Open Journal of Optimization, 2016, 5, 140-162
http://www.scirp.org/journal/ojop
ISSN Online: 2325-7091
ISSN Print: 2325-7105

Some Explicit Results for the Distribution Problem of Stochastic Linear Programming

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How to cite this paper: Ansaripour, A., Mata, A., Nourazari, S. and Kumin, H. (2016) Some Explicit Results for the Distribution Problem of Stochastic Linear Programming. *Open Journal of Optimization*, **5**, 140-162.
http://dx.doi.org/10.4236/ojop.2016.54014

Received: October 30, 2016
Accepted: December 27, 2016
Published: December 30, 2016

Abstract

A technique is developed for finding a closed form expression for the cumulative distribution function of the maximum value of the objective function in a stochastic linear programming problem, where either the objective function coefficients or the right hand side coefficients are continuous random vectors with known probability distributions. This is the "wait and see" problem of stochastic linear programming. Explicit results for the distribution problem are extremely difficult to obtain; indeed, previous results are known only if the right hand side coefficients have an exponential distribution [1]. To date, no explicit results have been obtained for stochastic *c*, and no new results of any form have appeared since the 1970's. In this paper, we obtain the first results for stochastic *c*, and new explicit results if *b* an *c* are stochastic vectors with an exponential, gamma, uniform, or triangle distribution. A transformation is utilized that greatly reduces computational time.

Keywords

Stochastic Linear Programming, The Wait and See Problem, Mathematics Subject Classification

1. Introduction

Consider the linear programming problem,

$$\text{Max } z(x) = cx \tag{1}$$

$$s.t: (A, I)x = b \tag{2}$$

$$x \geq 0 \tag{3}$$

where c is an $1\times(m+n)$ vector whose j_{th} component is c_j (where, $c_j = 0$, for $j > n$) and b is an $m\times 1$ vector whose i_{th} component is b_i, $A = (a_{ij})$ is an $m\times n$ matrix, I is an $m\times m$ identity matrix and x is an $(m+n)\times 1$ vector. Further assume that b and c are random vectors with joint density functions $f(b)$ and $g(c)$ respectively. Next, consider the value of $z(x)$ by first observing the vector b or the vector c and then solving (1)-(3). This paper is interested in finding explicit expressions for the distribution of $\max z(x)$ if either b or c is random. This is called the distribution problem of stochastic linear programming.

Early work on the distribution problem can be found in Babbar [2], Bereanu [3] [4] [5] [6] [7], Hsia [8], Prekopa [9], Sengupta, Tintner, and Millham [10], Sengupta, Tintner, and Morrison [11], and Wets [12]. For additional references, see the bibliographies by Stancu-Minasian [13] and Van Der Vlerk [14]. Application of the distribution problem can be found in the areas of agriculture [15] and economic planning [10], [11]. Explicit results for the distribution of $\max z(x)$ are very difficult to obtain; indeed, most analyses rely on approximation techniques or simulation. (See, for example, Bracken and Soland [16], Sarper [15], or Dempster [17]). Bereanu [3] discovered that under certain assumptions, the sample space of the random coefficients allows a partition into non-overlapping sets, called decision regions, such that a basis of the linear programming problem can be assigned to each of the sets, and this basis remains optimal for all of its sample points. Ewbank, *et al.* [1] extended this theory using a Jacobian transformation to simplify the computational analysis. To date, we believe that an explicit expression for the distribution of $\max z(x)$ has only been obtained for stochastic b [1], and no explicit results have been obtained for stochastic c. In addition, no explicit results have been obtained for non-exponential distributions. In this paper, we obtain new explicit results for exponential, uniform, gamma, and triangle distributions with b or c random. These are the first explicit results for the case in which c is random.

2. Theory

Following [1], consider the linear programming problem (1)-(3). Let $x_B^i = (x_1^i, \cdots, x_m^i)$ be the vector of basic variables corresponding to the ith basis, and B_i is the $m\times m$ basis matrix whose columns are the columns of (A, I) corresponding to the elements of x_B^i. Let $c_B^i = (c_1^i, \cdots, c_m^i)$ be the vector of coefficients of the basic variables in i_{th} basis and let a_j be the j_{th} column of (A, I) corresponding to c_j. Also, $D_i = \{ j | x_j^i \text{ is a basic variable} \}$ and $E_i = \{ k | x_k^i \text{ is a nonbasic variable} \}$. B_i is an optimal basis if

$$c_B^i B_i^{-1} a_j - c_j \geq 0$$

For all

$$j = 1, \cdots, m+n \tag{4}$$

and is feasible if

$$B_i^{-1} b \geq 0 \tag{5}$$

For the case in which the b vector is random, let the probability space be defined by the m-tuple $(b_1,\cdots,b_m)$. Bereanu discovered that there exist non-overlapping regions

$$S_i = \left\{ b \middle| \left(B_i^{-1}b\right)_j \ge 0 \text{ for all } j = 1,\cdots,m \right\} \tag{6}$$

where

$$P\left(S_k \cap S_l\right) = 0 \text{ for } k \ne l \tag{7}$$

Thus,

$$\Pr\left\{\left[z(x) \le \varnothing\right]\right\} = \sum_i \Pr\left[z(x) \le \varnothing \middle| S_i\right]\Pr\left\{S_i\right\} \tag{8}$$

Now, let

$$V_i = \left\{ b \middle| b \in S_i \right\}$$

Since

$$\Pr\left\{S_i\right\} = \int_{V_i}\cdots\int f(b)\prod\nolimits_{i=1}^{m}\mathrm{d}b_i \tag{9}$$

Then

$$\Pr\left\{\left[z(x) \le \varnothing\right] \cap S_i\right\} = \Pr\left\{z(x) \le \varnothing \middle| S_i\right\}\Pr\left\{S_i\right\} \tag{10}$$

$$\Pr\left\{z(x) \le \varnothing \middle| S_i\right\} = \Pr\left\{\left[z(x) \le \varnothing\right] \cap S_i\right\} \big/ \Pr\left\{S_i\right\} \tag{11}$$

$$\Pr\left\{\left[z(x) \le \varnothing\right] \cap S_i\right\} = \int_{\left[c_B^i x_B^i \le \varnothing\right] \cap V_i}\cdots\int f(b)\prod\nolimits_{i=1}^{m}\mathrm{d}b_i \tag{12}$$

Thus,

$$\Pr\left\{\left[z(x) \le \varnothing\right]\right\} = \sum\nolimits_i \Pr\left\{\left[z(x) \le \varnothing\right] \cap S_i\right\} \tag{13}$$

Now, consider the case in which only the c vector is random. Let the probability space C be defined by the n-tuple $c = (c_1,\cdots,c_n)$. Bereanu [3] found that the space C is partitioned by the sets:

$$T_i = \left\{ c \middle| c_B^i B_i^{-1} a_j - c_j \ge 0 \text{ for all } j \text{ such that } x_j^i \in E^i \right\} \tag{14}$$

where i refers to the i_{th} basis. Further the set of points $\left\{ c \middle| c_B^i B_i^{-1} a_j - c_j = 0 \text{ for all } j \in E^i \right\}$ is of probability measure zero if the joint density function of i is continuous. Points in this set are such that alternate optimal basis give the same value of $z(x)$. Also,

$$\Pr\left\{T_k \cap T_l\right\} = 0 \text{ for } k \ne l \tag{15}$$

Thus,

$$\Pr\left\{z(x) \le \varnothing\right\} = \sum\nolimits_i \left\{\Pr\left[z(x) \le \varnothing\right] \cap T_i\right\} = \sum\nolimits_i \Pr\left[z(x) \le \varnothing \middle| T_i\right]\Pr\left\{T_i\right\} \tag{16}$$

where $\varnothing$ is an arbitrary constant.

To evaluate the right-hand side of equation Equation (15) let $U_i = \{c | c \in T_i\}$. By definition

$$\Pr\{T_i\} = \int_{U_i}\cdots\int f'(c)\prod_{i=1}^{n}\mathrm{d}c_i \tag{17}$$

Since

$$\Pr\{[z(x) \le \varnothing] \cap T_i\} = \Pr[z(x) \le \varnothing | T_i]\Pr\{T_i\} \tag{18}$$

Then

$$\Pr\{[z(x) \le \varnothing] | T_i\} = \Pr\{[z(x) \le \varnothing] \cap T_i\} / \Pr\{T_i\} \tag{19}$$

where

$$\Pr\{[z(x) \le \varnothing] \cap T_i\} = \int_{[c_B^i x_B^i \le \varnothing] \cap U_i}\cdots\int f'(c)\prod_{i=1}^{n}\mathrm{d}c_i \tag{20}$$

Thus, if only the c vector is random the distribution function of $\max z(x)$ can be found, in theory, by evaluating the integral in equation Equation (20). Given a basis Bi and sets U_i and V_i, the limits of the integral in Equation (12) and Equation (20) are the intersection of m or n hyperplanes (depending on whether the b vector or the c vector is stochastic). These limits are extremely difficult to obtain if the probability space has dimension greater than 3. Ewbank, *et al*., [1] developed a Jacobian transformation that greatly simplifies the computation of the integrals.

In the case of stochastic b, Let

$$B_i^{-1} b \ge 0 \tag{21}$$

By substituting for b we have:

$$B(B^{-1}b) = b = Br \tag{22}$$

The probability that a basis G remains feasible is

$$P = \int_{s}\cdots\int f(b)\prod_{i=1}^{m}\mathrm{d}b_i \tag{23}$$

where s is the set of b's defined in Equation (20), and by substituting Equation (21) in Equation (22), we have:

$$P = \int\int_{r \ge 0}\cdots\int f(Br | J_r)\prod_{i=1}^{m}\mathrm{d}r_i \tag{24}$$

where J_r is the Jacobian

$$J_r = \det[\partial b_k / \partial r_i]$$

Because $b_k = (Br)_k = \sum_{j=1}^{m} B_{kj} r_j$ and $\partial b_k / \partial r_i = B_{ki}$, this implies

$$J_r = \det[B_{ki}] = \det(B) \tag{25}$$

Note that since B is the basis matrix, its determinant is nonzero; thus J_r is also nonzero.

3. Computational Results

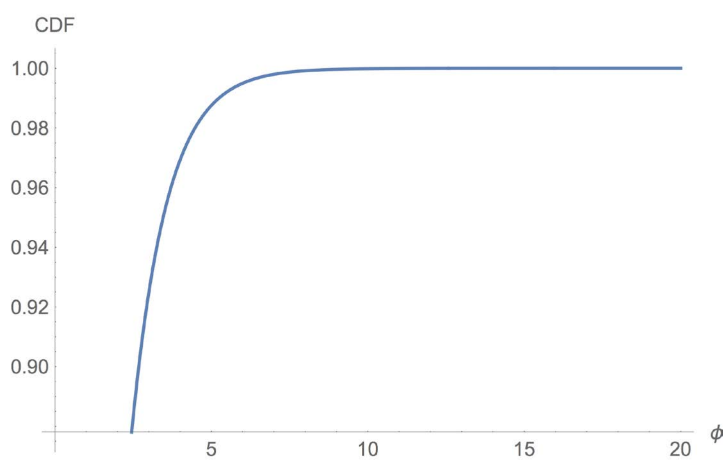
The problems were run using the Mathematica software version 8.0.1.0 utilizing the supercomputer at the University of Oklahoma.

- CPUs: All compute nodes have dual Intel Xeon E5-2650 "Sandy Bridge" oct core 2.0 GHz CPUs; there is also one "fat node" with quad Intel Xeon E7-4830 "Westmere" oct core 2.13 GHz CPUs.
- RAM: Most of the compute nodes have 32 GB of 1333 MHz RAM and 23 with 64 GB of 1333 MHz RAM; the one "fat node" has 1 TB of 1066 MHz RAM, which is called large memory.
- Accelerators: There are 18 NVIDIA Tesla M2075 cards, for an aggregate of an additional approximately 9 TFLOPs double precision.

In order to compare the run times, four types of distributions were considered as shown in Table 1. The coefficients were randomly generated in small interval, because large intervals led to computational results that had results with coefficients of the orders of 1020 or larger.

3.1. Results for Stochastic b with Exponential Distribution

3.1.1. Problem 1

Case	Case I. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$
B	$\{b_1, b_2\} \sim \text{Exponential iid}(\lambda = 1)$
C	$\{2,3\}$
CDF	$\begin{cases} \frac{1}{6}e^{\varnothing}\left(-6+6e^{\varnothing}-\varnothing\right) & \varnothing > 0 \\ 0 & \varnothing < 0 \end{cases}$
CDF Plot	
Time	2.808 seconds.

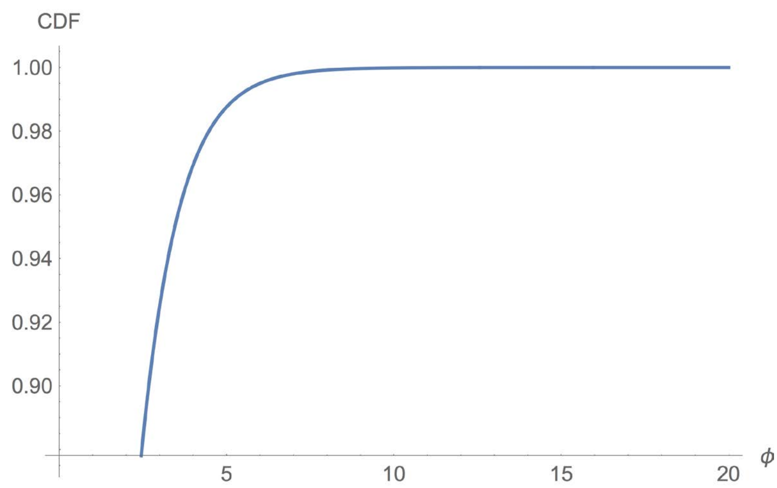
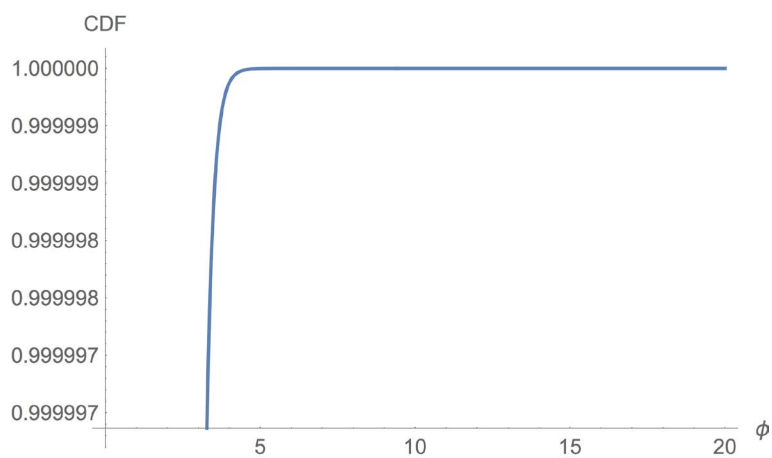
Case	Case I. Stochastic Resource Vector. Bereanu's model
A	$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$
B	$\{b_1, b_2\} \sim \text{Exponential iid}(\lambda = 1)$
C	$\{2, 3\}$
CDF	$\begin{cases} \frac{1}{6} e^{\varnothing}\left(-6 + 6e^{\varnothing} - \varnothing\right) & \varnothing > 0 \\ 0 & \varnothing < 0 \end{cases}$
CDF Plot	
Time	4.134 seconds.

Table 1. Equations of distribution.

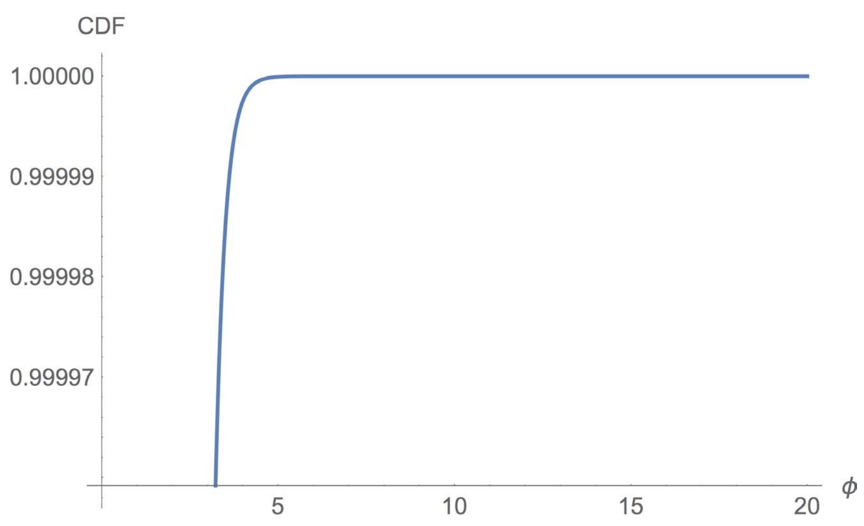
Distribution	Defined Equation	Parameters	pdf
Exponential	$f(x) = \lambda e^{-\lambda x}$	$\lambda = 1$	$f(x) = e^{-x}$
Uniform	$f(x) = \frac{1}{a - b}$	$b = 8$ $a = 0$	$f(x) = \frac{1}{8}$
Gamma	$f(x) = \frac{(x-\gamma)^{\alpha-1} \exp\left[-\frac{(x-y)}{\beta}\right]}{\beta^{\alpha}\Gamma(\alpha)}$	$\beta = 1$ $\alpha = 0$ $\gamma = 0$	$f(x) = xe^{-x}$
Triangular	$f(x) = f(x \mid a,b,c) = \begin{cases} 0 & x < a \\ \frac{2(x-a)}{(b-a)(c-a)} & a \le x \le c \\ \frac{2(b-a)}{(b-a)(b-c)} & c < C \le b \\ 0 & b < x \end{cases}$	$a = 0$ $c = 0.5$ $b = 1$	$\begin{cases} 0 & x < 0 \\ 4x & 0 \le x \le 0.5 \\ 4(1-x) & 0.5 \le x \le 1 \\ 0 & 1 < x \end{cases}$

3.1.2. Problem 2

Case	Case I. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 42 & 64 & 62 & 18 & 3 & 3 \\ 68 & 65 & 2 & 34 & 65 & 17 \\ 32 & 20 & 13 & 1 & 10 & 55 \\ 37 & 56 & 44 & 7 & 30 & 11 \\ 62 & 39 & 59 & 61 & 52 & 24 \\ 18 & 35 & 50 & 65 & 5 & 42 \end{bmatrix}$
B	$\{b_1, b_2, b_3, b_4, b_5, b_6\} \sim \text{Exponential iid}(\lambda = 1)$
C	$\{53, 57, 38, 7, 1, 32\}$
CDF	$\begin{cases} 1 + \frac{1126418971}{126591405630 e^{\frac{186\varnothing}{7}}} - \frac{174827706709}{334942876224 e^{\frac{115\varnothing}{19}}} - \frac{148007864405}{89303808 e^{\frac{93\varnothing}{19}}} + \frac{187191798507739}{105177626880 e^{\frac{259\varnothing}{53}}} - \frac{23288387}{189486 e^{\frac{19\varnothing}{4}}} & \varnothing > 0 \\ 0 & \varnothing < 0 \end{cases}$
CDF Plot	
Time	1335.08 Seconds.

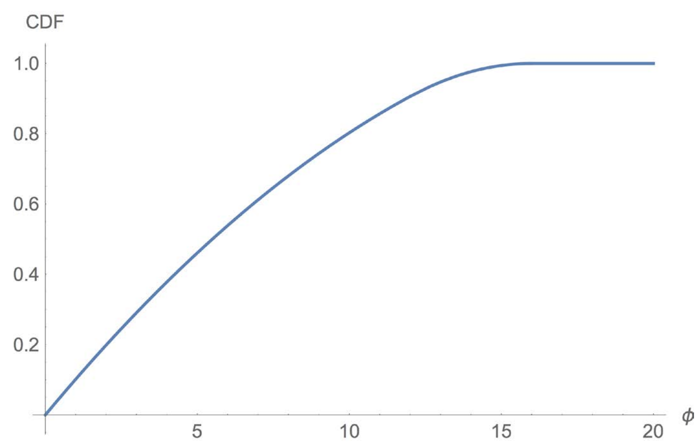
Case	Case I. Stochastic Resource Vector. Bereanu's model
A	$\begin{bmatrix} 42 & 64 & 62 & 18 & 3 & 3 \\ 68 & 65 & 2 & 34 & 65 & 17 \\ 32 & 20 & 13 & 1 & 10 & 55 \\ 37 & 56 & 44 & 7 & 30 & 11 \\ 62 & 39 & 59 & 61 & 52 & 24 \\ 18 & 35 & 50 & 65 & 5 & 42 \end{bmatrix}$
B	$\{b_1, b_2, b_3, b_4, b_5, b_6\} \sim \text{Exponential iid}(\lambda = 1)$
C	$\{1, 2, 3, 2, 1, 1\}$
CDF	No Result

3.1.3. Problem 3

Case	Case I. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 13 & 69 & 56 & 45 & 23 & 39 & 34 & 4 & 38 \\ 30 & 65 & 8 & 51 & 29 & 59 & 65 & 54 & 18 \\ 30 & 20 & 53 & 5 & 46 & 52 & 55 & 32 & 1 \\ 65 & 45 & 4 & 41 & 43 & 50 & 41 & 5 & 54 \\ 3 & 31 & 31 & 46 & 19 & 49 & 2 & 64 & 2 \\ 26 & 52 & 67 & 13 & 27 & 55 & 59 & 0 & 63 \\ 13 & 55 & 31 & 35 & 46 & 29 & 23 & 40 & 34 \\ 55 & 16 & 40 & 37 & 62 & 48 & 34 & 21 & 14 \\ 7 & 21 & 14 & 32 & 55 & 66 & 47 & 45 & 34 \end{bmatrix}$
B	$\{b_1, b_2, b_3, b_4, b_5, b_{6,,} b_7, b_8, b_9\} \sim \text{Exponential iid}(\lambda = 1)$
C	$\{63,\ 58,\ 33,\ 49,\ 50,\ 64,\ 47,\ 70,\ 40\}$
CDF	$\begin{cases} 1+\frac{923512167204243e^{\frac{-304\varnothing}{33}}}{112421428303294292}-\frac{4587824269069469e^{\frac{-360\varnothing}{47}}}{722317922166417500}-\frac{1829039266689190654e^{\frac{-257\varnothing}{40}}}{2336758228034180379}-\frac{727938601434779113e^{\frac{-365\varnothing}{58}}}{410788623816360000}-\frac{1665192956e^{\frac{-305\varnothing}{49}}}{1394020125}+\frac{14540635702595879e^{\frac{-55\varnothing}{14}}}{398109809908800}-\frac{264776529949169098094e^{\frac{-242\varnothing}{63}}}{7094601260302987937} & \varnothing > 0 \\ 0 & \varnothing < 0 \end{cases}$
CDF Plot	
Time	7792.77 Seconds.

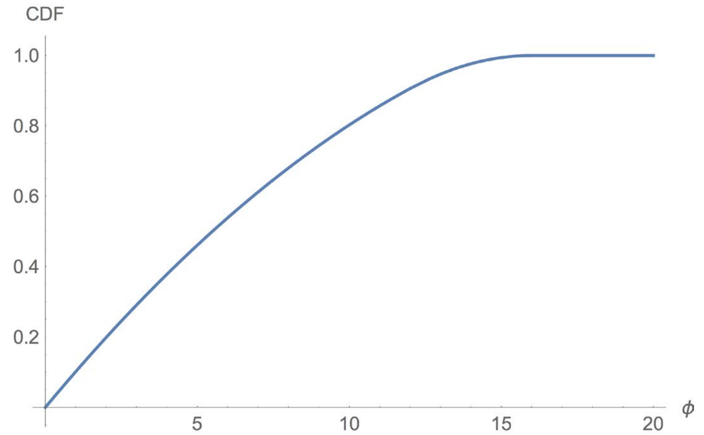
3.2. Results for Stochastic b with Uniform Distribution

3.2.1. Problem 1

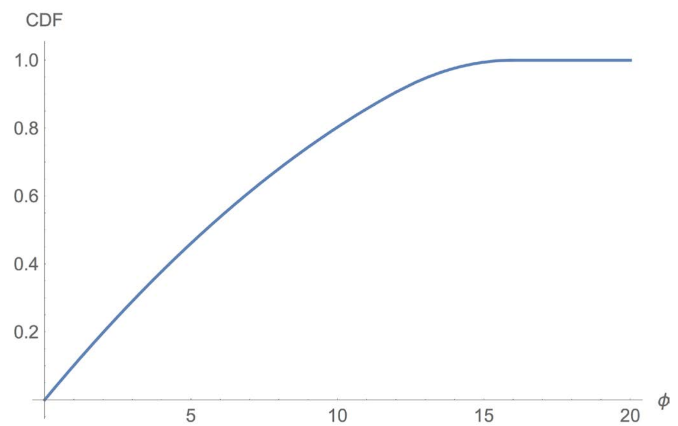
Case	Case I. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$
B	$\{b_1, b_2\} \sim \text{Uniform}(0,8)$
C	$\{2,3\}$
CDF	$\begin{cases} \frac{(480-11\varnothing)\varnothing}{460B} & 0<\varnothing<12 \\ \frac{1}{512}(-256+96\varnothing-3\varnothing^2) & 12<\varnothing<16 \\ 1 & \varnothing>16 \\ 0 & \varnothing<0 \end{cases}$
CDF Plot	
Time	1.36 Seconds.

Case	Case I. Stochastic Resource Vector. Bereanu's model
A	$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$
B	$\{b_1, b_2\} \sim \text{Uniform}(0,8)$
C	$\{2,3\}$
CDF	$\begin{cases} \frac{(480-11\varnothing)\varnothing}{460B} & 0<\varnothing<12 \\ \frac{1}{512}(-256+96\varnothing-3\varnothing^2) & 12<\varnothing<16 \\ 1 & \varnothing>16 \\ 0 & \varnothing<0 \end{cases}$

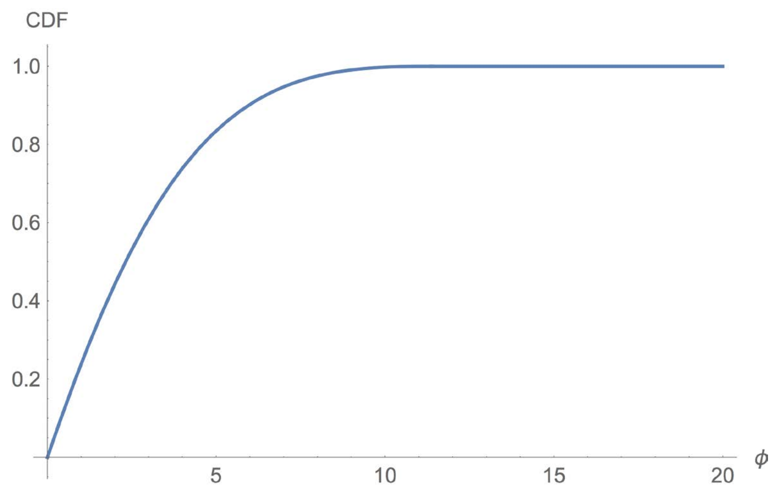
Continued

CDF Plot	
Time	1.36 Seconds.

3.2.2. Problem 2

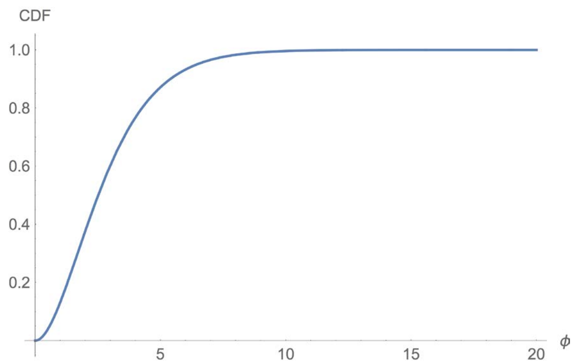
Case	Case I. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 2 & 6 & 25 \\ 26 & 17 & 22 \\ 39 & 51 & 42 \end{bmatrix}$
B	$\{b_1, b_2, b_3\} \sim \text{Uniform}(0,8)$
C	$\{4,3,46\}$
CDF	$\begin{cases} -\dfrac{\varnothing\left(-1472736+78775\varnothing+9299\varnothing^2\right)}{460B} & 0<\varnothing<\dfrac{32}{39} \\ \dfrac{-46970460160+5987642931648\varnothing-4615284022256\varnothing^2+114780919050\varnothing^3}{24705054176256} & \dfrac{32}{39}<\varnothing<\dfrac{184}{21} \\ 1 & \varnothing>\dfrac{184}{21} \\ 0 & \varnothing<0 \end{cases}$
CDF Plot	
Time	6.8 Seconds.

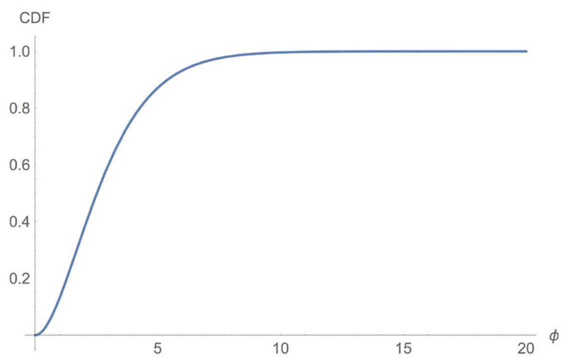
3.2.3. Problem 3

Case	Case I. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 4 & 10 & 3 & 5 & 1 & 4 \\ 5 & 5 & 10 & 6 & 4 & 4 \\ 2 & 8 & 8 & 5 & 1 & 5 \\ 7 & 9 & 5 & 2 & 8 & 3 \\ 2 & 6 & 4 & 8 & 1 & 2 \\ 8 & 6 & 7 & 6 & 8 & 6 \end{bmatrix}$
B	$\{b_1, b_2, b_3, b_4, b_5, b_6\} \sim \text{Uniform}(0,8)$
C	$\{10,\ 2,\ 2,\ 9,\ 4,\ 4\}$
CDF	$\begin{cases} \frac{\varnothing(946819694592000000 - 518719167512000000\varnothing - 623227239936000\varnothing^2 + 88202659881600\varnothing^3}{371504185344000000} + \frac{\varnothing(-365032235188\varnothing^4 + 513245190430\varnothing^5}{371504185344000000} & 0 < \varnothing < 4 \\ \frac{-819350510400000 + 667123354800000\varnothing - 588214667070000\varnothing^2 + 455829800400\varnothing^3)}{23219011584000000} + \frac{214304599260\varnothing^4 - 118583281368\varnothing^5 + 1973181073\varnothing^6}{2319011584000000} & 4 < \varnothing \le 9 \\ \frac{-346924362544798400000 + 421027877219066400000\varnothing - 562088033100693300000\varnothing^2}{911852659408896000000} + \frac{381486272393287600\varnothing^3 - 13806572388432060\varnothing^4 + 2514521036736408\varnothing^5 - 18027084023513\varnothing^6}{911852659408896000000} & 9 < \varnothing \le 10 \\ \frac{2050268425803784192 - 1109709749741822976\varnothing + 255630745634538240\varnothing^2 - 31026699217604480\varnothing^3}{72948212752711680} + \frac{2096997758693760\varnothing^4 - 74945946698532\varnothing^5 + 1107879173351\varnothing^6}{72948212752711680} & 10 < \varnothing < \frac{148}{13} \\ 113 & \varnothing \ge 148 \\ 0 & \varnothing < 0 \end{cases}$
CDF Plot	
Time	260.76 Seconds.

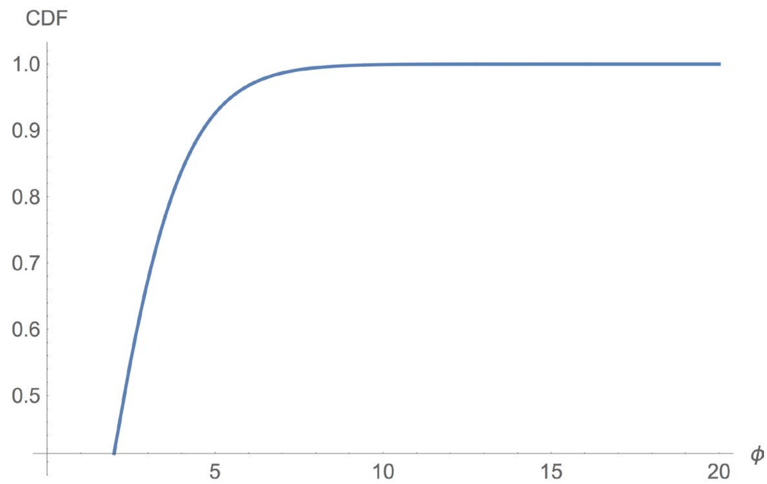
3.3. Results for Stochastic b with Gamma Distribution

3.3.1. Problem 1

Case	Case I. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$
B	$b_i \sim Gamma\left(\propto = 2, \beta = 1\right), i = 1, 2$
C	$\{2, 3\}$
CDF	$\begin{cases} \frac{1}{648} e^{-\varnothing}\left(-648 + 648e^{\varnothing} - 648\varnothing - 207\varnothing^2 - 26\varnothing^3\right) & \varnothing > 0 \\ 0 & \varnothing < 0 \end{cases}$
CDF Plot	
Time	1.685 Seconds

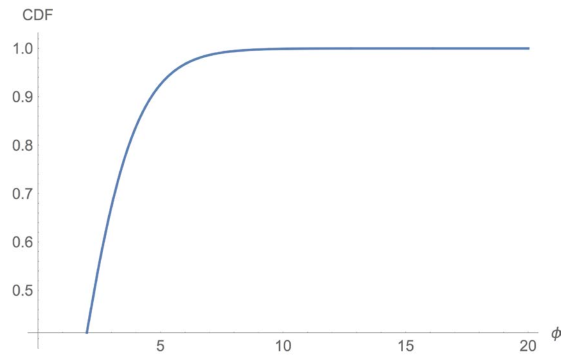
Case	Case I. Stochastic Resource Vector. Bereanu's model
A	$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$
B	$b_i \sim \text{Gamma}\left(\propto = 2, \beta = 1\right), i = 1, 2$
C	$\{2, 3\}$
CDF	$\begin{cases} \frac{1}{648} e^{-\varnothing}\left(-648 + 648e^{\varnothing} - 648\varnothing - 207\varnothing^2 - 26\varnothing^3\right) & \varnothing > 0 \\ 0 & \varnothing < 0 \end{cases}$
CDF Plot	
Time	2.761

3.3.2. Problem 2

Case	Case I. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 6 & 1 \end{bmatrix}$
B	$b_i \sim \text{Gamma}(\propto = 2, \beta = 1), i = 1,2,3$
C	$\{5,3,2\}$
CDF	$\begin{cases} \Big(e^{\frac{-8\varnothing}{3}}\Big(65884500e^{\frac{8\varnothing}{3}} - 243e^{\frac{22\varnothing}{15}}(243000+379225\varnothing+157905\varnothing^2+15972\varnothing^3)\Big) \\ +125(-54684+24519\varnothing+91355\varnothing^2+26620\varnothing^3)\Big)\Big)/6588400 & \varnothing > 0 \\ 0 & \varnothing < 0 \end{cases}$
CDF Plot	
Time	16.44

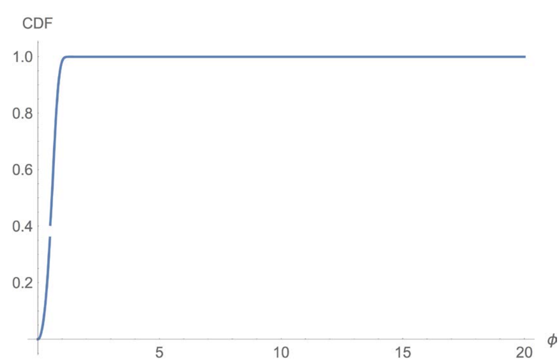
Case	Case I. Stochastic Resource Vector. Bereanu's model
A	$\begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 6 & 1 \end{bmatrix}$
B	$b_i \sim \text{Gamma}(\propto = 2, \beta = 1), i = 1,2,3$
C	$\{5,3,2\}$
CDF	$\begin{cases} \Big(e^{\frac{-8\varnothing}{3}}\Big(65884500e^{\frac{8\varnothing}{3}} - 243e^{\frac{22\varnothing}{15}}(243000+379225\varnothing+157905\varnothing^2+15972\varnothing^3)\Big) \\ +125(-54684+24519\varnothing+91355\varnothing^2+26620\varnothing^3)\Big)\Big)/6588400 & \varnothing > 0 \\ 0 & \varnothing < 0 \end{cases}$

Continued

CDF Plot	
Time	26.52

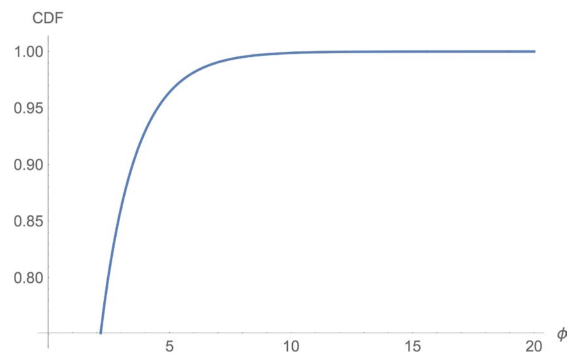
3.4. Results for Stochastic b with Triangle Distribution

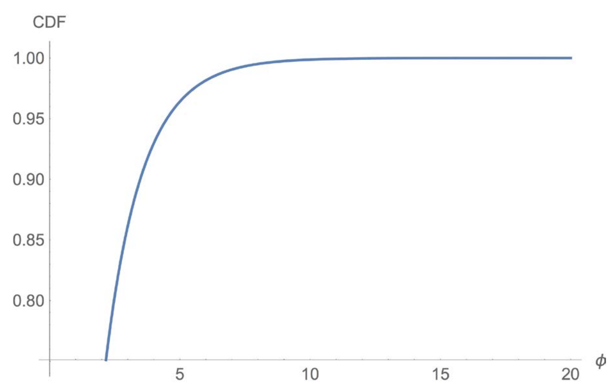
Problem 1

Case	Case I. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$
B	$\{b_1, b_2, b_3\} \sim \text{Triangular}(b \mid 0, 1, 0.5) = \begin{cases} 0 & b < 0 \\ 4b & 0 \le b \le 0.5 \\ 4(1-b) & 0.5 \le b \le 1 \\ 0 & 1 < b \end{cases}$
C	$\{3, 2, 2\}$
CDF	$\begin{cases} 1.222\varnothing^2 + 1.4537\varnothing^4 - 1.7222\varnothing^6 & 0 \le \varnothing \le 0.5 \\ 8.2003 - 52.7154\varnothing + 133.976\varnothing^2 - 164.267\varnothing^3 + 105.386\varnothing^4 - 33.8097\varnothing^5 + 4.21528\varnothing^6 & 0.5 \le \varnothing \le 1 \\ -13.4184 + 46.8147\varnothing - 49.5551\varnothing^2 + 5.9317\varnothing^3 + 24.9566\varnothing^4 - 17.3611\varnothing^5 + 3.6169\varnothing^6 & 1 < \varnothing \le 1.1 \\ -26.2336 + 116.715\varnothing - 208.42\varnothing^2 + 198.495\varnothing^3 - 106.337\varnothing^4 + 30.3819\varnothing^5 - 3.6169\varnothing^6 & 1.1 < \varnothing < 1.4 \\ 1 & \varnothing \ge 1.4 \\ 0 & \varnothing < 0 \end{cases}$
CDF Plot	
Time	410.189 Seconds

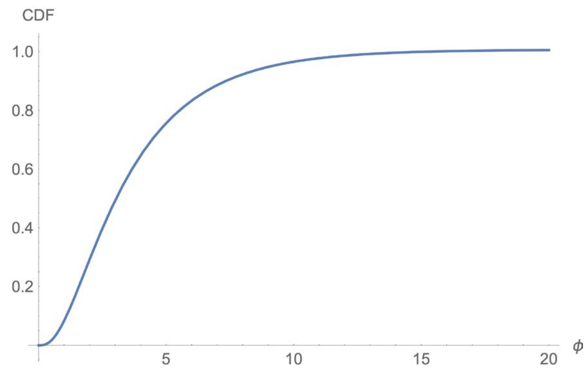
3.5. Results for Stochastic c with Exponential Distribution

3.5.1. Problem 1

Case	Case II. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 5 & 4 \\ 10 & 3 \end{bmatrix}$
C	$\{C_1, C_2\} \sim e^{(C_1+C_2)}$
B	$\{6,5\}$
CDF	$\begin{cases} 1+\frac{35}{33}e^{\frac{-13\varnothing}{5}}-e^{-2\varnothing}-\frac{2}{33}e^{\frac{-3\varnothing}{2}}-e^{\frac{-2\varnothing}{3}} & \varnothing>0 \\ 0 & \varnothing<0 \end{cases}$
CDF Plot	
Time	5.85 Seconds

Case	Case II. Stochastic Resource Vector. Bereanu's model
A	$\begin{bmatrix} 5 & 4 \\ 10 & 3 \end{bmatrix}$
C	$\{C_1, C_2\} \sim e^{(C_1+C_2)}$
B	$\{6,5\}$
CDF	$\begin{cases} 1+\frac{35}{33}e^{\frac{-13\varnothing}{5}}-e^{-2\varnothing}-\frac{2}{33}e^{\frac{-3\varnothing}{2}}-e^{\frac{-2\varnothing}{3}} & \varnothing>0 \\ 0 & \varnothing<0 \end{cases}$
CDF Plot	
Time	17.97 Seconds

3.5.2. Problem 2

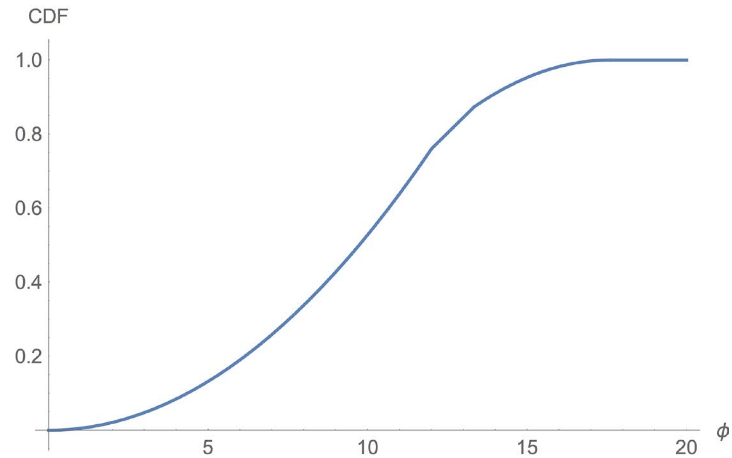
Case	Case II. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 10 & 2 & 5 \\ 1 & 3 & 2 \\ 6 & 3 & 4 \end{bmatrix}$
C	$\{C_1, C_2, C_3\} \sim \mathrm{e}^{(C_1+C_2+C_3)}$
B	$\{7,6,5\}$
CDF	$\begin{cases} \frac{23368}{23205}+\frac{205}{3744}\mathrm{e}^{\frac{-13\varnothing}{5}}-\frac{1757}{6120}\mathrm{e}^{\frac{-17\varnothing}{7}}+\frac{7}{69}\mathrm{e}^{\frac{-15\varnothing}{7}}+\frac{49}{132}\mathrm{e}^{\frac{-12\varnothing}{7}} & \\ +\frac{30}{119}\mathrm{e}^{\frac{-7\varnothing}{5}}+\frac{\mathrm{e}^{-\varnothing}}{160}+\frac{49}{85}\mathrm{e}^{\frac{-5\varnothing}{6}}-\frac{9}{44}\mathrm{e}^{\frac{-2\varnothing}{3}}-\frac{53e^{\frac{-\varnothing}{2}}}{69}-\mathrm{e}^{\frac{-\varnothing}{3}} & \varnothing>0 \\ 0 & \varnothing<0 \end{cases}$
CDF Plot	
Time	34.991 Seconds

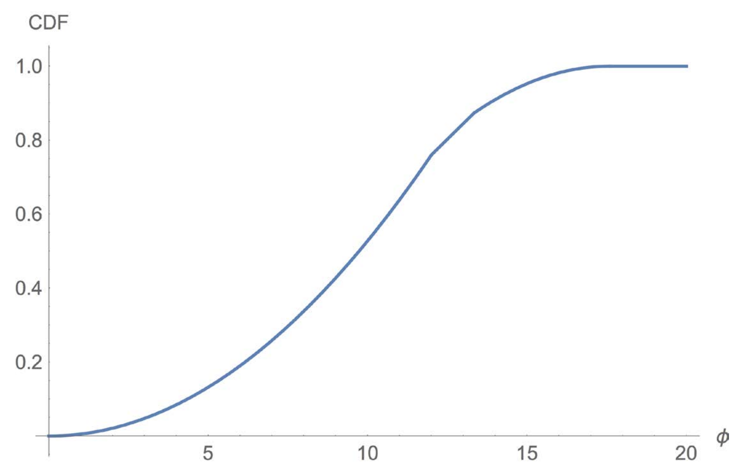
3.6. Results for Stochastic c with Uniform Distribution

3.6.1. Problem 1

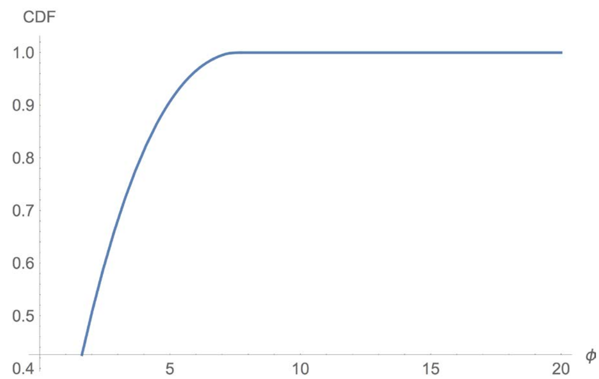
Case	Case II. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$
C	$\{C_1, C_2\} \sim \mathrm{uniform}(0,8)$
B	$\{10,10\}$
CDF	$\begin{cases} \frac{19\varnothing^2}{3600} & 0<\varnothing<12 \\ -\frac{2}{7}+\frac{5\varnothing}{56}-\frac{\varnothing^2}{5600} & 12<\varnothing<\frac{40}{3} \\ -\frac{5(832-176\varnothing+5\varnothing^2)}{3584} & \frac{40}{3}<\varnothing<\frac{88}{5} \\ 0 & 5\varnothing>88 \end{cases}$

Continued

CDF Plot	
Time	2.606 Seconds

Case	Case II. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$
C	$\{C_1, C_2\} \sim \text{uniform}(0,8)$
B	$\{10,10\}$
CDF	$\begin{cases} \frac{19\varnothing^2}{3600} & 0<\varnothing<12 \\ -\frac{2}{7}+\frac{5\varnothing}{56}-\frac{\varnothing^2}{5600} & 12<\varnothing<\frac{40}{3} \\ -\frac{5\left(832-176\varnothing+5\varnothing^2\right)}{3584} & \frac{40}{3}<\varnothing<\frac{88}{5} \\ 0 & 5\varnothing>88 \end{cases}$
CDF Plot	
Time	3.12 Seconds

3.6.2. Problem 2

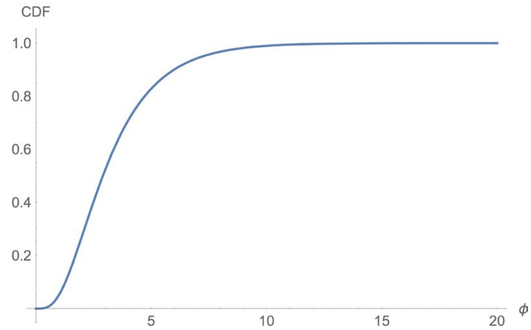
Case	Case II. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 20 & 38 & 23 \\ 40 & 36 & 26 \\ 24 & 30 & 29 \end{bmatrix}$
C	$\{C_1, C_2, C_3\} \sim \text{uniform}(0,8)$
B	$\{27, 22, 29\}$
CDF	$\begin{cases} \frac{\varnothing(16898367744 - 1625159\varnothing + 45699175\varnothing^2)}{54618872832} & 0 < \varnothing \le 6 \\ \frac{-885527518720 + 219442583488\varnothing - 214078289688\varnothing^2 + 8704702125\varnothing^3}{5682385695744} & 6 < \varnothing \le \frac{1096}{165} \\ \frac{-88146918912 + 222257865280\varnothing - 24372051240\varnothing^2 + 878254875\varnothing^3}{577406648320} & \frac{1096}{165} < \varnothing \le \frac{232}{33} \\ \frac{3(-4893594112 + 2398313408\varnothing - 307856280\varnothing^2 + 13171149\varnothing^3)}{4001054720} & \frac{232}{33} < \varnothing \le \frac{856}{11} \\ 1 & 111\varnothing > 856 \\ 0 & 0 < \varnothing \end{cases}$
CDF Plot	
Time	105.144 Seconds

3.7. Results for Stochastic c with Gamma Distribution

3.7.1. Problem 1

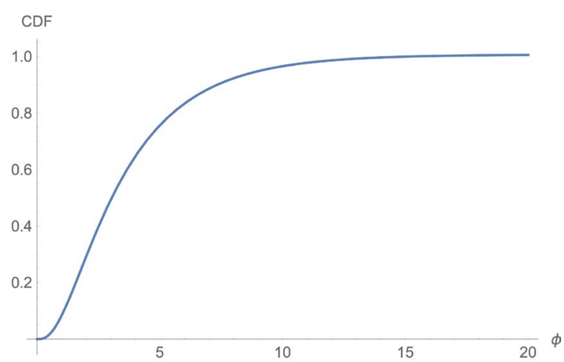
Case	Case II. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$
C	$\{C_1, C_2\} \sim C_1 \times C_2 e^{(C_1+C_2)}$
B	$\{3, 2\}$
CDF	$\begin{cases} \frac{1}{216} e^{-2\varnothing} \Big(216e^{2\varnothing} - 72e^{\frac{4\varnothing}{3}}(3+2\varnothing) - 108e^{\frac{\varnothing}{2}}(2+3\varnothing) \\ + e^{\varnothing}(20 - 50\varnothing - 8\varnothing^2) + 7(28 + 66\varnothing + 27\varnothing^2) \Big) & 0 > \varnothing \\ 0 & \varnothing < 0 \end{cases}$

Continued

CDF Plot	
Time	9.454 Seconds

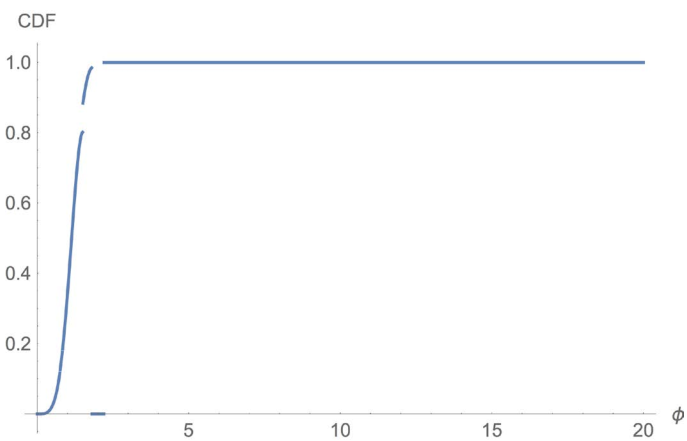
Case	Case II. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 2 & 5 \\ 2 & 4 \end{bmatrix}$
C	$\{C_1, C_2\} \sim C_1 \times C_2 e^{(C_1+C_2)}$
B	$\{3,1\}$
CDF	No Result

3.7.2. Problem 2

Case	Case II. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 5 & 1 & 5 \\ 2 & 4 & 3 \\ 1 & 1 & 5 \end{bmatrix}$
C	$\{C_1, C_2, C_3\} \sim \{C_1 \times C_2 \times C_3\} e^{-(C_1+C_2+C_3)}$
B	$\{3,2\}$
CDF	$\begin{cases} \frac{23368}{23205} - \frac{205e^{\frac{-13\varnothing}{5}}}{3744} - \frac{1757e^{\frac{-17\varnothing}{7}}}{6120} + \frac{7}{69}e^{\frac{-15e}{e}} + \frac{49}{132}e^{\frac{-12\varnothing}{7}} + \frac{30}{119}e^{\frac{-7\varnothing}{5}} + \frac{e^{\varnothing}}{160} + \frac{49}{85}e^{\frac{-5\varnothing}{6}} - \frac{9}{44}e^{\frac{-2\varnothing}{3}} - \frac{53}{69}e^{\frac{-\varnothing}{2}} - e^{\frac{-\varnothing}{3}} & \varnothing > 0 \\ 0 & \varnothing < 0 \end{cases}$
CDF Plot	
Time	91.385 Seconds

3.8. Results for Stochastic c with Triangle Distribution

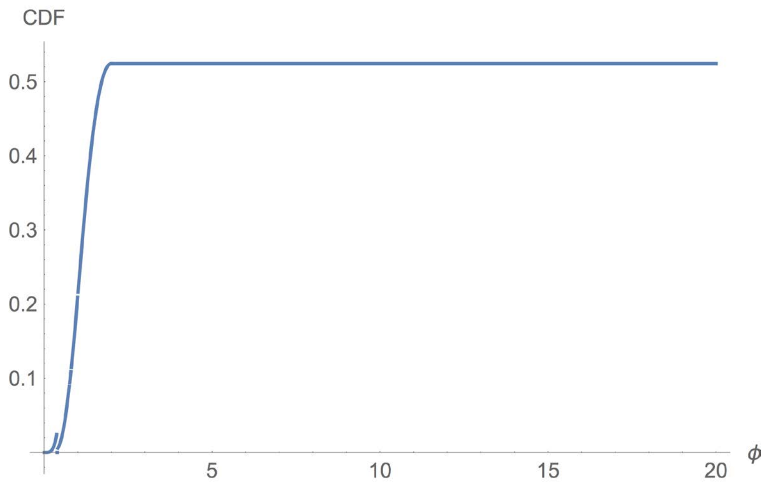
3.8.1. Problem 1

Case	Case II. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$
B	$\{C_1, C_2, C_3\} \sim \text{Triangular}(C \mid 0,1,0.5) = f(C \mid a,b,c) = \begin{cases} 0 & C<0 \\ 4C & 0 \le C \le 0.5 \\ 4(1-C) & 0.5 \le C \le 1 \\ 0 & 1<C \end{cases}$
C	$\{3,5\}$
CDF	$\begin{cases} 0.38\varnothing^4 & 0 \le \varnothing \le 0.75 \\ -0.027+0.27\varnothing-1.026\varnothing^2+1.53\varnothing^3-0.398\varnothing^4 & 0.75 \le \varnothing \le 0.83 \\ -0.28+1.73\varnothing-4.145\varnothing^2+4.49\varnothing^3-1.45\varnothing^4 & 0.83 \le \varnothing \le 1.5 \\ -2.33+3.76\varnothing-1.02\varnothing^2-0.012\varnothing^3+0.008\varnothing^4 & 1.5 \le \varnothing \le 1.66 \\ -0.29-2.16\varnothing+5.23\varnothing^2-2.98\varnothing^3-0.53\varnothing^4 & 1.66 < \varnothing \le 1.8 \\ -11.45+22.64\varnothing-15.43\varnothing^2+4.68\varnothing^3-0.53\varnothing^4 & 1.8 \le \varnothing \le 2.2 \\ 1 & 2.2 \le \varnothing \\ 0 & \varnothing < 0 \end{cases}$
CDF Plot	
Time	13.292 Seconds

3.8.2. Problem 2

Case	Case II. Stochastic Resource Vector. Ewbank's model
A	$\begin{bmatrix} 2 & 1 & 5 \\ 2 & 2 & 4 \\ 2 & 1 & 4 \end{bmatrix}$
C	$\{C_1, C_2, C_3\} \sim \text{Triangular}(C \mid 0,1,0.5) = \begin{cases} 0 & C<0 \\ 4C & 0 \le C \le 0.5 \\ 4(1-C) & 0.5 \le C \le 1 \\ 0 & 1<C \end{cases}$

Continued

B	$\{2,2,1\}$
CDF	$\begin{cases} 00.524755 & 2\le\varnothing \\ 1.32544\varnothing^{4} & 0\le\varnothing\le 0.375 \\ -0.228579-0.16\varnothing-1.12\varnothing^{2}-0.623333\varnothing^{3}+0.09875\varnothing^{4} & 15\le\varnothing\le 2 \\ 0.271421-1.49333\varnothing-2.45333\varnothing^{2}-1.21593\varnothing^{3}+0.197515\varnothing^{4} & 1\le\varnothing\le 1.5 \\ 0.792255-3.16\varnothing+4.45333\varnothing^{2}-2.29926\varnothing^{3}+0.426682\varnothing^{4} & 0.8\le\varnothing\le 1 \\ -50.3958-333.907\varnothing-918.213\varnothing^{2}+1341.7\varnothing^{3}-1098.47\varnothing^{4} & \\ +478.271\varnothing^{5}-86.5885\varnothing^{6} & 0.75<\varnothing\le 0.8 \\ -0.000798542-0.213349\varnothing^{4}+1.66836\varnothing^{5}-1.32544\varnothing^{6} & 0.4<\varnothing\le 0.75 \\ 0 & 0\ge\varnothing \end{cases}$
CDF Plot	

Case	Case II. Stochastic Resource Vector. Bereanu's model
A	$\begin{bmatrix} 2 & 1 & 5 \\ 2 & 2 & 4 \\ 2 & 1 & 4 \end{bmatrix}$
B	$\{C_1,C_2,C_3\}\sim \text{Triangular}(C\mid 0,1,0.5)=\begin{cases} 0 & C<0 \\ 4C & 0\le C\le 0.5 \\ 4(1-C) & 0.5\le C\le 1 \\ 0 & 1<C \end{cases}$
C	$\{2,2,1\}$
CDF	No Results

4. Computational Time Comparisons

The different distributions were solved using both Bereanu's method and the Ewbank, Foote and Kumin transformation method to compare the two. Table 2 and Table 3 compare the run times for both methods for case I and case II. The results show that

Table 2. Comparison between Bereanu and EFK method for case I.

	Size	Sample of number in result	Difference between run times
Exponential	2 × 2	2851	3.06
	3 × 3	10,071	7.71
	6 × 6	187,191,798,507,739	4.62
	9 × 9	264,776,529,949,169,000,000	11.00
Uniform	2 × 2	2,929,968	2.05
	3 × 3	46,970,460,160	2.06
	6 × 6	8,538,555,554,355,150,000	5.41
	9 × 9	844,697,996,409,499,233,632,305,152	86.61
Gamma	2 × 2	549,615,780	2.60
	3 × 3	15,629,133,492	1.41
	6 × 6	243,545,558,927,209,970,255,163,031,323,401,871,559	4.70

Table 3. Comparison between Bereanu and EFK method for case II.

	Dimention	Bereanu's Method	EFK Method
Exponential	2 × 2	2.386	1.747
	3 × 3	78.68	17.97
	6 × 6	No Result	9176.28
Uniform	2 × 2	3.12	2.606
	3 × 3	210.4	105.144
Gamma	2 × 2	No Result	11.544
	3 × 3	No Result	115.004
Triangular	2 × 2	No Result	13.292
	3 × 3	No Result	575.846

the EFK method substantially reduces the computational time. In addition, Bereanu's method is not able to solve some larger sizes of the problem. All times are measured in seconds.

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