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Bayes Prediction of Future Observables from Exponentiated Populations with Fixed and Random Sample Size

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Abstract

Bayesian predictive probability density function is obtained when the underlying pop-ulation distribution is exponentiated and subjective prior is used. The corresponding predictive survival function is then obtained and used in constructing $100(1 - \tau)\%$ predictive interval, using one- and two- sample schemes when the size of the future sample is fixed and random. In the random case, the size of the future sample is assumed to follow the truncated Poisson distribution with parameter λ. Special attention is paid to the exponentiated Burr type XII population, from which the data are drawn. Two illustrative examples are given, one of which uses simulated data and the other uses data that represent the breaking strength of 64 single carbon fibers of length 10, found in Lawless [40].

Keywords: Predictive Density And Survival Functions, One- And Two-Sample Schemes, Bayes Prediction, Exponentiated Population. Exponentiated Burr Type XII Distribution, Data Of Carbon Fibers

1. Introduction

The general problem of prediction may be described as that of inferring the values of unknown observables (future observations, known as *future sample*), or functions of such variables, from current available observations, known as *informative sample*. the problem of prediction can be solved fully within Bayes framework (Geisser [33]). Bernardo and Smith [22] stated that: "inference about parameters is thus seen to be a limiting form of predictive inference about observables".

Bayesian prediction bounds for order statistics of future observables from certain distributions, such as the exponential, Rayleigh, Weibull, Pareto and Lomax distributions, have been studied by several authors. See, for example, Dunsmore [27], Lingappaiah [44], Evans and Nigm [28]-[30], Sinha and Howlader [58], Howlader [36], Sinha [56], Raqab [54], Malik [46], Zellner [64], Lwin [43], Sinha and Howlader [57], Arnold and Press [20], Geisser [32], Nigm and Hamdy [50], AL-Hussaini and Jaheen [16], AL-Hussaini [5] and Nigm et al [51]. Prediction bounds for certain order statistics of samples from Burr type XII population, were obtained by Nigm [49], AL-Hussaini and Jaheen [14,15] and Ali Mousa and Jaheen [19]. Prediction bounds, based on heterogeneous populations that can be represented by finite mixtures were developed by Jaheen [37], AL-Hussaini [3], [6] among others.

Prediction was reviewed by Patel [52], Nagaraja [48], Kaminsky and Nelson [38] and AL-Hussaini [4].

Adding one, or more, parameters to a distribution makes it richer and more flexible for modeling data. There are several ways of adding one or more parameters to a distribution. A positive parameter was added to a general survival function (SF) by Marshall and Olkin [47]. In their consideration of a countable mixture of a Pascal(r,p) mixing proportion and positive integer powers of SFs, a SF with two extra parameters was obtained by AL-Hussaini and Ghitany [10]. A new family of distributions as a countable mixture with Poisson added parameter was constructed by AL-Hussaini and Gharib [9]. A simple way of adding a parameter to a distribution is by exponentiation. This goes back to Verhulst [62], who raised his 1838 [61] logistic cumulative distribution function (CDF) to a positive power. Ahuja and Nash [1] seemed to have been the first to raise Verhulst [63] exponential CDF to a positive power.

The exponentiated model is also known as proportional reversed hazard rate model, see Gupta and Gupta [35] or Lehman alternatives, when α is a positive integer, see Lehmann [41].

$$H(x) \equiv H(x;\underline{\theta}) = \left[G(x;\underline{\beta})\right]^{\alpha}, \quad (1.1)$$

where $G(x;\underline{\beta})$ is a CDF and α is a positive parameter, $\underline{\beta}$ may be a vector and $(\alpha,\underline{\beta}) \in \Omega$, where Ω is the parameter space. In this paper, we shall assume that G(x) is an absolutely continuous CDF defined on the positive half of the real line. So that, its corresponding probability density function (PDF) is $g(x),\ x > 0.$. If $h(x)$ is the PDF corresponding to H(x), then

$$h(x) = \alpha\left[G(x)\right]^{\alpha-1} g(x),\ x > 0. \quad (1.2)$$

AL-Hussaini [7] studied some of the properties of the exponentiated distributions where the baseline distribution G is a general CDF. AL-Hussaini [8] estimated the parameters, SF and hazard rate function (HRF) under the general exponentiated model, using maximum likelihood and Bayes methods. He also obtained prediction bounds of future observables based on the two-sample scheme under the exponentiated model.

AL-Hussaini and Hussein [11] estimated the parameters of the exponentiated Burr XII(α, β, γ) population when the data are subjected to type II censoring. Maximum likelihood (ML) and Bayes methods were used for estimation. The ML and Bayes estimates were compared when the Bayes estimates are based on square error and linear-exponential loss functions.

In Section 2, the one-sample scheme is used to predict future observables from exponentiated populations. In Section 3, future observables from exponentiated populations are predicted, using two-sample scheme, when the size of the future sample is fixed and when it is random. In Section 4 the results obtained in Sections 2 and 3 are applied to the exponentiated Burr type XII population. In Section 5, Numerical examples are given to illustrate our results.

2. One-Sample Prediction

Suppose that $X_1 < \cdots < X_r$ are the first r ordered life times in a random sample of n components whose failure times are identically distributed as a random variable X having the exponentiated distribution, given by (1.1). Bayesian one-sample prediction is made for some order statistics of the remaining n-r life times. For such remaining n-r components, let $Y_s \equiv X_{r+s}$, denote the life time of the sth component to fail, where $1 \le s \le n-r$. Write $f_r(y_s \mid \underline{\theta})$ to denote the conditional PDF of the sth component to fail, given that r components had already failed. Then

$$f_r(y_s \mid \underline{\theta}) \propto \left[H(y_s \mid \underline{\theta}) - H(x_r \mid \underline{\theta})\right]^{s-1} \left[1 - H(y_s \mid \underline{\theta})\right]^{n-r-s} \left[R_H(x_r \mid \underline{\theta})\right]^{-(n-r)} h(y_s \mid \underline{\theta}),$$

The binomial expansion of each of the first three terms on the right hand side then yields

$$f_r(y_s \mid \underline{\theta}) \propto \sum_{j_1=0}^{s-1}\sum_{j_2=0}^{n-r-s}\sum_{j_3=0}^{\infty} C_{j_1j_2j_3}\left[H(y_s \mid \underline{\theta})\right]^{s-1-j_1} \left[H(x_r \mid \underline{\theta})\right]^{j_1}\left[H(y_s \mid \underline{\theta})\right]^{j_2}\left[H(x_r \mid \underline{\theta})\right]^{j_3} h(y_s \mid \underline{\theta}),$$

where

$$C_{j_1j_2j_3} = (-1)^{j_1+j_2}\binom{s-1}{j_1}\binom{n-r-s}{j_2}C_{j_3} \quad (2.1)$$

$$C_{j_3} = \frac{(n-r)(n-r+1)\cdots(n-r+j_3-1)}{j_3!}. \quad (2.2)$$

Substituting $H(y_s \mid \underline{\theta}) = \left[G(y_s \mid \underline{\beta})\right]^{\alpha}$ and $h(y_s \mid \underline{\theta}) = \alpha\left[G(y_s \mid \underline{\beta})\right]^{\alpha-1} g(y_s \mid \underline{\beta})$ we then have

$$f_r(y_s \mid \underline{\theta}) \propto \alpha\ \lambda_G(y_s \mid \underline{\beta}) \sum{}^{*} C_{j_1j_2j_3}\left[G(y_s \mid \underline{\beta})\right]^{\alpha(s-j_1+j_2)}\left[G(x_r \mid \underline{\beta})\right]^{\alpha(j_1+j_3)},$$

where

$$\lambda_G(y_s \mid \underline{\beta}) = \left[\frac{g(y_s \mid \underline{\beta})}{G(y_s \mid \underline{\beta})}\right],\quad \sum{}^{*} = \sum_{j_1=0}^{s-1}\sum_{j_2=0}^{n-r-s}\sum_{j_3=0}^{\infty}. \quad (2.3)$$

We shall rewrite $f_r(y_s \mid \underline{\theta})$ in the form

$$f_r(y_s \mid \underline{\theta}) \propto \alpha\ \lambda_G(y_s \mid \underline{\beta})\sum{}^{*} C_{j_1j_2j_3}\exp\left[-\alpha T_{j_1j_2j_3}(y_s;\underline{\beta})\right], \quad (2.4)$$

where

$$T_{j_1j_2j_3}(y_s;\underline{\beta}) = -\left[(s-j_1+j_2)\ln G(y_s \mid \underline{\beta}) + (j_1+j_3)\ln G(x_r \mid \underline{\beta})\right]. \quad (2.5)$$

The predictive PDF of Y_s is defined by

$$f_r^{*}(y_s \mid \underline{x}) \propto \int f_r(y_s \mid \underline{\theta})\,\pi(\underline{\theta} \mid \underline{x})\, d\underline{\theta},\quad y_s > x_r \quad (2.6)$$

where $\pi(\underline{\theta} \mid \underline{x})$ is the posterior PDF of $\underline{\theta} = (\theta_1,\cdots,\theta_k)$, given $\underline{x} = (x_1, x_2,\cdots,x_r)$. The posterior PDF is such that

$$\pi(\underline{\theta} \mid \underline{x}) \propto L(\underline{\theta};\underline{x})\,\pi(\underline{\theta}), \quad (2.7)$$

where $L(\underline{\theta};\underline{x})$ is the likelihood function and $\pi(\underline{\theta})$ is a prior PDF of $\underline{\theta}$.

Substitution of (2.4) and (2.7) in (2.6), then yields the predictive PDF of the future order statistic Y_s, $s = 1,\cdots,n-r$.

Suppose that the CDF $G(x)$ depends on an unknown k-dimensional vector of parameters $\underline{\beta}=(\beta_1,\cdots,\beta_k)$, , so that $H(x)\equiv H(x;\underline{\theta})=\left[G(x;\underline{\beta})\right]^{\alpha}$ depends on the (k + 1) unknown parameters $(\alpha,\underline{\beta})$, . Suppose that α and $\underline{\beta}$ are independent so that the prior belief of the experimenter is given by

$$\pi(\underline{\theta})\equiv\pi(\alpha,\underline{\beta})=\pi_1(\alpha)\pi_2(\underline{\beta}) \tag{2.8}$$

where $\pi_1(\alpha)$ is gamma (b_1,b_2) distribution and $\pi_2(\underline{\beta})$ is a k-variate PDF.

The likelihood function is given by:

$$L(\underline{\theta};\underline{x})\propto\left[\prod_{i=1}^{r}h(x_i\mid\underline{\theta})\right]\left[1-H(x_r\mid\underline{\theta})\right]^{n-r}$$

Substitution of (1.1) and (1.2) then yields

$$L(\underline{\theta};\underline{x})\propto\sum_{j_4=0}^{n-r}C_{j_4}\alpha^{r}e^{-\alpha T_{j_4}(\underline{\beta})-T_0(\underline{\beta})},.$$

where

$$C_{j_4}=(-1)^{j_4}\binom{n-r}{j_4},$$

$$T_{j_4}(\underline{\beta})=-\left[\sum_{i=1}^{r}\ln G(x_i\mid\underline{\beta})+j_4\ln G(x_r\mid\underline{\beta})\right], \tag{2.9}$$

$$T_0(\underline{\beta})=\sum_{i=1}^{r}\ln G(x_i\mid\underline{\beta})-\sum_{i=1}^{r}\ln g(x_i\mid\underline{\beta}). \tag{2.10}$$

The posterior PDF of $\underline{\theta}=(\alpha,\underline{\beta})$ given the data, is then given by

$$\pi(\underline{\theta}\mid\underline{x})\propto L(\underline{\theta};\underline{x})\pi(\underline{\theta})\propto\sum_{j_4=0}^{n-r}C_{j_4}\alpha^{r+b_1-1}e^{-\alpha T_{0j_4}(\underline{\beta})-T_0(\underline{\beta})}\pi_2(\underline{\beta}) \tag{2.11}$$

where

$$T_{0j_4}(\underline{\beta})=b_2+T_{j_4}(\underline{\beta}) \tag{2.12}$$

Therefore, the predictive PDF of Y_s is given by

$$f^*(y_s\mid\underline{x})\propto\sum\nolimits_1^*C^*\int_{\underline{\beta}}\int_0^{\infty}\alpha^{r+b_1}\exp\left[-\alpha\left(T_{0j_4}(\underline{\beta})+T_{j_1j_2j_3}(y_s;\underline{\beta})\right)-T_0(\underline{\beta})\right]\left[\frac{g(y_s\mid\underline{\beta})}{G(y_s\mid\underline{\beta})}\right]\mathrm{d}\alpha\,\mathrm{d}\underline{\beta}$$

$$=A\sum\nolimits_1^*C^*\;I(y_s),\qquad y_s>x_r.$$

where

$$\left.\begin{aligned}&\sum\nolimits_1^*=\sum\nolimits^*\sum_{j_4=0}^{n-r}=\sum_{j_1=0}^{s-1}\sum_{j_2=0}^{n-r-s}\sum_{j_3=0}^{\infty}\sum_{j_4=0}^{n-r},\\&C^*=C_{j_1j_2j_3}C_{j_4}=(-1)^{j_1+j_2+j_4}\binom{s-1}{j_1}\binom{n-r-s}{j_2}\binom{n-r}{j_4}C_{j_3},\end{aligned}\right\} \tag{2.13}$$

C_{j_3} is given by (2.2), and

$$I(y_s)=\int_{\underline{\beta}}\frac{\exp\left[-T_0(\underline{\beta})\right]\pi_2(\underline{\beta})}{\left[T_{0j_4}(\underline{\beta})+T_{j_1j_2j_3}(y_s;\underline{\beta})\right]^{r+b_1+1}}\left[\frac{g(y_s\mid\underline{\beta})}{G(y_s\mid\underline{\beta})}\right]\mathrm{d}\underline{\beta}$$

It then follows that the predictive SF of Y_s is given by

$$P[Y_s>v\mid\underline{x}]=A\sum\nolimits_1^*C^*\int_v^{\infty}I(y_s)\,\mathrm{d}y_s=\frac{A}{(r+b_1)}S(v),\quad v>x_r.$$

where

$$S(v)=\sum\nolimits_1^*C^*\left[\frac{1}{s-j_1+j_2}\int_{\underline{\beta}}\left\{\frac{1}{\left[T_{0j_4}(\underline{\beta})+(j_1+j_3)\ln G(x_r\mid\underline{\beta})\right]^{r+b_1}}-\frac{1}{\left[T_{0j_4}(\underline{\beta})+T_{j_1j_2j_3}(y_s;\underline{\beta})\right]^{r+b_1}}\right\}\exp\left(-T_0(\underline{\beta})\right)\pi_2(\underline{\beta})\,\mathrm{d}\underline{\beta}\right],$$

$T_{j_1j_2j_3}(y_s;\beta,\gamma)$, $T_0(\underline{\beta})$ and $T_{0j_4}(\underline{\beta})$ are given, respectively, by (2.5) , (2.10) and (2.12).

Notice that:

$$1=P[Y_s>x_r\mid\underline{x}]=\frac{A}{(r+b_1)}S(x_r)\;\Rightarrow\;A=\frac{r+b_1}{S(x_r)}.$$

So that

$$P[Y_s>v\mid\underline{x}]=\frac{S(v)}{S(x_r)},\quad v>x_r. \tag{2.14}$$

A two sided 100(1- _predictive interval for Y_s is given by $L<Y_s<U$, where L and U are the solution of the equations

$$S(L)-\left(1-\frac{\tau}{2}\right)S(x_r)=0, \tag{2.15}$$

$$S(U)-\left(\frac{\tau}{2}\right)S(x_r)=0. \tag{2.16}$$

REMARKS

1) A one-sided 100(1- _predictive interval of the form $Y_s > L$ is such that L is the solution of (2.15) after replacing $\tau/2$ by τ. Similarly, a one-sided 100(1 – τ)% predictive interval of the form $Y_s < U$ is such that U is the solution of (2.16) after replacing $\tau/2$ by τ.

2) Equations (2.15) and (2.16) are generally not obtainable analytically and some iteration scheme is used for their solution.

3. Two-Samples Prediction

Suppose that $X_1 < \cdots < X_r$ are the first r ordered life times in a random sample of n components (type II censoring) whose failure times are identically distributed as a random variable *X* having the exponentiated distribution, given by (1.1). Bayesian prediction is made for Y_l, the l th ordered lifetime in a future sample of size m (two-sample prediction), $\text{l} = 1, 2, \cdots, m$. The sample size m is assumed to be fixed or random.

4.1. Fixed Sample Size

Suppose that the CDF $G(x)$ depends on an unknown k-dimensional vector of parameters $\underline{\beta} = (\beta_1, \cdots, \beta_k)$, so that $H(x) \equiv H(x;\underline{\theta}) = \left[G(x;\underline{\beta})\right]^\alpha$, depends on the (k + 1) unknown parameters $(\alpha, \underline{\beta})$. Suppose that α and $\underline{\beta}$ are independent so that the prior belief of the experimenter is given by (2.8). Then the predictive PDF and SF of the future Y_l, $\text{l} = 1, \cdots, m$ are given, respectively, by

$$f_1^*(y_\text{l} \mid \underline{x}) = \frac{(r+b_1) S_{k+4}^*}{S_{02}} \tag{3.1}$$

$$P\left[Y_\text{l} > \nu \mid \underline{x}\right] = \frac{S_{k+5}^*(\nu)}{S_{02}}, \tag{3.2}$$

where

$$\left.\begin{aligned} S_{02} &= \sum_{j_4=0}^{n-r} \sum_{j_5=0}^{m-\text{l}} \left[\frac{C_{j_4} C_{j_5}}{\text{l} + j_5}\right] I_{0 j_4}, \quad S_{k+4}^* = \sum_{j_4=0}^{n-r} \sum_{j_5=0}^{m-\text{l}} C_{j_4} C_{j_5} I_{k+4, j_4, j_5}, \\ S_{k+5}^*(\nu) &= \sum_{j_4=0}^{n-r} \sum_{j_5=0}^{m-\text{l}} C_{j_4} C_{j_5} I_{k+5, j_4, j_5}(\nu), \end{aligned}\right\} \tag{3.3}$$

C_{j_4} is given by (2.9) ,

$$C_{j_5} = (-1)^{j_5} \binom{m-\text{l}}{j_5}, \quad I_{0 j_4} = \int_{\underline{\beta}} \frac{\exp\left\{-T_0(\underline{\beta})\right\} \pi_2(\underline{\beta})}{\left\{T_{0 j_4}(\underline{\beta})\right\}^{r+b_1}} \mathrm{d}\underline{\beta}, \quad I_{k+4, j_4, j_5} = \int_{\underline{\beta}} \left[\frac{g(y_\text{l} \mid \underline{\beta})}{G(y_\text{l} \mid \underline{\beta})}\right] \frac{\exp\left\{-T_0(\underline{\beta})\right\} \pi_2(\underline{\beta})}{\left\{T_{j_4, j_5}(\underline{\beta})\right\}^{r+b_1+1}} \mathrm{d}\underline{\beta}$$

$$I_{k+5, j_4, j_5}(\nu) = \int_{\underline{\beta}} \left[\frac{1}{\left\{T_{0 j_4}(\underline{\beta})\right\}^{r+b_1}} - \frac{1}{\left\{T_{0 j_4}(\underline{\beta}) - (\text{l} + j_5) \ln G(\nu \mid \underline{\beta})\right\}^{r+b_1}}\right] \exp\left\{-T_0(\underline{\beta})\right\} \pi_2(\underline{\beta}) \, \mathrm{d}\underline{\beta}$$

$$T_{j_4, j_5}(\underline{\beta}) = T_{0 j_4}(\underline{\beta}) - (\text{l} + j_5) \ln G(y_\text{l} \mid \underline{\beta}) \tag{3.4}$$

$T_{j_4}(\underline{\beta}), T_0(\underline{\beta})$ and $T_{0 j_4}(\underline{\beta})$ are given, respectively, by (2.9), (2.10) and (2.12).

It follows, from (3.2), that a two-sided predictive interval for Y_l is given by $L < Y_\text{l} < U$, where L and U are the solution of the two equations

$$\left.\begin{aligned} 0 &= \sum_{j_4=0}^{n-r} \sum_{j_5=0}^{m-\text{l}} \left[\frac{C_{j_4} C_{j_5}}{\text{l} + j_5}\right] \left[I_{k+5, j_4, j_5}(L) - (1 - \tau/2) I_{0 j_1}\right], \\ 0 &= \sum_{j_4=0}^{n-r} \sum_{j_5=0}^{m-\text{l}} \left[\frac{C_{j_4} C_{j_5}}{\text{l} + j_5}\right] \left[I_{k+5, j_4, j_5}(U) - (\tau/2) I_{0 j_1}\right]. \end{aligned}\right\} \tag{3.5}$$

For proof, see AL-Hussaini [8].

3.2. Random Sample Size

If the sample size m of the future sample is random, Gupta and Gupta (1984) suggested the use of the predictive PDF of Y_l to be given in the form

$$f_2^*(y_\text{l} \mid \underline{x}) = \frac{1}{p(m \ge \text{l})} \sum_{m=\text{l}}^{\infty} p(m) f_1^*(y_\text{l} \mid \underline{x}) \tag{3.6}$$

where p(m) is the probability mass function (PMF) of the random variable m and $f_1^*(y_\text{l} \mid \underline{x})$ is the predictive PDF of Y_l when m is fixed.

Consider the case when m has a truncated Poisson (λ), with PMF

$$p(m) = \frac{\lambda^m e^{-\lambda}}{m!\left(1 - e^{-\lambda}\right)}, \quad m = 1, 2, 3, \cdots \tag{3.7}$$

The predictive PDF and SF of Y_1 when m is random can be written as

$$f_2^*(y_1 \mid \underline{x}) = \frac{1}{p(m \ge 1)} \sum_{m=1}^{\infty} p(m) \frac{(r+b_1) S_{k+4}^*}{S_{02}}, \quad y_1 > 0, \tag{3.8}$$

$$P_2[Y_1 > \nu \mid \underline{x}] = \frac{\sum_{m=1}^{\infty} p(m)\left[S_{k+5}^*(\nu)/S_{02}\right]}{\sum_{m=1}^{\infty} p(m)}, \tag{3.9}$$

where $p(m)$ is given by (3.7), S_{02} and $S_{k+5}^*(.)$ (which are functions of m), are given by (3.3).

4. Application To Exponentiated Burr Type Xii Population

Burr [23] suggested a differential equation of a CDF of a random variable *X*, whose solution depends on a 'general' function of x. Burr chose twelve specific forms for this function to obtain the well-known twelve types of Burr distributions. See Burr [23]. In a different direction, Takahasi [60], compounded the Weibull distribution with the gamma distribution to obtain a 3-parameter Burr XII distribution. Baharith [21], studied the 4-parameter Burr XII $(\beta,\gamma,\delta,\xi)$ whose CDF, for $x > 0$, $\beta,\gamma,\delta,\xi > 0$, is of the form

$$G(x) = 1 - \left[1 + \delta(x-\xi)^{\beta}\right]^{-\gamma}.$$

The two-parameter Burr XII (β,γ) distribution has CDF and PDF of the form

$$G(x) = 1 - \left(1 + x^{\beta}\right)^{-\gamma}, \tag{4.1}$$

$$g(x) = \gamma \beta x^{\beta-1} \left(1 + x^{\beta}\right)^{-\gamma-1}. \tag{4.2}$$

Among the 12 distributions of Burr, the 2-parameter Burr XII (β,γ) has found numerous applications. It was proposed as a life time model and its properties were studied by Burr and Cislak [24], Dubey [25,26], Tadikamalla [59] and Lewis [42], among others. Inferences, based on the Burr XII (β,γ) distribution and some of its testing measures were made by Papadopoulos [53], Evans and Ragab [31], Lingappaiah [45], Shah and Gekhale [55], AL-Hussaini and Jaheen [12,13]. Khan and Khan [39] and AL-Hussaini [3] characterized the Burr XII (β,γ) distribution. Nigm [49] and AL-Hussaini and Jaheen [14] predicted observables based on the Burr XII (β,γ) model.

An exponentiated Burr type XII distribution with parameters α,β,γ, denoted by EBurr XII(α,β,γ), has CDF and PDF of the forms (1.1) and (1.2), where G and g are given by (4.1) and (4.2), respectively. We shall obtain the predictive PDF, and hence predictive bounds of Y_s when the unknown parameters are α,β and γ.

Suppose that the baseline distribution *G* is Burr XII with two unknown parameters β and γ. It is assumed that α is independent of (β,γ) and that $\alpha \sim \text{gamma}(b_1, b_2)$, $\gamma \mid \beta \sim \text{gamma}(b_3, \beta)$ and $\beta \sim \text{gamma}(b_4, b_5)$, so that the prior PDF of $\underline{\theta} = (\alpha,\beta,\gamma)$ is given by

$$\pi(\underline{\theta}) \equiv \pi(\alpha,\beta,\gamma) = \pi_1(\alpha)\pi_2(\beta,\gamma) = \pi_1(\alpha)\pi_3(\gamma \mid \beta)\pi_4(\beta) \propto \alpha^{b_1-1}\beta^{b_3+b_4-1}\gamma^{b_3-1}e^{-b_2\alpha-\beta(b_5+\gamma)} \tag{4.3}$$

4.1. One-Sample Prediction

The predictive PDF and SF of Y_s are given, respectively, by

$$f^*(y_s \mid \underline{x}) \propto \sum\nolimits_1^* C^* \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \alpha^{r+b_1}\beta^{b_3+b_4-1}\gamma^{b_3-1} exp\left[-\alpha\left(T_{0j_4}(\beta,\gamma) + T_{j_1j_2j_3}(y_s;\beta,\gamma)\right)\right]\exp\left[-T^*(\beta,\gamma)\right]\left[\frac{g(y_s \mid \beta,\gamma)}{G(y_s \mid \beta,\gamma)}\right] \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma$$

$$= A\sum\nolimits_1^* C^* I(y_s), \quad y_s > x_r.$$

$$P[Y_s > \nu \mid \underline{x}] = A\sum\nolimits_1^* C^* \int_{\nu}^{\infty} I(y_s)\, dy_s = \frac{A}{(r+b_1)} S(\nu), \; \nu > x_r.$$

where

$$I(y_s) = \int_0^{\infty}\int_0^{\infty} \frac{\beta^{b_3+b_4-1}\gamma^{b_3-1}\exp\left[-T^*(\beta,\gamma)\right]}{\left[T_{0j_4}(\beta,\gamma) + T_{j_1j_2j_3}(y_s;\beta,\gamma)\right]^{r+b_1+1}}\left[\frac{g(y_s \mid \beta,\gamma)}{G(y_s \mid \beta,\gamma)}\right] \mathrm{d}\beta\, \mathrm{d}\gamma$$

$$S(\nu) = \sum\nolimits_1^* C^* \left[\frac{1}{s-j_1+j_2}\int_0^{\infty}\int_0^{\infty}\left\{\frac{1}{\left[T_{0j_4}(\beta,\gamma) + (j_1+j_3)\ln G(x_r \mid \beta,\gamma)\right]^{r+b_1}} - \frac{1}{\left[T_{0j_4}(\beta,\gamma) + T_{j_1j_2j_3}(y_s;\beta,\gamma)\right]^{r+b_1}}\right\}\beta^{b_3+b_4-1}\gamma^{b_3-1}\exp\left(-T^*(\beta,\gamma)\right) \mathrm{d}\beta\, \mathrm{d}\gamma\right],$$

$$T^*(\beta,\gamma)=\beta(b_5+\gamma)+T_0(\beta,\gamma) \tag{4.4}$$

$T_{j_1j_2j_3}(y_s;\beta,\gamma)$, $T_0(\beta,\gamma)$, $T_{0j_4}(\beta,\gamma)$ and C^* are given, respectively, by (2.5), (2.10), (2.12) and (2.13).

4.2. Two-Sample Prediction

In the following two subsections, it is assumed that the two independent samples of sizes n and m are drawn from an exponentiated population. The size m of the future sample is assumed to be fixed, in subsection 4.2.1 and random in subsection 4.2.2.

4.2.1. Fixed Sample Size

The predictive PDF and SF of the future Y_l, $l=1,\cdots,m$ are given by

$$f_l^*(y_l\mid \underline{x})=\frac{(r+b_l)S_{k+4}^*}{S_{02}},\quad P[Y_l>\nu\mid \underline{x}]=\frac{S_{k+5}^*(\nu)}{S_{02}},$$

where

$$S_{02}=\sum_{j_4=0}^{n-r}\sum_{j_5=0}^{m-l}\left[\frac{C_{j_4}C_{j_5}}{l+j_5}\right]I_{0j_4},\ S_{k+4}^*=\sum_{j_4=0}^{n-r}\sum_{j_5=0}^{m-l}C_{j_4}C_{j_5}I_{k+4,j_4,j_5},\quad S_{k+5}^*(\nu)=\sum_{j_4=0}^{n-r}\sum_{j_5=0}^{m-l}C_{j_4}C_{j_5}I_{k+5,j_4,j_5}$$

C_{j_4} is given by (2.9),

$$C_{j_5}=(-1)^{j_5}\binom{m-l}{j_5},\ I_{0j_4}=\int_0^\infty\int_0^\infty\frac{\beta^{b_3+b_4-1}\gamma^{b_3-1}\exp\{-T^*(\beta,\gamma)\}}{\{T_{0j_4}(\beta,\gamma)\}^{r+b_l}}\,\mathrm{d}\beta\mathrm{d}\gamma$$

$$I_{k+4,j_4,j_5}=\int_0^\infty\int_0^\infty\left[\frac{g(y_l\mid\beta,\gamma)}{G(y_l\mid\beta,\gamma)}\right]\frac{\beta^{b_3+b_4-1}\gamma^{b_3-1}\exp\{-T^*(\beta,\gamma)\}}{\{T_{j_4,j_5}(\beta,\gamma)\}^{r+b_l+1}}\,\mathrm{d}\beta\mathrm{d}\gamma$$

$$I_{k+5,j_4,j_5}=\int_0^\infty\int_0^\infty\left[\frac{1}{\{T_{0j_4}(\beta,\gamma)\}^{r+b_l}}-\frac{1}{\{T_{0j_4}(\beta,\gamma)-(l+j_5)\ln G(\nu\mid\beta,\gamma)\}^{r+b_l}}\right]\times\beta^{b_3+b_4-1}\gamma^{b_3-1}\exp\{-T^*(\beta,\gamma)\}\,\mathrm{d}\beta\,\mathrm{d}\gamma,$$

$$T_{j_4,j_5}(\beta,\gamma)=T_{0j_4}(\beta,\gamma)-(l+j_5)\ln G(y_l\mid\beta,\gamma)$$

$T_{j_4}(\beta,\gamma), T_0(\beta,\gamma), T_{0j_4}(\beta,\gamma)$ and $T^*(\beta,\gamma)$ are given by (2.9), (2.10), (2.12) and (4.4).

A two-sided predictive interval for Y_l is given by $L<Y_l<U$, where L and U are the solution of (3.5). For proof, see [8].

4.2.2. Random Sample Size

The predictive PDF and SF of the future Y_l, $l=1,\cdots,m$, when m is random, are given by (3.8) and (3.9), using $f_l^*(y_l\mid\underline{x})$ and $P[Y_l>\nu\mid\underline{x}]$ defined in section 4.2.1. It follows, from (3.9), that a two-sided predictive interval for Y_l is given by $L<Y_l<U$, where L and U are the solution of the two equations

$$0=\sum_{m=l}^{\infty}p(m)\left[S_{k+5}^*(L)/S_{02}\right]-(1-\tau/2)\sum_{m=l}^{\infty}p(m)$$

$$0=\sum_{m=l}^{\infty}p(m)\left[S_{k+5}^*(U)/S_{02}\right]-(\tau/2)\sum_{m=l}^{\infty}p(m)$$

5. Numerical Computations

The numerical examples given here are to illustrate the use of the results obtained.

5.1. Example 1

Twenty observations are generated from the EBurr XII ($\alpha=2.5, \beta=1.5, \gamma=2$) according to the expression:

$$X=\left[\left(1-U^{1/\alpha}\right)^{-1/\gamma}-1\right]^{1/\beta},\quad \text{where } U\sim Unif(0,1).$$

The observations are ordered and only the first 15 out of the 20 observations are assumed to be known. The observations are given as

0.31631	0.37028	0.46877	0.53664	0.54401	0.64687	0.66259
0.80129	0.82068	0.84947	0.90148	1.0728	1.2344	1.2932
1.351	**1.7918**	**1.8123**	**2.6583**	**2.7362**	**5.0043**	

Table 1. Lower and Upper Limits of Some Predicted Values.

	One-Sample			Two-Sample					
				m fixed			m random		
	Y_1	Y_2	Y_{n-r}	Y_1	Y_2	Y_{10}	Y_1	Y_2	Y_{10}
L	1.56	1.62	1.88	0.13	0.25	0.37	0.14	0.29	0.38
U	2.59	3.18	6.99	0.82	0.94	1.64	0.89	1.24	1.87

Table 2. Lower and Upper Limits of Some Predicted Values.

	One-Sample			Two-Sample					
				m fixed			m random		
	Y_1	Y_2	Y_{n-r}	Y_1	Y_2	Y_{10}	Y_1	Y_2	Y_{10}
L	3.63	3.66	4.41	0.69	1.15	1.63	0.57	1.26	1.59
U	4.14	4.20	5.03	2.32	3.12	3.52	2.41	2.98	3.18

When the hyper-parameters are $b_1 = 0.6$, $b_2 = 0.6$, $b_3 = 2$, $b_4 = 2$, $b_5 = 3$, the lower and upper limits for the 95% predictive intervals of $X_{r+1} \equiv Y_1$, $X_{r+2} \equiv Y_2$ and $X_n \equiv Y_{n-r}$, based on the one-sample scheme, are given in **Table 1**. On the other hand, when predicti-on is based on the two-sample scheme, Table 1 also shows the lower and upper limits for the 95% predictive intervals of three failure times of a future sample of size m, Y_1, Y_2 and Y_{10}, of size m, when m is fixed (=10) and when m has a truncated Poisson with $\lambda = 10$.

5.2. Example 2 (Real Life Data)

The following data (Lawless 2003, p.573, [40]) represent the breaking strength of *n* = 64 single carbon fibers of length 10, given as ordered,

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, **3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.**

It is assumed that only 55 (= r) observations are known. AL-Hussaini and Hussein [11], showed that the vector of hyper-parameters ($b_1 = 180$, $b_2 = 0.6$, $b_3 = 2$, $b_4 = 2$, $b_5 = 3$) are appropriate for the prior density of α, β, γ so that the EBurr XII distribution fits the set of data using Kolmogorov Smirnov test.

The lower and upper limits for the 95% predictive intervals of $X_{r+1} \equiv Y_1$, $X_{r+2} \equiv Y_2$ and $X_n \equiv Y_{n-r}$, are given in **Table 2**, when based on the one-sample scheme, where *n* = 64 and *r* = 55. On the other hand, when prediction is based on the two-sample scheme, with *n* = 64, *r* = 55 and m = 10, the lower and upper limits for the 95% predictive intervals of three failure times of a future sample of size m, Y_1, Y_2 and Y_{10}, when m is fixed (= 10) and when m has a truncated Poisson with $\lambda = 10$, are displayed in **Table 2.**

6. Concluding Remarks

Bayes 100(1 – τ)% predictive interval for future observables is obtained when data are order statistics of a random sample drawn from a population having the exponentiated distribution and only the first r observations are known. The one- and two-sample schem-

es are used in prediction and a subjective prior is used as the prior belief of the experimenter. In the two-sample case, the Bayes intervals are obtained when the size of the future sample is assumed to be fixed and to be random. In the random case, the size of the future sample follows the Poisson distribution with parameter λ. Results are applied to the exponentiated Burr type XII population. Two examples are given: one uses simulated data and the other uses data representing the breaking strength of 64 single carbon fibers of length 10, found in Lawless [40].

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