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A View on Stochastic Finite Element and Geostatistics for Resource Parameters Estimation

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ABSTRACT

The resource parameter estimation using stochastic finite element, geostatistics etc. is a key point on uncertainty, risk analysis, optimization [1-5] etc. In this view, the paper presents some consideration on: 1) Stochastic finite element estimation. The concept of random element is simplified as a stochastic finite element (SFE) taking into account a parallelepiped element with eight nodes in which are given the probability density functions (pdf) on its point supports. In this context it is shown: a—the stochastic finite element is a linear interpolator, related to the distributions given at each nodes; b—the distribution pdf in whatever point $x \in$ V; c—the estimation of the mean value of $Z(x)$; 2) Volume integrals calculus; 3) SFE in geostatistics approaches; 4) SFE in PDE solution. Finally, some conclusions are presented underlying the importance of SFE applications

Keywords: Parameter Estimation; SFE; Geostatistics; Kriging; Risk Analysis; Optimisation

1. Introduction

Many physical phenomena and processes are mathematically modeled by partial differential equations (PDE). The data required by PDE's models as resource and material parameters are in practice subject to uncertainty due to different errors or modeling assumptions, the lack of knowledge and information. In this view the parameters are (not deterministic) stochastic ones [6].

The considerable attention that stochastic finite element (SFE) received over the last decade [7-9] is mainly attributed to the spectacular growth of computing power, rendering possible the efficient treatment of large scale problems in dynamics of processes etc.

Fundamental issue in SFE is the parameter estimation and reserves. The most outstanding method for the approximate solution of a SPDE is the MONTE CARLO method [10]. On the other hand, the geostatistics is a useful discipline to make the inference about the spatial risk phenomenon (processes) [11].

2. A View on the Random Element

Let's be defined a fixed probability space (Ω, $\mathcal{A}$, P) [7], where Ω is a nonempty set of "outcomes" or elementary events", $\mathcal{A}$ is a σ algebra of subsets of Ω (the "random events") and P is a probability measure on the measurable space (Ω, $\mathcal{A}$) If (χ, S_x) is another measurable space, then a random element X in χ is a measurable mapping from (Ω, $\mathcal{A}$, P) into (χ, S_x) *i.e.* it holds $X:\Omega \to \chi$ with:

$$X^{-1}(B) := \{X \in B\} := \{\omega \in \Omega : \chi(\omega) \in B\} \in A, \ \forall B \in S_x$$

with each random element X: $\Omega \to \chi$, P_x is a probability measure of (χ, S_x) connected with the distribution of random elements. It is defined by:

$$P_x'(B) := P\{X \in B\} := P\{\omega \in \Omega : \chi(\omega) \in B\}, (B \in S_x)$$

A random element X with values in X is called a simple random element if the range is a finite nonempty set in X, where exists a partition [4,12] of the probability space $\Omega = \bigcup_{k=1}^{N} \Omega_k$ with measurable sets

$$\Omega_k \in A \quad k = 1, 2, \cdots, N \ (N \in \mathbf{N})$$

such like: $X(\omega) = x_k$ for $\omega \in \Omega_k$.
The corresponding probabilities are:

$$P(\Omega_k) = p_k, \ p_k \ge 0, \ k = 1, 2, \cdots, N$$

$$\sum_{k=1}^{N} p_k = 1$$

The distribution of a simple random element is a discrete probability measure on (X, S_x) that might be written as: $P_x = \sum_{k=1}^{N} p_k \delta_{xk}$,

where: δ_{xk} the Dirac measure

$$\delta_{xk}(B) = \begin{cases} 1 & \text{if } x_k \in B \\ 0 & \text{otherwise} \end{cases}$$

3. Stochastic Finite Element [13]

Even though this is a general concept [7,14] we will present some considerations in the viewpoint its applications in the parameter estimation of different phenomena and processes.

Let's consider a zone $V \subset R^3$ and a random function $Z(x)$, $x \in V$. The zone V is sorted out into blocks v_i by a parallelepiped grid:

$$V = \bigcup v_i \quad (1)$$

where: v_i is a parallelepiped element with eight nodes.

At each node, the random function $Z(x)$ is known, in other words is given the probability density function (pdf) on its point support (**Figure 1**). It is required:

The distribution pdf in whatever point $x \in V$.

The estimation of the mean value

$$z_{vi} = \frac{1}{v}\int_V Z(x)\mathrm{d}x \text{ over the domain } v \quad (2)$$

We define a stochastic element as a block, with the random function $Z(x)$, $x \in v_i$.

Let us consider a reference element w_i in the co-ordinate system s_1 s_2 s_3. If we choose an incomplete base [15]:

$$P(s) = \langle 1, s_1, s_2, s_3, s_1s_2, s_2s_3, s_3s_1, s_1s_2s_3 \rangle \quad (3)$$

Then the function $Z(x)$ could be presented as a linear combination :

$$Z(x) = Z(s_1\, s_2\, s_3) = \langle P(s)\rangle [P_8]^{-1} \{Z_s^8\} = \langle N(s)\rangle \{Z_s^8\} \quad (4)$$

where:

$[P_8]^{-1}$—is the matrix, whose elements are the polynomials base values at the nodes

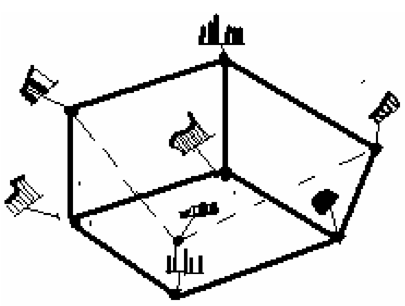

Figure 1. Parallelpiped element.

$\{Z_s^8\}$—is the vector of the distributions of the nodes;

$\langle N(s)\rangle$—is the vector of the shape functions;

$$N_i,\ i = 1, 2, \cdots, 8$$

$$N(s) = \langle N_i(s)\rangle = \langle N_1(s), N_2(s), \cdots, N_8(s)\rangle$$

$$N_i(s) = (1 + s_1 s_1^i)(1 + s_2 s_2^i)(1 + s_3 s_3^i) \quad i = 1, 2, \cdots, 8 \quad (5)$$

In the formula (5), "the exponent *i*" is not a variable. It indicates only the sign within the parentheses.

3.1. The Mean Value

To calculate the mean value $z_{vi} = 1/v \int_V Z(x)\mathrm{d}x$, we consider the deterministic transformation :

$$X_i(s) = \langle N_i(s)\rangle \langle x_i^8 \rangle \quad i = 1, 2, \cdots, 8 \quad (6)$$

Therefore [13]

$$\begin{aligned} Z_{vi} &= \frac{1}{v}\int_v Z\left(x_1(S_1S_2S_3), x_2(S_1S_2S_3), x_3(S_1S_2S_3)\right) \det J \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \\ &= \sum_{i=1}^{8} H_i Z_i(x) \end{aligned}$$

$$H_i = f(a_{ij}, b_{ij}, c_{ij}, d_{ij})$$

The coefficients a_{ij}, b_{ij}, c_{ij}, d_{ij}, $i,j = 1, 2, 3$ are depend only on the node coordinates. Knowing the above coefficient we can calculate [13] the weight coefficient H_i $(i = 1, \cdots, 8)$ as for example for H_2:

$$\begin{aligned} H_2 &= 8/3\, c_{21}a_{32}c_{13} + 8/9\, d_{21}a_{32}a_{13} + 8/27\, c_{21}c_{32}d_{13} \\ &+ 8/27\, d_{21}c_{32}b_{13} + 8/9\, a_{21}b_{32}c_{13} + 8/9\, b_{21}b_{32}a_{13} \\ &+ 8/27\, a_{21}d_{32}d_{13} + 8/27\, b_{21}d_{32}b_{13} + 8/9\, c_{11}a_{22}d_{33} \\ &+ 8/9\, d_{11}a_{22}a_{33} + 8/9\, a_{11}b_{22}c_{33} + 8/9\, b_{11}b_{22}a_{33} \\ &+ 8/27\, c_{11}a_{22}d_{33} + 8/27\, d_{11}c_{22}b_{33} + 8/27\, a_{11}d_{22}d_{33} \\ &+ 8/27\, b_{11}d_{22}b_{33} + 8/9\, a_{12}a_{23}d_{31} + 8/9\, b_{21}a_{23}b_{31} \\ &+ 8/27\, c_{12}b_{23}d_{31} + 8/27\, d_{12}b_{23}b_{31} + 8/9\, a_{12}c_{32}c_{31} \\ &+ 8/9\, b_{12}c_{23}a_{31} + 8/27\, a_{21}d_{32}c_{31} + 8/27\, d_{12}d_{23}a_{31} \\ &+ 8/9\, a_{13}a_{22}d_{31} + 8/9\, c_{13}a_{22}c_{31} + 8/9\, a_{13}b_{13}b_{31} \\ &+ 8/9\, c_{13}b_{22}a_{31} + 8/9\, b_{13}d_{22}d_{31} + 8/27\, d_{13}c_{22}c_{31} \\ &+ 8/27\, b_{13}d_{13}b_{31} + 8/27\, d_{13}d_{22}a_{31} + 8/9\, c_{21}a_{12}c_{33} \\ &+ 8/9\, d_{21}a_{12}a_{33} + 8/9\, a_{12}b_{12}c_{33} + 8/9\, b_{21}b_{12}a_{33} \\ &+ 8/27\, c_{21}c_{12}d_{33} + 8/27\, a_{13}a_{22}d_{31} + 8/27\, a_{12}d_{12}d_{33} \\ &+ 8/27\, b_{21}d_{12}b_{33} + 8/9\, c_{11}a_{32}c_{23} + 8/9\, d_{11}a_{32}a_{23} \\ &+ 8/9\, a_{32}b_{32}c_{23} + 8/27\, b_{11}b_{32}a_{13} + 8/9\, c_{11}c_{32}d_{23} \\ &+ 8/27\, d_{11}c_{32}b_{23} + 8/27\, a_{32}d_{32}d_{23} + 8/27\, b_{11}d_{32}b_{23} \end{aligned} \quad (7)$$

It is known that:

$$\sum_{i=1}^{8} H_i = 1 \tag{8}$$

Thus, the coefficients H_i are the distribution weights. In other words they make the weighted average of the given distributions at the nodes.

Thus, the mentioned stochastic finite element estimator is a linear interpolator, regarding to the distributions given at its nodes [13].

Taking into account that averaging process is one of the most frequently employed concept in computational techniques at finite element and geostatistics, below are presented two integral estimation procedures, which are key points on the estimation of the stiffness matrices in SFE and kriging, cokriging, covariance matrices in geostatistics [11,16,17].

4. Volume Integrals within Polyedras [18]

Let's take a function u (x_1, x_2, x_3) in a coordinate system x_1, x_2, x_3. The integral of volume V will be estimated:

$$\int_v u\left(x_1, x_2, x_3\right) \mathrm{d}v \tag{9}$$

We will construct a vector $\hat{\phi}$ (x_1, x_2, x_3) that will satisfy :

$$u = div\hat{\Phi} \tag{10}$$

where:

$$\varphi = \varphi_1 \hat{i}_1 + \varphi_2 \hat{i}_2 + \varphi_3 \hat{i}_3 \tag{11}$$

i_1, i_2, i_3 is the system of the unit vector along the coordinate directions.

Let's suppose that the boundary surface S of the volume V is composed of k plane polygonal faces S_i (i = 1, 2 ,⋯, k). Applying the divergence theorem we find:

$$\int_v u\left(x_1, x_2, x_3\right) \mathrm{d}v = \sum_{j=1}^{k} \iint u\left(x_1, x_2, x_3\right) \mathrm{d}x_1 \mathrm{d}S_j^1 \tag{12}$$

where: the projected area $\mathrm{d}\,S_j^1$ is perpendicular to $\hat{i}$ and lies in the (x_2, x_3) plane.

The equation of the plane face $\mathrm{d}S_j$ can be expressed as:

$$x_1 = x_1^{(j)}\left(x_2, x_3\right) = \alpha_1^{(j)} + \alpha_2^{(j)} x_2 + \alpha_3^{(j)} x_3$$

so the right-hand side of Equation (12) can be simplified to be :

$$\int_V u\left(x_1, x_2, x_3\right) \mathrm{d}V = \sum_{j=1}^{k} \int_{S_j} \varphi^{(j)}\left(x_1, x_2\right) \mathrm{d}S_j \tag{13}$$

where: the surface S_j^1 is a polygon in the (x_2, x_3), in which the function $\varphi^{(j)}$ is to be integrated for $\overline{j}$ = 1,2 ,⋯, k.

In this way, the computation of the volume integral is a procedure to integrate an arbitrary function within a polygon. Further repeating the above mentioned procedure we could find:

$$\begin{aligned}\int_\Omega V\left(x_1, x_2\right) \mathrm{d}\Omega &= \int_T \Psi\left(x_1, x_2\right) n_1 \mathrm{d}T = \sum_{j=1}^{k} \int_T \Psi\left(x_1, x_2\right) n_1 \mathrm{d}T \\ &= \sum_{j=1}^{k} \int_{\Gamma_j} \Psi\left(x_1^{(1)}, x_2\right) n_j^{(j)} \mathrm{d}T\end{aligned} \tag{14}$$

where, the perimeter T is a collection of the straight lines

$$T_j, \quad j = 1, \cdots, k,$$

while,

$$x_1 = x_j^{(j)}\left(x_2\right) \tag{15_1}$$

$$n_1^{(j)} \mathrm{d}T = \mathrm{d}x_2 \tag{15_2}$$

Let the x_2 coordinates for j^{th} side x_2^{js} and x_2^{je}. So:

$$\begin{aligned}\int_{T_j} \Psi_1\left(x_1, x_2\right) n_1 \mathrm{d}T &= \int_{x_2^{js}}^{x_2^{je}} \Psi\left(x_1^{(j)}\left(x_2\right), x_2\right) \mathrm{d}x_2 \\ &= \int_{x_2^{js}}^{x_2^{je}} \gamma_j\left(x_2\right) \mathrm{d}x_2\end{aligned} \tag{16}$$

Finally the above integral could be estimated by the Gaussian scheme quadrature. It is to be noted that volume integral is a deterministic procedure, but if the ω = v and $X(\omega) = u$, then it could be estimated as a stochastic finite element using Monte-Carlo method.

Parallelly if $u(x), x \in R^n$ is a random function (RF) then the integral $1/v \int_V u(x) \mathrm{d}v$ could be treated in the geo-statistical view as a mean value.

5. Geostatistical Approach

5.1. Variograms

Geostatistics are based on the theory of the regionalized variables [2] with assumption that data are observations of stochastic variables. The central tool of geostatistics is the variogram or semivariance function which is a structure describing the spatial dependence of the spatial variable [11].

The following formula is the most frequently used for the variogram (semivariance) calculations:

$$\gamma(h) = \frac{1}{2N} \sum_{i=1}^{N} \left[Z\left(x_i\right) - Z\left(x_i + h\right)\right]^2 \tag{17_1}$$

where:

x_i is a data location, h is a log vector, $z(x_i)$ is the data value at location x_i, N is the number of data pairs spaced a distance and direction h units apart

Semivariance calculations can also be performed with data from RS images for example as a cross variogram. It is defined as half of the average product of the log distance relative to the two variables Z and Y.

$$\gamma_{zy}(h) = \frac{1}{2N}\sum_{i=1}^{n(h)}\left\{\left[Z(x_i)-Z(x_i+h)\right]\left[Y(x_i)-Y(x_i+h)\right]\right\} \quad (17_2)$$

where:

$Z(x_1)$ and $Y(x_1)$ are the data value in point x_1 for two bands (profiles);

N is the number of data separated by length of the vector h;

A variogram usually is characterized by three parameters [2]:

- Sill—the platean that the semivariogram reaches;
- Range—the distance at which two data points are uncorrelated;
- Nugget—the vertical discontinuity at the origin.

Usually the application of the semivariograms requires that the data accomplish the intrinsic hypothesis for a regionalized variable. In other words a random function $Z(x)$ is said to be intrinsic when:

the mathematical expectation exists and does not depend on the support x

$$E\{Z(x)\} = m \;\forall(x) \quad (18)$$

for all vectors h the incerement $Z(x+h) - Z(x)$ has a finite variance which does not depend on x

$$Var\left[Z(x+h)-Z(x)\right] = E\left\{\left[Z(x+h)-Z(x)\right]^2\right\} \;\forall(x) \quad (19)$$

where:

$Z(x)$ is a random function *i.e.* locally at a point x_1, $Z(x_1)$ is a random variable and $Z(x_1)$ and $Z(x_1+h)$ are generally independent but are related by a correlation expressing the spatial structure of the initial regionalized variable $Z(x)$. Experimental variogrames are approximated by different models like: spherical, exponential, Gaussian, circular, tetraspherical, pentaspherical, Hole effect, K—Bessel etc. [2,16,18].

5.2. Kriging in SFE View [13]

Let be $Z(x)$ the random function and the estimation of the mean value:

$$Z_V = \frac{1}{v}\int_V Z(x)\,\mathrm{d}x \quad (20)$$

over a given domain v is required knowing a support of discrete values $Z_\alpha, \alpha = 1,\cdots,n$.

According to the Kriging approach [2] the linear estimator Z_k^* of the n data values is considered:

$$Z_k^* = \sum_{\alpha=1}^{n}\lambda_\alpha Z_\alpha \qquad Z_\alpha = \frac{1}{v_\alpha}\int_{v_\alpha} Z(x)\,\mathrm{d}x \quad (21)$$

The n weights λ_α are calculated under the classic hypothesis of the moments:

$$\begin{aligned} &E\{Z(x)\} = m \\ &E\{Z(x+h)Z(x)\} - m^2 = C(h) \text{ or} \\ &E\{Z(x+h)-Z(x)\}^2 = 2\gamma(h) \end{aligned} \quad (22)$$

We must be assure that the estimator is unbiased as well as the variance is minimal. Let us suppose that one (or both) of two hypotheses are not accomplished and both the expectation of $Z(x)$ and the covariance depend on x:

$$E\{Z(x)\} = m(x)$$

$$C(x,h) = E\{Z(x+h)Z(x)\} - m(x+h)m(x) \quad (23)$$

Before taking into consideration this hypothesis, it should be underlined, whatever the moment functions are going to be, they should always lead to a positive variance. Also, we will show the calculation of Kriging solution using SFE but without considering its existence and uniqueness (It is not the aim of this paper). To ensure that estimator is unbiased we impose the condition:

$$\sum_{\alpha=1}^{n}\lambda_\alpha m_\alpha - m_v = 0 \quad (24)$$

With

$$m_v = E\{Zv(x)\} = E\left\{\frac{1}{v}\int_V Z(x)\,\mathrm{d}x\right\},$$

$$m_\alpha = E\{Z(v_\alpha)\} = E\left\{\frac{1}{v_\alpha}\int_{V_\alpha} Z(x)\,\mathrm{d}x\right\} \quad \alpha = 1,\cdots,n \quad (25)$$

The estimation variance is:

$$E\left\{\left[Z_v - Z_k^*\right]^2\right\} = E\{Z_v^2\} - 2E\{Z_v Z_k^*\} + E\{Z_k^*\} \quad (26)$$

Taking into account the expression of $E\{Z_v^2\}$ we have:

$$\begin{aligned} E\{Z_v^2\} &= E\left\{\frac{1}{v^2}\int \mathrm{d}x \int Z(x)Z(y)\,\mathrm{d}y\right\} \\ &= \sum_{i=1}^{8}\sum_{i=1}^{8} c_{i,j} E\{Z_v(x_i)Z_y(y_i)\} \end{aligned} \quad (27)$$

Also

$$\begin{aligned} &\sum_{i=1}^{8}\sum_{i=1}^{8} c_{i,j}\cdot E\{Z_v(x_i)Z_y(y_i)\} \\ &= \sum_{i=1}^{8}\sum_{i=1}^{8}\left\{c_{i,j}\left[C(v_i,v_j) + m_{vi}m_{vj}\right]\right\} \\ &= \overset{v}{C}(v,v) \end{aligned} \quad (28)$$

where:

m_{vi} is the expectation of $Z(x_i)$ at the node $\boldsymbol{i}$,

$\overset{v}{C}(v,v)$ is the covariation depending not only by the distance $\boldsymbol{h}$, but also on $\boldsymbol{x}$.

Carrying out other means and substituting to the estimated variance we obtain:

$$E\left\{\left[Z_v - Z_k^*\right]^2\right\} = \overset{v}{C}(v,v) - 2\sum_{\alpha=1}^{n}\lambda_\alpha \overset{v}{C}(v,v_\alpha) + \sum_{\alpha=1}^{n}\lambda_\alpha \sum_{\beta=1}^{n}\lambda_\beta \overset{v}{C}(v_\beta, v_\alpha) \quad (29)$$

Now the problem is to find the weights λ_α, $\alpha = 1,\cdots,k$ which minimize the estimation under non-bias conditions:

$$\sum_{\alpha=1}^{n}\left(\lambda_\alpha - \frac{1}{n}\right)m_\alpha = 0 \quad (30)$$

For this reason, we use the Lagrange multiplier's method, according to which we need to take the derivatives of:

$$F = \overset{v}{C}(v,v) - 2\sum_{\alpha=1}^{n}\lambda_\alpha \overset{v}{C}(v,v_\alpha) + \sum_{\alpha=1}^{n}\lambda_\alpha \sum_{\beta=1}^{n}\lambda_\beta (v_\beta, v_\alpha) + 2\mu\sum_{\alpha=1}^{n}\left(\lambda_\alpha - \frac{1}{n}\right)m_\alpha \quad (31)$$

This procedure provides the Kriging system of $n + 1$ linear equation equations in (λ_α, μ):

$$\sum_{\alpha=1}^{n}\lambda_\alpha \overset{v}{C}(v_\beta, v_\alpha) - \mu m_\alpha = \overset{v}{C}(v, v_\alpha)$$

$$\sum_{\alpha=1}^{n}\lambda_\alpha m_\alpha = e, \qquad e = \sum_{\alpha=1}^{n}\frac{1}{n}m_\alpha \quad (32)$$

which can be expressed in matrix form:

$$[K]\{\lambda_\alpha\} = \{M\}$$

$$[K] = \begin{bmatrix} c(v_1v_1) & c(v_1v_2) & \cdots & c(v_1v_n) \\ c(v_2v_1) & c(v_2v_2) & \cdots & c(v_2v_n) \\ \vdots & \vdots & \ddots & \vdots \\ c(v_nv_1) & c(v_nv_2) & \cdots & c(v_nv_n) \end{bmatrix}$$

$$\{\lambda\} = \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{Bmatrix} \quad \{M\} = \begin{Bmatrix} c(v_1v) \\ c(v_2v) \\ \vdots \\ c(v_nv) \end{Bmatrix} \quad (33)$$

Let us suppose that solution of system (33) exists and it is unique. In this situation, it is quite clear that system (33) is general, in the sense of so-called Kriging system.

Example 1

In **Figure 2** it is shown a structure with 3 blocs: $v_1 = 1 \times 1$, $v_x = 1 \times 1$, $v_2 = 2 \times 2$ in a contaminated (radioactive, oil, gas etc.) zone.

The equation of the variogram is $\gamma(x) = 4h$ and the means of the parameter measured in the blocs v_1, v_2 are respectively:

$$E(Z_1(x)) = 0.590 \quad E(Z_2(x)) = 0.409.$$

Let's estimate the parameter $Z(x)$ in the block v_x resolving the Kriging system using finite element.

According to Kriging approach we have:

$Z = \lambda_1 Z_1 + \lambda_2 Z_2$ where λ_1, λ_2 parameters of the Kriging system:

$$\begin{bmatrix} \bar{\gamma}(v_1,v_1) & \bar{\gamma}(v_1,v_2) & 1 \\ \gamma(v_2,v_1) & \gamma(v_2,v_2) & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \gamma(x_1x_x) \\ \gamma(x_2x_x) \\ 1 \end{bmatrix}$$

The solution is $\lambda_1 = 0.5906$, $\lambda_2 = 0.409$, $\lambda_3 \approx 0$.

Therefore,

$$Z = Z = \lambda_1 Z_1 + \lambda_2 Z_2 = 0.5906 \times 5 + 0.409 \times 7 = 5.81.$$

Example 2

In the **Figure 3**, it is presented a profile in a waste zone in which a parameter has been measured using a constant step h.

The respective variogram shown in **Figure 4** has been approximated by a spheric model:

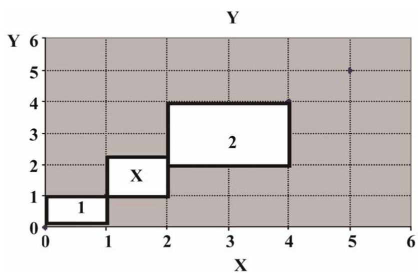

Figure 2. Contaminated zone with three blocks.

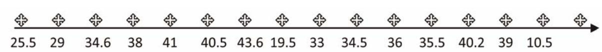

Figure 3. Profile of measured parameter.

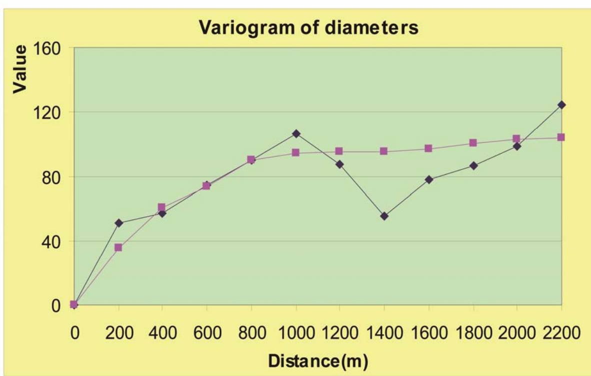

Figure 4. Variogram of diameters depending on distance between sample plots.

$$\gamma(h)=\begin{cases} c\left(\dfrac{3h}{2a}-\dfrac{1}{2}\dfrac{h^3}{a^3}\right) & h\le a \\ 0 & h\ge a \end{cases}$$

with $c = 1$ and $a = 4h$ (the range).

The parabolic form of the variogram around the origin shows it is homogeneous [2].

6. SFE in Partial Differential Equations

The parameters of partial differential equations in many cases are not deterministic but stochastic ones. In this view let's have a look on a PDE. First our starting point is the second order elliptic boundary value problem:

$$\begin{aligned} &-\bar{V}*\left(T\bar{V}p\right)=F && \text{in } D \\ &p=g && \text{on } \partial D_D\ne 0 \\ &n*\left(T\bar{V}p\right)=0 && \text{on } \partial D_N \end{aligned} \tag{34}$$

posed on a bounded polygonal domain $D\subset R^2$, whose boundary is divided into two parts, $\partial D=\partial D_D\bigcup\partial D_N$ (Dirichle and Neumman). This steady state diffusion problem can be reformatted by introducing the variable $u=-T\bar{V}p$ as:

$$\begin{aligned} &T^{-1}u+\bar{V}p=0 \\ &\bar{V}*u=F && \text{in } D \\ &p=g && \text{on } \partial D_D \\ &n*u=0 && \text{on } \partial D_D \end{aligned} \tag{35}$$

In the context of groundwater flow modeling the variable p is the hydraulic head and $\boldsymbol{u}$ is the volumetric flux, respectively.

In many applications, only limited information about the diffusion coefficient T or the source term F is available.

We assume $T = t(x, \omega)$ (and $F = F(x, \omega)$ to be random fields, *i.e*. a family of random variables $T(x, \omega)$ with index variable $x\in\bar{D}$. Each random variable takes on values in $\mathbb{R}$ and is defined on a complete probability space $(\Omega, \mathfrak{A}, P)$, where Ω denotes the set of elementary events, $\mathfrak{A}$ is a σ—algebra on Ω generated by the random variables $T(x,)$, (and $F(x,)$) and P is a probability measure.

A consequence of the randomness in the diffusion coefficient or source term is that the output variables p and, if present, $\boldsymbol{u}$ are random fields as well. The primal formulation [12] transforms to the problem of finding a random field:

$u=u(x,\omega)$, $p=p(x,\omega)$, such that, P almost surely

$$\begin{aligned} &\bar{V}*\left(T(x,\omega)\bar{V}p(x,\omega)\right)=F(x,\omega) && \text{in } D\times\Omega \\ &p(x,\omega)=g(x) && \text{on } \partial D_D\times\Omega \\ &n*\left(T(x,\omega)\bar{V}p(x,\omega)\right)=0 && \text{on } \partial D_N\times\Omega \end{aligned} \tag{36}$$

Analogously in the mixed formulation [12] we now look for random fields $u=u(x,\omega)$ and $p=p(x,\omega)$ such that: p—almost surely (as):

$$\begin{aligned} &T_{-1}(x,\omega)u(x,\omega)+\bar{V}p(x,\omega)=0 \\ &\bar{V}u(x,\omega)=F(x,\omega) && \text{in } D\times\Omega \\ &p(x,\omega)=g(x) && \text{on } \partial D_D\times\Omega \\ &n*u(x,\omega)=0 && \text{on } \partial D_N\times\Omega \end{aligned} \tag{37}$$

As a simple example let's take a glance at the stochastic finite element on diffusion-convection equation [5,6,12,19]:

$$\begin{aligned} &\frac{\partial}{\partial x}\left(k_x\frac{\partial\varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial\varphi}{\partial y}\right) \\ &+V_x\frac{\partial\varphi}{\partial x}+V_y\frac{\partial\varphi}{\partial y}+Q=C\frac{\partial\varphi}{\partial t} \end{aligned} \tag{38}$$

using the Crack-Nickolson algorithm with $0<\theta\le 1$:

$$\begin{aligned} &\varphi_{i,j}^{n+1}a_1\varphi_{i-1,j}^{n+1}+a_2\varphi_{i,j}^{n+1}+a_3\varphi_{i+1,j}^{n+1} \\ &+a_4\varphi_{i,j-1}^{n+1}+a_5\varphi_{i,j+1}^{n+1}=\varphi_{i,j}^{n}+bQ_{i,j}^{n} \end{aligned} \tag{39}$$

where :

C —the solute concentration, x, y—spatial co-ordinate, t—time coordinate, V—the flow velocity vector with its components V_x, V_y, D—the diffusion coefficient, a_i, $i=\overline{1,5}$ and b are the coefficients depending on the mentioned coefficients, $\Delta x,\Delta y$ spatial steps, Δt time step. Below we are presenting a river plane zone contaminated by a point pollutant source **Figure 5**, placed in the left side of the node 13.

In this scheme, it was operated with mean values of the random diffusion convection parameters, resulting from their synthetic and real distributions.

The components V_x and V_y has been measured in an interval of time. The component of V_y is positive over the line 13 - 18 and negative under this one.

To illustrate the idea, it is shown below a partial solution of the contaminant concentration in the step $st = 5$ for a simple non stationary flow problem (Dirichle—Newman conditions). Using $q = 1$, $K_x = 1$, $K_y = 1$, $V_x = 1$,

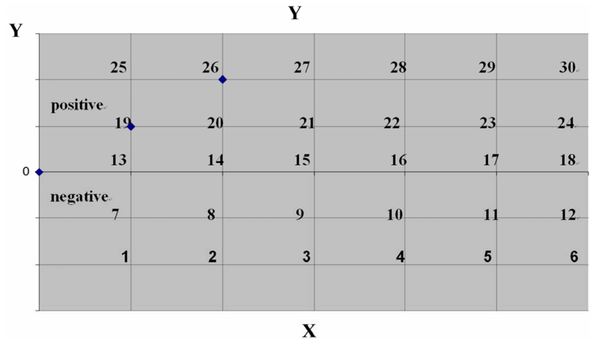

Figure 5. A contaminated river zone.

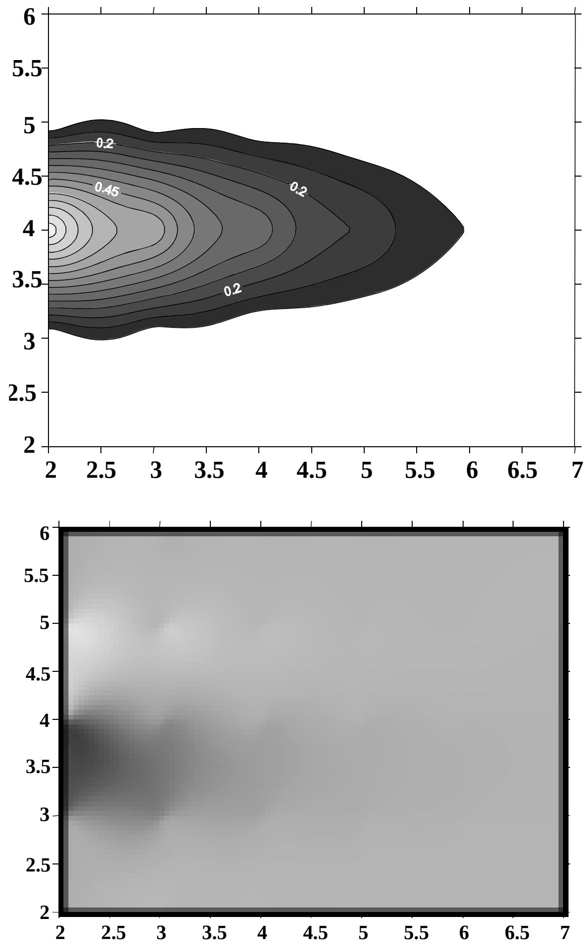

Figure 6. The contaminant concentration dynamic.

V_y = 0.01, the following dimensionless means values by a Monte Carlo procedure [10] resulted:

xx [1] = 0.0012	xx [2] = 0.003	xx [3] = 0.004
xx [4] = 0.0043	xx [5] = 0.0036	xx [6] = 0.0029
xx [7] = 0.0410	xx [8] = 0.069	xx [9] = 0.074
xx [10] = 0.073	xx [11] = 0.051	xx [12] = 0.053
xx [13] = 0.9000	xx [14] = 0.770	xx [15] = 0.590
xx [16] = 0.450	xx [17] = 0.250	xx [18] = 0.290
xx [19] = 0.0410	xx [20] = 0.069	xx [21] = 0.074
xx [22] = 0.073	xx [23] = 0.051	xx [24] = 0.053
xx [25] = 0.0012	xx [26] = 0.003	xx [27] = 0.0041
xx [28] = 0.0043	xx [29] = 0.0036	xx [30] = 0.0029

In **Figure 6**, it is presented the contaminant concentration dynamic for different times of the flow.

As it was expected the solution is symmetric.

There are simple resemblances between different concepts and operators in geostatistics and SFE as for example: blocs, interpolation operator, minimization of the variance (energy).

7. Conclusion

SFE and Geostatistic applications are of the great importance in environmental resources, nuclear and renewable energy, ecology, forestry, geology, climate, water and air pollution, mapping as well as on their uncertainty, risk analysis and optimization [1,5,14,15,20,21].

REFERENCES

[1] C. Busby, *et al*., "2010 ECRR Recommendations of the European Committee on Radiation Risk Brussels," BELGIUM, 2010.

[2] A. G. Journel and Ch. J. Huijbreght, "Mining Geostatistics," Academic Press, London, 1979.

[3] E. Guastaldy, "Geostatistical Modeling of Uncertainty for the Risk Analysis of a Contaminated Site in the Northern Italy," *Geophysical Research Abstract*, *European Geoscience Union*, Vol. 7, 2005.

[4] M. Kleiber and T. D. Hien, "The Stochastic Finite Element Method," Wiley, Chichester, 1992.

[5] S. Osmani, "Some Consideration on Finite Element Solution of Diffusion—Convection Equation and Its Application on Water Pollution," *Proceedings of Internaional Conference* on *Water Problem in the Mediterranean Countries*, North Cyprus, 1997.

[6] S. Osmani and M. Qirko, "Stochastic Finite Elements in a Complex System: Nonstationary Fluid Flow, Mass Transport, Heat Conduction and Vibrating," *International Conference in Scaling up in Fluid Flow in Porous Medium CROATIA 2008 as well as* (*Second Part*) *in 5th Asian Rock*, *Symposium Tehran*, Tehran, 2008.

[7] R. Ghanem and P. Spanos, "Stochastic Finite Elements: A Spectral Approach," Springer-Verlag, Berlin, 1991.

[8] R. Ghanem, "Ingredients for a General Purpose Stochastic Finite Element Formulation," *Computer Methods in Applied Mechanics and Engineering*, Vol. 168, No. 19, 1999.

[9] A. Keese, "A Review of Recent Developments in the Numerical Solution of Stochastic Partial Differential Equations (Stochastic Finite Elements)," Institut für Wissenschaftliches Rechnen, Brunswick, 2003.

[10] K. Saberfield, "Monte Carlo Methods for Stochastic PDEs," Institute for Applied Analysis and Stochastic, Berlin, 2007.

[11] J. P. Chiles and P. Delfiner, "Geostatistics, Modeling Spatial Uncertainty," Wiley, New York, 1999. doi:10.1002/9780470316993

[12] A. Bode and S. Osmani, "Using Stochastic Finite Element in Geostatistics and Difusion Convection Equations. MAMERNO9," *Third International Conference on Approximation Methods and Numerical Modeling in Environment and Natural Resources*, Pau, 8-11 June 2009.

[13] S. Osmani, "Energy Distribution Estimation Using Stochastic Finite Element," *Renewable Energy*, Elsevier, London, 2002.

[14] G. Stefanou, "The Stochastic Finite Element Method: Past, Present and Future," *Computer Methods in Applied Mechanics and Engineering*, Elsevier, London, 2009.

[15] O. C. Zienkiewitz and R. L. Taylor, "The Finite Element

Method," 5th Edition, *The Basic*, Elsevier, Oxford, 2000.

[16] P. Govartes, "Geostatistics for Natural Resources Evaluation," Oxford University, New York, 1997.

[17] S. Osmani, "Finite Element on Geostatistique Calculations," In: A. G. Fabri et Royer, Ed., 3*rd International Codata Conference in Geomathematics and Geostatistics Science de la Terre*, Nancy, 1994.

[18] G. Dasgupta, "Integration within Polygonal Finite Elements," *Journal of Aerospace Engineering*, Vol. 16, No. 1, 2003, pp. 9-18. doi:10.1061/(ASCE)0893-1321(2003)16:1(9)

[19] S. Osmani, "Finite Element Method (in Albanian)," Tirana, 2009.

[20] Nuclear Energy, "The Future Climat," The Royal Society, Portsmuth, 1999.

[21] S. Osmani, "A Model of Convection Diffusion Equation in 3D with Finite Elements," 17*th International Biennal Conference Numerical Analysis*, Dundee, 1997.

