1}X^{**'}A_2X^{**},$$

and

$$F_T = \frac{1}{d}\left(X^{**'}\left(\Sigma^{**}\right)^{-1}X^{**}\right)^{-1}X^{**'}A_3X^{**} = I - F_C - F_B$$

Proof:

From Equation (10), it comes that

$$X^{**'}\left(\Sigma^{**}\right)^{-1}y^{**} = \left(X^{**'}\left(\Sigma^{**}\right)^{-1}X^{**}\right)\eta_{GLS}$$

with

$$X^{**'}\left(\Sigma^{**}\right)^{-1}y^{**} = \frac{1}{\sigma_1^2}X^{**'}A_1y^{**} + X^{**'}A_2y^{**} + \frac{1}{d}X^{**'}A_3y^{**}$$

By definition, the estimators η_C, η_B and η_T are respectively such that

$$X^{**'}A_1y^{**} = \left(X^{**'}A_1X^{**}\right)\eta_C,$$

$$X^{**'}A_2y^{**} = \left(X^{**'}A_2X^{**}\right)\eta_B,$$

and

$$X^{**'}A_3y^{**} = \left(X^{**'}A_3X^{**}\right)\eta_T.$$

Therefore,

$$X^{**'}\left(\Sigma^{**}\right)^{-1}y^{**} = \frac{1}{\sigma_1^2}\left(X^{**'}A_1X^{**}\right)\eta_C + \left(X^{**'}A_2X^{**}\right)\eta_B + \frac{1}{d}\left(X^{**'}A_3X^{**}\right)\eta_T$$

Or,

$$\left(X^{**'}\left(\Sigma^{**}\right)^{-1}X^{**}\right)\eta_{GLS} = \frac{1}{\sigma_1^2}\left(X^{**'}A_1X^{**}\right)\eta_C + \left(X^{**'}A_2X^{**}\right)\eta_B + \frac{1}{d}\left(X^{**'}A_3X^{**}\right)\eta_T$$

Thus,

$$\begin{aligned}\eta_{GLS} &= \frac{1}{\sigma_1^2}\left(X^{**'}\left(\Sigma^{**}\right)^{-1}X^{**}\right)^{-1}\left(X^{**'}A_1X^{**}\right)\eta_C \\ &+\left(X^{**'}\left(\Sigma^{**}\right)^{-1}X^{**}\right)^{-1}\left(X^{**'}A_2X^{**}\right)\eta_B \\ &+\frac{1}{d}\left(X^{**'}\left(\Sigma^{**}\right)^{-1}X^{**}\right)^{-1}\left(X^{**'}A_3X^{**}\right)\eta_T \\ &= F_C\eta_C + F_B\eta_B + F_T\eta_T\end{aligned}$$

with F_C, F_B and F_T defined according to Equation (11).

We should also note that the three estimators η_C, η_B and η_T are uncorrelated. In fact,

$$A_1\Sigma^{**}A_2 = \frac{\sigma_1^2}{N}\left(E_NJ_N \otimes K_TDS_T\right) = 0$$

and

$$A_1\Sigma^{**}A_3 = \sigma_1^2\left(E_N \otimes K_TDL_T\right) = 0$$

because $K_Ti_T^{\lambda} = 0 = E_Ni_N$, while $A_2\Sigma^{**}A_3 = 0$ since $J_NE_N = 0$. As a result,

$$\operatorname{cov}\left(\eta_C, \eta_B\right) = \operatorname{cov}\left(\eta_C, \eta_T\right) = \operatorname{cov}\left(\eta_B, \eta_T\right) = 0 \quad (12)$$

Moreover, following [1], the fact that

$$\begin{aligned}&\operatorname{rank}\left(M_1\right) + \operatorname{rank}\left(M_2\right) + \operatorname{rank}\left(M_3\right) \\ &= \left(N-1\right)\left(T-1\right) + T + \left(N-1\right) = NT\end{aligned} \quad (13)$$

gives evidence on the use of all available information from the sample. The estimators η_C, η_B and η_T together use up the entire set of information to build the GLS estimator η_{GLS} with no loss at all.

5. Asymptotic Properties

Under regular assumptions, the GLS and the three pseudo estimators of the coefficient vector, say η_{GLS}, η_C, η_B and η_T are all consistent and asymptotically equivalent. It is a result similar to the one obtained in the classical two-way error component model (see [15]).

5.1. Assumptions

We assume that the $x_{it}s$ are weakly non-stochastic, *i.e.* do not repeat in repeated samples. We also state that the following matrices exist and are positive definite:

$$(a_1)\,\text{plim}\left(\frac{X^{**'}A_1X^{**}}{NT}\right)=\text{plim}\left(\frac{X^{**'}\left(E_N\otimes K_T\right)X^{**}}{NT}\right)$$

for the first transformation;

$$(b_1)\,\text{plim}\left(\frac{X^{**'}A_2X^{**}}{T}\right)=\text{plim}\left(\frac{X^{**'}\left(\frac{1}{N^2}J_N\otimes S_T\right)X^{**}}{T}\right)$$

for the second transformation; and

$$(c_1)\,\text{plim}\left(\frac{X^{**'}A_3X^{**}}{NT}\right)=\text{plim}\left(\frac{X^{**'}\left(E_N\otimes L_T\right)X^{**}}{NT}\right)$$

for the third transformation. Furthermore, in the straight line of [1], we also assume that,

$$(a_2)\,\text{plim}\left(\frac{X^{**'}A_1u^{**}}{NT}\right)=\text{plim}\left(\frac{X^{**'}\left(E_N\otimes K_T\right)u^{**}}{NT}\right)=0$$

for the first transformation;

$$(b_2)\,\text{plim}\left(\frac{X^{**'}A_2u^{**}}{T}\right)=\text{plim}\left(\frac{X^{**'}\left(\frac{1}{N^2}J_N\otimes S_T\right)u^{**}}{T}\right)=0$$

for the second transformation;

$$(c_2)\,\text{plim}\left(\frac{X^{**'}A_3u^{**}}{NT}\right)=\text{plim}\left(\frac{X^{**'}\left(E_N\otimes L_T\right)u^{**}}{NT}\right)=0$$

for the third transformation. In addition, $\lim_{T\to\infty}\left(i_T^{\lambda'}D^{-1}i_T^{\lambda}\right)=\infty$, so that the variance-components quantity $\left(i_T^{\lambda'}D^{-1}i_T^{\lambda}\right)\sigma_2^2+\sigma_1^2$ denotes by d remains infinite as $T\to\infty$. The limits and probabilities are taken as $T\to\infty$ and $N\to\infty$. All along this section, following [1], we consider the "usual" assumptions regarding the error vector u^{**}, as stated in [16] and [17], which ensures the asymptotic normality.

5.2. Asymptotic Property of the Covariance Estimator

Proposition 3:

The covariance estimator η_C is consistent.

Proof:

Since,

$$\eta_C=\eta+\left(X^{**'}\left(E_N\otimes K_T\right)X^{**}\right)^{-1}X^{**'}\left(E_N\otimes K_T\right)u^{**}$$

Hence,

$$\text{plim}\left(\eta_C-\eta\right)=\left(\text{plim}\left(\frac{X^{**'}\left(E_N\otimes K_T\right)X^{**}}{NT}\right)\right)^{-1}\times\text{plim}\left(\frac{X^{**'}\left(E_N\otimes K_T\right)u^{**}}{NT}\right)=0$$

Making use of assumptions (a1) and (a2), we establish the consistency of the covariance estimator, $\text{plim}\left(\eta_C\right)=\eta$.

Proposition 4:

The covariance estimator η_C has an asymptotic normal distribution given by,

$$\eta_C\xrightarrow{a}N\left(\eta,\frac{1}{NT\sigma_1^2}\left(\text{plim}\left(\frac{X^{**'}\left(E_N\otimes K_T\right)X^{**}}{NT}\right)\right)^{-1}\right)\tag{14}$$

Proof:

Under the M_1-transformation, we have

$$E\left[\left(E_N\otimes K_T\right)u^{**}\right]=\left(E_N\otimes K_T\right)E\left(u^{**}\right)=0.$$

Moreover its variance is given by $V\left[\left(E_N\otimes K_T\right)u^{**}\right]=\sigma_1^2\left(E_N\otimes K_T\right)$ and its inverse is equal to $\frac{1}{\sigma_1^2}\left(E_N\otimes D\right)$ while assumption (a2) states the absence of correlation between regressors and disturbances under the M_1 transformation. We have

$$\frac{X^{**'}\left(E_N\otimes K_T\right)u^{**}}{\sqrt{NT}}\xrightarrow{d}N\left(0,\sigma_1^2\text{plim}\left(\frac{X^{**'}\left(E_N\otimes K_T\right)X^{**}}{NT}\right)\right)\tag{15}$$

and,

$$\sqrt{NT}\left(\eta_C-\eta\right)=\left(\frac{X^{**'}\left(E_N\otimes K_T\right)X^{**}}{NT}\right)^{-1}\left(\frac{X^{**'}\left(E_N\otimes K_T\right)u^{**}}{\sqrt{NT}}\right)$$

from which we deduce that

$$\sqrt{NT}\left(\eta_C-\eta\right)\xrightarrow{d}N\left(0,\sigma_1^2\left(\text{plim}\left(\frac{X^{**'}\left(E_N\otimes K_T\right)X^{**}}{NT}\right)\right)^{-1}\right).$$

Thus, the asymptotic normality of the covariance estimator immediately follows,

$$\eta_C\xrightarrow{a}N\left(\eta,\frac{1}{NT\sigma_1^2}\left(\text{plim}\left(\frac{X^{**'}\left(E_N\otimes K_T\right)X^{**}}{NT}\right)\right)^{-1}\right).$$

5.3. Asymptotic Property of the Between-Time Estimator

Proposition 5:

The between time estimator η_B is consistent.

Proof:

Since,

$$\eta_B = \eta + \left(X^{**'} A_2 X^{**}\right)^{-1} X^{**'} A_2 u^{**}$$

Hence, according to assumptions (b1) and (b2),

$$\operatorname{plim}\left(\eta_B - \eta\right) = \left(\operatorname{plim}\left(\frac{X^{**'}\left(\frac{J_N}{N^2}\otimes S_T\right)X^{**}}{T}\right)\right)^{-1} \times \operatorname{plim}\left(\frac{X^{**'}\left(\frac{J_N}{N^2}\otimes S_T\right)u^{**}}{T}\right) = 0$$

Making use of assumptions (b1) and (b2), we establish the consistency of the between time estimator,

$$\operatorname{plim}\left(\eta_B\right) = \eta .$$

Proposition 6:

The between-time estimator η_B has an asymptotic normal distribution given by,

$$\eta_B \xrightarrow{a} N\left(\eta, \frac{1}{T}\left(p\lim\left(\frac{X^{**'}\left(\frac{1}{N^2}J_N\otimes S_T\right)X^{**}}{T}\right)\right)^{-1}\right) \quad (16)$$

Proof:

Under the M_2-transformation, we get

$$E\left[\left(\frac{i_N^{'}}{N}\otimes I_T\right)u^{**}\right] = \left(\frac{i_N^{'}}{N}\otimes I_T\right)E\left(u^{**}\right) = 0$$

The variance of this error term is written as

$$V\left[\left(\frac{i_N^{'}}{N}\otimes I_T\right)u^{**}\right] = \sigma_1^2 S^{-1} + \frac{\sigma_2^2}{N}\left(i_T \lambda i_T^{\lambda'}\right)$$

Its inverse is S_T. Again, assumption (b2) states the absence of correlation between regressors and disturbances under the M_2 transformation. We get

$$\frac{X^{**'}\left(\frac{J_N}{N^2}\otimes S_T\right)u^{**}}{T} \xrightarrow{d} N\left(0, \operatorname{plim}\left(\frac{X^{**'}\left(\frac{J_N}{N^2}\otimes S_T\right)X^{**}}{T}\right)\right)$$

In addition, we have

$$\sqrt{T}\left(\eta_B - \eta\right) = \left(\frac{X^{**'}\left(\frac{J_N}{N^2}\otimes S_T\right)X^{**}}{T}\right)^{-1} \times \left(\frac{X^{**'}\left(\frac{J_N}{N^2}\otimes S_T\right)u^{**}}{\sqrt{T}}\right)$$

from which we deduce that

$$\sqrt{T}\left(\eta_B - \eta\right) \xrightarrow{d} N\left(0, \left(\operatorname{plim}\left(\frac{X^{**'}\left(\frac{J_N}{N^2}\otimes S_T\right)X^{**}}{T}\right)\right)^{-1}\right)$$

Thus, the asymptotic normality of the between-time estimator immediately follows,

$$\eta_B \xrightarrow{a} N\left(\eta, \frac{1}{T}\left(\operatorname{plim}\left(\frac{X^{**'}\left(\frac{1}{N^2}J_N\otimes S_T\right)X^{**}}{T}\right)\right)^{-1}\right).$$

5.4. Asymptotic Property of the Within-Individual Estimator

Proposition 7:

The within individual estimator η_T is a consistent estimator.

Proof:

Since,

$$\eta_T = \eta + \left(X^{**'} A_3 X^{**}\right)^{-1} X^{**'} A_3 u^{**}$$

Hence,

$$\operatorname{plim}\left(\eta_T - \eta\right) = \left(\operatorname{plim}\left(\frac{X^{**'}\left(E_N\otimes L_T\right)X^{**}}{NT}\right)\right)^{-1} \times \operatorname{plim}\left(\frac{X^{**'}\left(E_N\otimes L_T\right)u^{**}}{NT}\right) = 0$$

Making use of assumptions (c1) and (c2), we establish the consistency of the covariance estimator,

$$\text{plim}(\eta_T) = \eta$$

Proposition 8:

The within individual estimator η_T has an asymptotic normal distribution given by,

$$\eta_T \xrightarrow{a} N\left(\eta, \frac{d}{NT}\left(\text{plim}\left(\frac{X^{**'}(E_N \otimes L_T)X^{**}}{NT}\right)\right)^{-1}\right) \quad (17)$$

Proof:

Under the M_3-transformation, we obtain

$$E\left[(E_N \otimes L_T)u^{**}\right] = (E_N \otimes L_T)E(u^{**}) = 0$$

The variance of $(E_N \otimes L_T)u^{**}$ is obtained as

$$V\left[(E_N \otimes L_T)u^{**}\right] = d(E_N \otimes L_T)$$

The inverse of this matrix is $\frac{1}{d}(E_N \otimes L_T)$. Assumption (c2) states the absence of correlation between regressors and disturbances under the M_3 transformation. We have

$$\frac{X^{**'}(E_N \otimes L_T)u^{**}}{\sqrt{NT}} \xrightarrow{d} N\left(0, d\ \text{plim}\left(\frac{X^{**'}(E_N \otimes L_T)X^{**}}{NT}\right)\right)$$

and,

$$\sqrt{NT}(\eta_T - \eta) = \left(\frac{X^{**'}(E_N \otimes L_T)X^{**}}{NT}\right)^{-1} \times \left(\frac{X^{**'}(E_N \otimes L_T)u^{**}}{\sqrt{NT}}\right)$$

from which we deduce that

$$\sqrt{NT}(\eta_T - \eta) \xrightarrow{d} N\left(0, d\left(\text{plim}\left(\frac{X^{**'}(E_N \otimes L_T)X^{**}}{NT}\right)\right)^{-1}\right)$$

Thus, the asymptotic normality of the within individual estimator immediately follows,

$$\eta_T \xrightarrow{a} N\left(\eta, \frac{d}{NT}\left(\text{plim}\left(\frac{X^{**'}(E_N \otimes L_T)X^{**}}{NT}\right)\right)^{-1}\right)$$

5.5. Asymptotic Property of the GLS Estimator

Proposition 9:

The GLS estimator η_{GLS} is asymptotically equivalent to the covariance estimator η_C and therefore,

$$\eta_{GLS} \xrightarrow{a} N\left(\eta, \frac{1}{NT\sigma_1^2}\left(p\lim\left(\frac{X^{**'}(E_N \otimes K_T)X^{**}}{NT}\right)\right)^{-1}\right) \quad (18)$$

Proof:

From Equation (10), we get

$$\sqrt{NT}(\eta_{GLS} - \eta) = \left(\frac{X^{**'}(\Sigma^{**})^{-1}X^{**}}{NT}\right)^{-1}\left(\frac{X^{**'}(\Sigma^{**})^{-1}u^{**}}{\sqrt{NT}}\right)$$

On the one hand, we have

$$\frac{X^{**'}(\Sigma^{**})^{-1}X^{**}}{NT} = \frac{1}{\sigma_1^2}\frac{X^{**'}(E_N \otimes K_T)X^{**}}{NT} + \frac{1}{d}\frac{X^{**'}(E_N \otimes L_T)X^{**}}{NT} + \frac{1}{N}\frac{X^{**'}(J_N \otimes S_T)X^{**}}{N^2T}$$

where $d = \left(i_T^{\lambda'} D^{-1} i_T^{\lambda}\right)\sigma_2^2 + \sigma_1^2 \to \infty$, as $T \to \infty$. Therefore, from assumption (a1), we find that $\frac{1}{d}\frac{X^{**'}(E_N \otimes L_T)X^{**}}{NT} \to 0$, when $N, T \to \infty$. Likewise, assumption (a2) leads us to $\frac{1}{N}\frac{X^{**'}(J_N \otimes S_T)X^{**}}{N^2T} \to 0$, when $N, T \to \infty$. Hence,

$$\text{plim}\left(\frac{X^{**'}(\Sigma^{**})^{-1}X^{**}}{NT}\right) = \frac{1}{\sigma_1^2}\text{plim}\left(\frac{X^{**'}(E_N \otimes K_T)X^{**}}{NT}\right)$$

On the other hand, we can write

$$\frac{X^{**'}(\Sigma^{**})^{-1}u^{**}}{\sqrt{NT}} = \frac{1}{\sigma_1^2}\frac{X^{**'}(E_N \otimes K_T)u^{**}}{\sqrt{NT}} + \frac{1}{d}\frac{X^{**'}(E_N \otimes L_T)u^{**}}{\sqrt{NT}} + \frac{1}{N^2}\frac{X^{**'}(J_N \otimes S_T)u^{**}}{\sqrt{NT}}.$$

Under the M_1 and M_2 transformations, we get

$$\text{plim}\left(\frac{1}{d}\frac{X^{**'}(E_N \otimes L_T)u^{**}}{\sqrt{NT}}\right) = \text{plim}\left(\frac{1}{N^2}\frac{X^{**'}(J_N \otimes S_T)u^{**}}{\sqrt{NT}}\right) = 0$$

leading to

$$\text{plim}\left(\frac{X^{**'}\left(\Sigma^{**}\right)^{-1}u^{**}}{\sqrt{NT}}\right)=\frac{1}{\sigma_1^2}\text{plim}\left(\frac{X^{**'}\left(E_N\otimes K_T\right)u^{**}}{\sqrt{NT}}\right).$$

As a result,

$$\begin{aligned}&\text{plim}\left[\sqrt{NT}\left(\eta_{GLS}-\eta\right)\right]\\&=\text{plim}\left[\left(\frac{X^{**'}\left(E_N\otimes K_T\right)X^{**}}{NT}\right)^{-1}\right]\\&\times\text{plim}\left[\left(\frac{X^{**'}\left(E_N\otimes K_T\right)u^{**}}{\sqrt{NT}}\right)\right]\end{aligned}$$

i.e.,

$$\text{plim}\left[\sqrt{NT}\left(\eta_{GLS}-\eta\right)\right]=\text{plim}\left[\sqrt{NT}\left(\eta_C-\eta\right)\right]$$

Finally, $\sqrt{NT}\left(\eta_{GLS}-\eta\right)$ has the same limiting distribution as $\sqrt{NT}\left(\eta_C-\eta\right)$. This shows the asymptotic equivalence of the two estimators η_{GLS} and η_C. We then deduce that,

$$\eta_{GLS}\xrightarrow{a}N\left(\eta,\frac{1}{NT\sigma_1^2}\left(\text{plim}\left(\frac{X^{**'}\left(E_N\otimes K_T\right)X^{**}}{NT}\right)\right)^{-1}\right).$$

Thus, the GLS estimator suggested under the double autocorrelation error structure has the desired asymptotic properties.

6. FGLS Estimation

In practice, the variance-covariance matrix is unknown, as well as all the parameters involved in its determination. Therefore, a FGLS approach is required. The method used consists in removing the time specific effect to obtain a one-way error component model where only ν_{it} carries the serial correlation (see [18] and [3]). This method has been directly applied to AR(1) and MA(1) processes in separate subsections.

6.1. Feasible Double AR(1) Model

We assume that $\nu_{it}=\rho_\nu\nu_{i,t-1}+e_{it}$, $\lambda_t=\rho_\lambda\lambda_{t-1}+\varepsilon_t$, $\left|\rho_\nu\right|<1$, $\left|\rho_\lambda\right|<1$, $e_{it}\sim IID\left(0,\sigma_e^2\right)$, $\varepsilon_t\sim IID\left(0,\sigma_\varepsilon^2\right)$. The within error term is,

$$\tilde{u}=\left(E_N\otimes I_T\right)u=\left(E_N\mu\otimes i_T\right)+\left(E_N\otimes I_T\right)\nu=\tilde{\mu}\otimes i_T+\tilde{\nu}\tag{19}$$

The associated variance-covariance matrix is,

$$\tilde{\Sigma}=E\left(\tilde{u}\tilde{u}'\right)=\sigma_\mu^2\left(E_N\otimes i_Ti_T'\right)+\sigma_\nu^2\left(E_N\otimes\Gamma_\nu\right)\tag{20}$$

Since ν_{it} follows an AR(1) process of parameter ρ_ν, we define the matrix C_ν as the familiar [19] transformation matrix with parameter ρ_ν. This matrix is such that,

$$C\left(\sigma_\nu^2\Gamma_\nu\right)C'=\left(1-\rho_\nu^2\right)\sigma_\nu^2I_T=\sigma_e^2I_T$$

The resulting GLS estimator is given by

$$\eta_W=\left(X^{*'}X^*\right)^{-1}X^{*'}y^*\tag{21}$$

where $y^*=\left(I_N\otimes C\right)\tilde{y}$ and $X^*=\left(I_N\otimes C\right)\tilde{X}$.
The covariance matrix of $u^*=\left(I_N\otimes C\right)\tilde{u}$, using [20] trick, is

$$\Sigma^*=\left(\sigma_\mu^2\left(1-\rho_\nu\right)d_\alpha^2+\sigma_e^2\right)\left(E_N\otimes\bar{J}_T^\nu\right)+\sigma_e^2\left(E_N\otimes E_T^\nu\right)\tag{22}$$

where

$$\begin{aligned}i_T^\nu&=Ci_T=\left(1-\rho_\nu\right)\left(\sqrt{\frac{1+\rho_\nu}{1-\rho_\nu}}\quad 1\quad\cdots\quad 1\right)'\\&=\left(1-\rho_\nu\right)\left(\alpha\quad 1\quad\cdots\quad 1\right)'=\left(1-\rho_\nu\right)i_T^\alpha\end{aligned}$$

$$\bar{J}_T^\nu=\frac{1}{d_\nu^2}i_T^\nu i_T^{\nu'},\quad E_T^\nu=I_T-\bar{J}_T^\nu$$

and

$$\begin{aligned}d_\nu^2&=i_T^{\nu'}i_T^\nu=\left(1-\rho_\nu\right)^2i_T^{\alpha'}i_T^\alpha\\&=\left(1-\rho_\nu\right)^2\left[\alpha^2+\left(T-1\right)\right]=\left(1-\rho_\nu\right)^2d_\alpha^2\end{aligned}$$

Following [21], another GLS estimator can be derived. We label this estimator the within-type estimator and is given by

$$\eta_{WGLS}=\left(X^{**'}X^{**}\right)^{-1}X^{**'}y^{**}\tag{23}$$

with $y^{**}=\hat{\sigma}_\nu\hat{\Sigma}^{*-\frac{1}{2}}y^*$ and $X^{**}=\hat{\sigma}_\nu\hat{\Sigma}^{*-\frac{1}{2}}X^*$. In order to get the estimates of numerous parameters involved in the model, we first need an estimate of the correlation coefficient ρ_ν. The autocorrelation function of the error term $\tilde{u}$ is given by

$$\tilde{\gamma}\left(h\right)=E\left(\tilde{u}_{it}\tilde{u}_{i,t-h}\right)=\left(\frac{N-1}{N}\right)\left(\sigma_\mu^2+\rho_\nu^h\sigma_\nu^2\right)\quad\text{for } h=0,1,\cdots,t\tag{24}$$

We deduce from it that $\rho_\nu=\dfrac{\tilde{\gamma}(1)-\tilde{\gamma}(2)}{\tilde{\gamma}(0)-\tilde{\gamma}(1)}$. It then leads to a convergent estimator of ρ_ν (see [7]), *i.e.*,

$$\hat{\rho}_\nu = \frac{\hat{\hat{\gamma}}(1) - \hat{\hat{\gamma}}(2)}{\hat{\hat{\gamma}}(0) - \hat{\hat{\gamma}}(1)}$$

where $\hat{\hat{\gamma}}(h) = \frac{1}{N(T-h)} \sum_{i=1}^{N} \sum_{t=h+1}^{T} \hat{\hat{u}}_{it} \hat{\hat{u}}_{i,t-h}$ with $\hat{\hat{u}}_{it}$ defined as the OLS residuals of the within equation $\tilde{y} = \tilde{X}\eta + \tilde{u}$. Hence, we get

$$\hat{\alpha} = \sqrt{\frac{1+\hat{\rho}_\nu}{1-\hat{\rho}_\nu}} \quad (25a)$$

and

$$\hat{d}_\nu^2 = \left(1-\hat{\rho}_\nu\right)^2 \left[\hat{\alpha}^2 + (T-1)\right] \quad (25b)$$

Furthermore, the BQU estimate of σ_e^2 is also available as

$$\hat{\sigma}_e^2 = \frac{\hat{u}^{*'} \left(E_N \otimes E_T^\nu\right) \hat{u}^*}{(N-1)(T-1)} \quad (26)$$

$\hat{u}^*$ being the OLS estimate of u^*. As a consequence, we get

$$\hat{\sigma}_\nu^2 = \frac{\hat{\sigma}_e^2}{1-\hat{\rho}_\nu^2} \quad (27a)$$

and

$$\hat{\sigma}_\mu^2 = \left(\frac{N}{N-1}\right) \hat{\hat{\gamma}}(0) - \hat{\sigma}_\nu^2 \quad (27b)$$

We now need to find $\hat{\sigma}_\lambda^2$ and $\hat{\rho}_\lambda$. The autocovariance function of the initial error term u is given

$$\gamma(h) = \rho_\nu^h \sigma_\nu^2 + \sigma_\mu^2 + \rho_\nu^h \sigma_\lambda^2 = \left(\frac{N}{N-1}\right) \tilde{\gamma}(h) + \rho_\nu^h \sigma_\lambda^2$$

for $h = 0, 1, \cdots, t$.

It comes that,

$$\hat{\sigma}_\lambda^2 = \hat{\gamma}(0) - \left(\frac{N}{N-1}\right) \hat{\hat{\gamma}}(0) = \hat{\gamma}(0) - \hat{\sigma}_\mu^2 - \hat{\sigma}_\nu^2 \quad (28)$$

We immediately deduce a convergent estimator of the second correlation coefficient, *i.e.*,

$$\hat{\rho}_\lambda = \frac{\left[\hat{\gamma}(1) - \hat{\gamma}(2)\right] - \left(\frac{N}{N-1}\right)\left[\hat{\hat{\gamma}}(1) - \hat{\hat{\gamma}}(2)\right]}{\left[\hat{\gamma}(0) - \hat{\gamma}(1)\right] - \left(\frac{N}{N-1}\right)\left[\hat{\hat{\gamma}}(0) - \hat{\hat{\gamma}}(1)\right]} \quad (29)$$

where $\hat{\hat{\gamma}}(h) = \frac{1}{N(T-h)} \sum_{i=1}^{N} \sum_{t=h+1}^{T} \hat{\hat{u}}_{it} \hat{\hat{u}}_{i,t-h}$ with $\hat{u}_{it}$ denoting the OLS residuals of $y = X\eta + u$. The variances σ_e^2 is estimated by,

$$\hat{\sigma}_e^2 = \left(1-\hat{\rho}_\lambda^2\right)\hat{\sigma}_\lambda^2 \quad (30)$$

In addition to the GLS estimators mentioned in Section 4, other GLS estimators such as the within estimator η_W and the within-type estimator η_{WGLS} can all be performed as well. Actually, the knowledge of the AR(1) parameters ρ_ν and ρ_λ entitles us to build the matrices involved in the determination of $\left(\Sigma^{**}\right)^{-1}$, say matrices C_ν, C_λ, C, P, Λ, D, K_T, L_T, S and S_T.

6.2. Feasible Double MA(1) Model

We now state that $\nu_{it} = e_{it} - \rho_\nu e_{i,t-1}$, with $|\rho_\nu| < 1$ and $e_{it} \sim IID\left(0, \sigma_e^2\right)$. Again, deviations from individual means lead to the model

$$\tilde{y} = \tilde{X}\eta + \tilde{u} \quad \text{with} \quad \tilde{u}_{it} = \tilde{\mu}_i + \tilde{\nu}_{it}.$$

The variance-covariance matrix of $\tilde{u}$ is still given by Equation (20), with now

$$\Gamma_\nu = \text{Toeplitz}\left(1, r_\nu = \frac{-\rho_\nu}{1+\rho_\nu^2}, 0, \cdots, 0\right) \quad (31)$$

Here, we set $C = C_T$, C_T denoting the correlation correction matrix as defined by [8] in their orthogonalizing algorithm. We then transform the within model by $\left(I_N \otimes C\right)$. The new error term u^* has the following covariance matrix,

$$\Sigma^* = \left(\sigma_\mu^2 d_\nu^2 + \sigma_\nu^2\right)\left(E_N \otimes \bar{J}_T^\nu\right) + \sigma_\nu^2 \left(E_N \otimes E_T^\nu\right) \quad (32)$$

Because of the moving average nature of the process, linear estimation of the correlation parameter ρ_ν is not easily obtainable. Instead, $r_\nu = \frac{-\rho_\nu}{1+\rho_\nu^2}$ proves useful. The autocorrelation function of the within error term $\tilde{u}_{it}$ is given by,

$$\tilde{\gamma}(h) = E\left(\tilde{u}_{it}\tilde{u}_{i,t-h}\right) = \left(\frac{N-1}{N}\right)\left[\sigma_\mu^2 + \gamma_\nu(h)\right] \quad (33)$$
$$\text{for } h = 0, 1, \cdots, t$$

with $\gamma_\nu(h)$ denoting the autocovariance function of $\tilde{\nu}_{it}$. As a consequence, $\hat{\sigma}_\mu^2 = \left(\frac{N}{N-1}\right)\hat{\hat{\gamma}}(j)$ for some $j \geq 2$ and

$$\hat{\sigma}_\nu^2 = \hat{\gamma}_\nu(0) = \left(\frac{N}{N-1}\right)\hat{\hat{\gamma}}(0) - \hat{\sigma}_\mu^2 \quad (34)$$

where $\hat{\hat{\gamma}}(h) = \frac{1}{N(T-h)} \sum_{i=1}^{N} \sum_{t=h+1}^{T} \hat{\hat{u}}_{it} \hat{\hat{u}}_{i,t-h}$ is the empirical autocovariance function and $\hat{\hat{u}}_{it}s$ are the OLS residuals of the within equation. We also get, for some $j \geq 2$,

$$\hat{r}_{\nu} = \frac{N}{(N-1)\hat{\sigma}_{\nu}^{2}} \left[\hat{\hat{\gamma}}(1) - \hat{\hat{\gamma}}(j) \right] \quad (35)$$

We then apply the [8] matrix C_T to the data (for instance to the within transformed dependent vector $\tilde{y}$). Moreover, C_T will be applied to the vector of constants to get estimates of the $\alpha_t s$. We have, in the straight line of [8], the following steps:

Step 1: Compute $y_{i1}^{*} = \frac{y_{i1}}{\sqrt{\hat{g}_{\nu,1}}}$ and

$$y_{it}^{*} = \frac{y_{it} - \frac{\hat{r}_{\nu} y_{i,t-1}^{*}}{\sqrt{\hat{g}_{\nu,t-1}}}}{\sqrt{\hat{g}_{\nu,t}}} \quad \text{for } t = 2, \cdots, T$$

where $\hat{g}_{\nu,t} = 1 - \frac{\hat{r}_{\nu}^{2}}{\hat{g}_{\nu,t-1}}$ for $t = 2, \cdots, T$.

Step 2: Compute $y^{**} = \hat{\sigma}_{\nu} \hat{\Sigma}^{*-\frac{1}{2}} y^{*}$ knowing that $i_T^{\nu} = C_T i_T = (\alpha_1\ \alpha_2\ \cdots\ \alpha_T)'$. The estimates of the $\alpha_t s$ are obtained as $\hat{\alpha}_1 = 1$ and

$$\hat{\alpha}_t = \frac{1 - \frac{\hat{r}_{\nu}}{\sqrt{\hat{g}_{\nu,t-1}}}}{\sqrt{\hat{g}_{\nu,t}}} \quad \text{for } t = 2, \cdots, T.$$

We then obtain the estimate of d_{ν}^{2} as $\hat{d}_{\nu}^{2} = \sum_{t=1}^{T} \hat{\alpha}_t^{2}$.

The autocovariance function $\gamma(h) = E(u_{it} u_{i,t-h}) = \gamma_{\nu}(h) + \sigma_{\mu}^{2} + \gamma_{\lambda}(h)$ of the initial composite error term u and its empirical counterpart $\gamma(h) = \frac{1}{N(T-h)} \sum_{i=1}^{N} \sum_{t=h+1}^{T} \hat{u}_{it} \hat{u}_{i,t-h}$, ($\hat{u}_{it}$ being the OLS residuals of the initial two-way model) permit the estimation of $\hat{\sigma}_{\lambda}^{2}$ and $\hat{r}_{\lambda}$,

$$\hat{\sigma}_{\lambda}^{2} = \hat{\gamma}(0) - \hat{\sigma}_{\nu}^{2} - \hat{\sigma}_{\mu}^{2} \quad (36a)$$

and

$$\hat{r}_{\lambda} = \frac{\hat{\gamma}(1) - \hat{r}_{\nu} \sigma_{\nu}^{2} - \hat{\sigma}_{\mu}^{2}}{\hat{\sigma}_{\lambda}^{2}} \quad (36b)$$

The within estimator η_W and the within-type one η_{WGLS} are now obtainable. However, the GLS estimator η_{GLS} can be estimated, provided the MA(1) parameters ρ_{ν} and ρ_{λ} are known, especially under the conditions $\Delta_{\lambda} = 1 - 4\hat{r}_{\lambda}^{2} \geq 0$ and $\Delta_{\nu} = 1 - 4\hat{r}_{\nu}^{2} \geq 0$. In other words, the estimates $\hat{r}_{\lambda}$ and $\hat{r}_{\nu}$ should both lie inside the open interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ as a pre-requisite to a direct estimation of η_{GLS}, η_C, η_B and η_T.

7. Final Remarks

This paper has considered a complex but realistic correlation structure in the two-way error component model: the double autocorrelation case. It dealt with some parsimonious models, especially the AR(1) and MA(1) ones, as well as the general framework. Through a precise formula of the variance-covariance matrix of the errors, we derived the GLS estimator and related asymptotic properties. An investigation of the FGLS is also considered in the paper.

8. References

[1] N. S. Revankar, "Error Component Models with Serial Correlated Time Effects," *Journal of the Indian Statistical Association*, Vol. 17, 1979, pp. 137-160.

[2] B. H. Baltagi and Q. Li, "A Transformation that will Circumvent the Problem of Autocorrelation in an Error Component Model," *Journal of Econometric*, Vol. 48, No. 3, 1991, pp. 385-393. doi:10.1016/0304-4076(91)90070-T

[3] B. H. Baltagi and Q. Li, "Prediction in the One-Way Error Component Model with Serial Correlation," *Journal of Forecasting*, Vol. 11, No. 6, 1992, pp. 561-567. doi:10.1002/for.3980110605

[4] B. H. Baltagi and Q. Li, "Estimating Error Component Models with General MA(q) Disturbances," *Econometric Theory*, Vol. 10, No. 2, 1994, pp. 396-408. doi:10.1017/S026646660000846X

[5] J. W. Galbraith and V. Zinde-Walsh, "Transforming the Error Component Model for Estimation with general ARMA Disturbances," *Journal of Econometrics*, Vol. 66, No. 1-2, 1995, pp. 349-355. doi:10.1016/0304-4076(94)01621-6

[6] P. Balestra and M. Nerlove, "Pooling Cross-Section and Time-Series Data in the Estimation of a Dynamic Model: The Demand for Natural Gas," *Econometrica*, Vol. 34, No. 3, 1966, pp. 585-612. doi:10.2307/1909771

[7] B. H. Baltagi, "Econometric Analysis of Panel Data," 3rd Edition, John Wiley and Sons, New York, 2008.

[8] C. Hsiao, "Analysis of Panel Data," Cambridge University Press, Cambridge, 2003.

[9] G. S. Maddala, "Limited Dependent and Qualitative Variables in Econometrics," Cambridge University Press,

Cambridge, 1983.

[10] G. S. Maddala, "The Econometrics of Panel Data," Vols I and II, Edward Elgar Publishing, Cheltenham, 1983.

[11] M. H. Pesaran, "Exact Maximum Likelihood Estimation of a Regression Equation with a First Order Moving Average Errors," *The Review of Economic Studies*, Vol. 40, No. 4, 1973, pp. 529-538.

[12] P. A. V. B. Swamy and S. S. Arora, "The Exact Finite Sample Properties of the Estimators of Coefficients in the Error Components Regression Models," *Econometrica*, Vol. 40, No. 2, 1972, pp. 261-275. doi:10.2307/1909405

[13] M. Nerlove, "A Note on Error Components Models," *Econometrica*, Vol. 39, No. 2, 1971, pp. 383-396. doi:10.2307/1913351

[14] G. S. Maddala, "The Use of Variance Components Models in Pooling Cross Section and Time Series Data," *Econometrica*, Vol. 39, No. 2, 1971, pp. 341-358. doi:10.2307/1913349

[15] T. Amemiya, "The Estimation of the Variances in a Variance-Components Model," *International Economic Review*, Vol. 12, No. 1, 1971, pp. 1-13. doi:10.2307/2525492

[16] H. Theil, "Principles of Econometrics," John Wiley and Sons, New York, 1971.

[17] T. D. Wallace and A. Hussain, "The Use of Error Components Models in Combining Cross-Section and Time Series Data," *Econometrica*, Vol. 37, No. 1, 1969, pp. 55-72. doi:10.2307/1909205

[18] T. A. MaCurdy, "The Use of Time Series Processes to Model the Error Structure of Earnings in a Longitudinal Data Analysis," *Journal of Econometrics*, Vol. 18, No. 1, 1982, pp. 83-114. doi:10.1016/0304-4076(82)90096-3

[19] S. J. Prais and C. B. Winsten, "Trend Estimators and Serial Correlation," Unpublished Cowles Commission Discussion Paper: Stat No. 383, Chicago, 1954.

[20] W. A. Fuller and G. E. Battese, "Estimation of Linear Models with Cross-Error Structure," *Journal of Econometrics*, Vol. 2, No. 1, 1974, pp. 67-78. doi:10.1016/0304-4076(74)90030-X

[21] T. J. Wansbeek and A. Kapteyn, "A Simple Way to obtain the Spectral Decomposition of Variance Components Models for Balanced Data," *Communications in Statistics*, Vol. 11, No. 18, 1982, pp. 2105-2112.

Appendix: Computing the Inverse of Σ^{**}

We established that

$$\Sigma^{**} = \sigma_1^2 \left(I_N \otimes D\right) + \sigma_2^2 \left(I_N \otimes i_T^{\lambda} i_T^{\lambda'}\right) + \sigma_3^2 \left(i_N i_N' \otimes \Lambda\right)$$

with $D = diag\left(d_1, \cdots, d_t\right)$ and $\Lambda = diag\left(\Lambda_1, \cdots, \Lambda_T\right)$. Setting $G = \sigma_1^2 \left(I_N \otimes D\right) + \sigma_3^2 \left(i_N i_N' \otimes \Lambda\right)$, we can re-write the variance covariance matrix as

$$\Sigma^{**} = G + \sigma_2^2 \left(I_N \otimes i_T^{\lambda}\right)\left(I_N \otimes i_T^{\lambda'}\right) = G + \sigma_2^2 JJ'$$

where $J = \left(I_N \otimes i_T^{\lambda}\right)$. By the means of an update formula, we deduce an expression of the inverse of Σ^{**},

$$\left(\Sigma^{**}\right)^{-1} = G^{-1} - G^{-1} J \left(J' G^{-1} J + \frac{1}{\sigma_2^2} I_N\right)^{-1} J' G^{-1}$$

We need to obtain G^{-1} and the inverse of the bracketed expression. On the one hand,

$$G = \left(I_N \otimes D^{\frac{1}{2}}\right)\left[\sigma_1^2 I_{NT} + \sigma_3^2 \left(i_N i_N'\right) \otimes \Lambda D^{-1}\right] \times \left(I_N \otimes D^{\frac{1}{2}}\right)$$

Let H denote the matrix $\sigma_1^2 I_{NT} + \sigma_3^2 \left(i_N i_N'\right) \otimes \Lambda D^{-1}$. At this step, the inverse of H is required. Let $C' = \left(\frac{i_N}{\sqrt{N}}, C_a\right)$ be a $N \times N$ orthogonal matrix. Then,

$$\left(C' \otimes I_T\right) H \left(C \otimes I_T\right) = \sigma_1^2 I_{NT} + \sigma_3^2 \left(C' i_N i_N' C\right) \otimes \Lambda D^{-1} = \sigma_1^2 \left(I_N \otimes I_T\right) + \sigma_3^2 B$$

Therefore,

$$\left(C' \otimes I_T\right) H \left(C \otimes I_T\right) = diag\left(\sigma_1^2 + \frac{N\Lambda_1 \sigma_3^2}{d_1}, \cdots, \sigma_1^2 + \frac{N\Lambda_T \sigma_3^2}{d_T}, \sigma_1^2, \cdots, \sigma_1^2\right)$$

with

$$B = \begin{pmatrix} N\Lambda D^{-1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It is worth mentioning that $C_a' \frac{1}{\sqrt{N}} i_N = 0$ for C_a and $\frac{1}{\sqrt{N}} i_N$ are different columns of the same diagonal matrix. It is therefore obvious that H has already been diagonalized. As a consequence, the inverse of H is given by,

$$H^{-1} = \left(C \otimes I_T\right) \times \text{diag}\left(\frac{d_1}{d_1\sigma_1^2 + N\Lambda_1\sigma_3^2}, \cdots, \frac{d_T}{d_T\sigma_1^2 + N\Lambda_T\sigma_3^2}, \frac{1}{\sigma_1^2}, \cdots, \frac{1}{\sigma_1^2}\right) \times \left(C' \otimes I_T\right)$$

$$H^{-1} = \frac{1}{\sigma_1^2}\left(C_a C_a'\right) \otimes I_T + \left(\frac{1}{\sqrt{N}} i_N \otimes I_T\right) A \left(\frac{1}{\sqrt{N}} i_N' \otimes I_T\right)$$

where

$$A = \text{diag}\left(\frac{d_1}{d_1\sigma_1^2 + N\Lambda_1\sigma_3^2}, \cdots, \frac{d_T}{d_T\sigma_1^2 + N\Lambda_T\sigma_3^2}\right)$$

Since $C_a' i_N = 0$ and $C_a' C_a = I_{N-1}$, we have

$$C_a C_a' = I_N - \frac{1}{N} i_N i_N' = E_N.$$

Therefore,

$$H^{-1} = \frac{1}{\sigma_1^2}\left(E_N \otimes I_T\right) + \left(\frac{1}{\sqrt{N}} i_N \otimes I_T\right) A \left(\frac{1}{\sqrt{N}} i_N' \otimes I_T\right)$$

It then follows that,

$$G^{-1} = \left(I_N \otimes D^{-\frac{1}{2}}\right) \times \left[\frac{1}{\sigma_1^2}\left(E_N \otimes I_T\right) + \left(\frac{1}{\sqrt{N}} i_N \otimes I_T\right) A \left(\frac{1}{\sqrt{N}} i_N' \otimes I_T\right)\right] \times \left(I_N \otimes D^{-\frac{1}{2}}\right)$$

$$G^{-1} = \frac{1}{\sigma_1^2}\left[\left(E_N \otimes D^{-1}\right) + \left(\frac{1}{N^2} i_N i_N' \otimes S\right)\right]$$

in which $S = diag\left(s_1, \cdots, s_T\right)$ with $s_t = \frac{N\sigma_1^2}{d_t\sigma_1^2 + N\Lambda_t\sigma_3^2}$, $t = 1, \cdots, T$.

On the other hand, the matrix $\left(J' G^{-1} J + \frac{1}{\sigma_2^2} I_N\right)$ has to be determined. We get,

$$J' G^{-1} J = \frac{1}{\sigma_1^2}\left(I_N \otimes i_T^{\lambda'}\right) \times \left[\left(E_N \otimes D^{-1}\right) + \left(\frac{1}{N^2} i_N i_N' \otimes S\right)\right] \times \left(I_N \otimes i_T^{\lambda}\right)$$

or,

$$J'G^{-1}J = \frac{1}{\sigma_1^2}\left[E_N \otimes i_T^{\lambda'} D^{-1} i_T^{\lambda} + \frac{1}{N^2} i_N i_N' \otimes i_T^{\lambda'} S i_T^{\lambda}\right]$$

Thus,

$$J'G^{-1}J = \frac{1}{\sigma_1^2}\left[\left(i_T^{\lambda'} D^{-1} i_T^{\lambda}\right) E_N + \left(i_T^{\lambda'} S i_T^{\lambda}\right)\frac{1}{N^2} i_N i_N'\right]$$

Hence,

$$J'G^{-1}J + \frac{1}{\sigma_2^2} I_N = \frac{1}{\sigma_1^2}\left[I_N\left(i_T^{\lambda'} D^{-1} i_T^{\lambda}\right) + i_N i_N'\left(\frac{i_T^{\lambda'} S i_T^{\lambda}}{N^2} - \frac{i_T^{\lambda'} D^{-1} i_T^{\lambda}}{N}\right)\right]$$

and,

$$J'G^{-1}J + \frac{1}{\sigma_2^2} I_N = aI_N + bi_N i_N'$$

where

$$a = i_T^{\lambda'} D^{-1} i_T^{\lambda} + \frac{1}{\sigma_2^2} \quad \text{and} \quad b = \frac{i_T^{\lambda'} S i_T^{\lambda}}{\sigma_1^2 N^2} - \frac{i_T^{\lambda'} D^{-1} i_T^{\lambda}}{\sigma_1^2 N}.$$

Since

$$\left(aI_N + bi_N i_N'\right)^{-1} = \frac{1}{a}\left(I_N - \frac{b}{a+bN} i_N i_N'\right),$$

we deduce $\left(J'G^{-1}J + \frac{1}{\sigma_2^2} I_N\right)^{-1}$.

We are now interested in the expression

$$G^{-1}J\left(J'G^{-1}J + \frac{1}{\sigma_2^2} I_N\right)^{-1} J'G^{-1}.$$

We have,

$$G^{-1}J\left(J'G^{-1}J + \frac{1}{\sigma_2^2} I_N\right)^{-1} J'G^{-1} = \frac{1}{a}\left(G^{-1}JJ'G^{-1} - \frac{b}{a+bN} G^{-1}Ji_N i_N' J'G^{-1}\right)$$

From the definitions of the matrices G and J, we can write

$$G^{-1}J = \frac{1}{\sigma_1^2}\left[E_N \otimes D^{-1} i_T^{\lambda} + \left(\frac{1}{N^2} i_N i_N' \otimes S i_T^{\lambda}\right)\right],$$

and

$$G^{-1}Ji_N = \frac{i_N}{N\sigma_1^2} \otimes S i_T^{\lambda},$$

so that

$$G^{-1}JJ'G^{-1} = \frac{1}{\sigma_1^4}\left[\left(E_N \otimes D^{-1} i_T^{\lambda} i_T^{\lambda'} D^{-1}\right) + \left(\frac{1}{N^2} i_N i_N' \otimes S i_T^{\lambda} i_T^{\lambda'} S\right)\right]$$

and lastly

$$G^{-1}Ji_N i_N' J'G^{-1} = \frac{1}{N^2\sigma_1^4} i_N i_N' \otimes S i_T^{\lambda} i_T^{\lambda'} S$$

It then comes that

$$\begin{aligned} &G^{-1}J\left(J'G^{-1}J + \frac{1}{\sigma_2^2} I_N\right)^{-1} J'G^{-1} \\ &= \frac{1}{a\sigma_1^4}\left(E_N \otimes D^{-1} i_T^{\lambda} i_T^{\lambda'} D^{-1}\right) + \frac{1}{a\sigma_1^4}\left(\frac{1}{N^2} i_N i_N' \otimes S i_T^{\lambda} i_T^{\lambda'} S\right) \\ &+ \frac{1}{a\sigma_1^4}\left(\frac{1}{N^2} i_N i_N' \otimes S i_T^{\lambda} i_T^{\lambda'} S\right) \\ &- \frac{1}{a\sigma_1^4}\left(\frac{b}{N^2(a+bN)} i_N i_N' \otimes S i_T^{\lambda} i_T^{\lambda'} S\right) \end{aligned}$$

In other words,

$$\begin{aligned} &G^{-1}J\left(J'G^{-1}J + \frac{1}{\sigma_2^2} I_N\right)^{-1} J'G^{-1} \\ &= \frac{1}{a\sigma_1^4} E_N \otimes \left(D^{-1} i_T^{\lambda} i_T^{\lambda'} D^{-1}\right) \\ &+ \frac{1}{a\sigma_1^4 N^3}\left(\frac{a}{a+bN}\right)\left(i_N i_N' \otimes S i_T^{\lambda} i_T^{\lambda'} S\right) \end{aligned}$$

Finally, the inverse of Σ^{**} can be derived as

$$\begin{aligned} \left(\Sigma^{**}\right)^{-1} &= E_N \otimes \left(\frac{1}{\sigma_1^2} D^{-1} - \frac{1}{a\sigma_1^4} D^{-1} i_T^{\lambda} i_T^{\lambda'} D^{-1}\right) \\ &+ J_N \otimes \left(\frac{1}{\sigma_1^2 N^2} S - \frac{1}{\sigma_1^4 N^3 (a+bN)} S i_T^{\lambda} i_T^{\lambda'} S\right) \end{aligned}$$

with $J_N = i_N i_N'$. An alternative expression for $\left(\Sigma^{**}\right)^{-1}$ is available. Setting $K_T = D^{-1} - L_T$, and $L_T = \frac{1}{i_T^{\lambda'} D^{-1} i_T^{\lambda}} D^{-1} i_T^{\lambda} i_T^{\lambda'} D^{-1}$, we get

$$\begin{aligned} \left(\Sigma^{**}\right)^{-1} &= \frac{1}{\sigma_1^2} E_N \otimes \left(D^{-1} - L_T + L_T - \frac{1}{a\sigma_1^2} D^{-1} i_T^{\lambda} i_T^{\lambda'} D^{-1}\right) \\ &+ J_N \otimes \left(\frac{1}{\sigma_1^2 N^2} S - \frac{1}{\sigma_1^2 N^3 (a+bN)} S i_T^{\lambda} i_T^{\lambda'} S\right) \end{aligned}$$

$$\begin{aligned} \left(\Sigma^{**}\right)^{-1} &= \frac{1}{\sigma_1^2}\left(E_N \otimes K_T\right) + \frac{1}{\left(i_T^{\lambda'} D^{-1} i_T^{\lambda}\right)\sigma_2^2 + \sigma_1^2}\left(E_N \otimes L_T\right) \\ &+ \frac{1}{N^2}\left(J_N \otimes S_T\right) \end{aligned}$$

where

$$S_T = \frac{1}{\sigma_1^2} S - \frac{1}{\sigma_1^4 N\left(a+bN\right)} S i_T^{\lambda} i_T^{\lambda}{}' S$$

i.e.,

$$S_T = \frac{1}{\sigma_1^2} S - \frac{\sigma_2^2}{\sigma_1^4 N + \sigma_1^2 \sigma_2^2 \left(i_T^{\lambda}{}' S i_T^{\lambda}\right)} S i_T^{\lambda} i_T^{\lambda}{}' S$$

Hence, we finally get

$$\left(\Sigma^{**}\right)^{-1} = \frac{1}{\sigma_1^2}\left(E_N \otimes K_T\right) + \frac{1}{d}\left(E_N \otimes L_T\right) + \frac{1}{N^2}\left(J_N \otimes S_T\right)$$

where $d = \left(i_T^{\lambda}{}' D^{-1} i_T^{\lambda}\right)\sigma_2^2 + \sigma_1^2$