

# Infinite Sets of Solutions and Almost Solutions of the Equation $N \cdot M = \text{reversal}(N \cdot M)$

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## Abstract

Motivated by their intrinsic interest and by applications to the study of numeric palindromes and other sequences of integers, we discover infinite sets of solutions and almost solutions of the equation  $N \cdot M = \text{reversal}(N \cdot M)$ . Most of our results are valid in a general numeration base.

## Keywords

Sequences of Integers, Infinite Sets

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## 1. Introduction

In this paper, motivated by their intrinsic interest and by applications to the study of numeric palindromes and other sequences of integers, we discover infinite sets of solutions and almost solutions of the equation

$$N \cdot M = \text{reversal}(N \cdot M). \quad (1)$$

An *almost solution* of (1) is a pair of integers  $(M, N)$  for which the equality (1) holds up to a string of digits for which we understand the position and all entries. Most of our results are valid in a general numeration base. Recently one of us showed in Nitica [1] that, in any numeration base  $b$ , for any integer  $N$  not divisible by  $b$ , the equation (1) has an infinity of solutions  $(N, M)$ . Nevertheless, as one can see from [3], finding explicit values for  $M$  can be difficult from a computational point of view, even for small values of  $N$ , e.g.  $N = 81$ . We show here explicit infinite families of solutions of (1) that are valid in all numeration bases.

Another application of our results may appear in the study of the classes of multiplicative and additive Ramanujan-Hardy numbers, recently introduced in Nitica [2]. The first class consists of all integers  $N$  for which there exists an in-

teger  $M$  such that  $s_b(N)$ , the sum of base  $b$ -digits of  $N$ , times  $M$ , multiplied by the reversal of the product, is equal to  $N$ . The second class consists of all integers  $N$  for which there exists an integer  $M$  such that  $s_b(N)$ , times  $M$ , added to the reversal of the product, is equal to  $N$ . As showed in Nitica [1] [2], the solutions of Equation (1) for which we can compute the sum of digits of  $s_b(N) \cdot M + \text{reversal}(s_b(N) \cdot M)$  or of  $s_b(N) \cdot M \times \text{reversal}(s_b(N) \cdot M)$ , can be used to find infinite sets of above numbers.

## 2. Statements of the Main Results

Let  $b \geq 2$  be a numeration base. If  $x$  is a string of digits, let  $(x)^k$  denote the base 10 integer obtained by repeating  $x$   $k$ -times. Let  $[x]_b$  denote the value of the string  $x$  in base  $b$ .

**Theorem 1.** a) Let  $b \geq 3$  a numeration base. Let  $k \geq n \geq 1$ ,  $k, n$  be integers. Then the product

$$(b^{2n-1} + b^n - 1)(b^k - 1) \tag{2}$$

is a palindrome.

b) Let  $b = 2$ . Then the product (2) is a palindrome if  $k \geq 2n - 1$  and almost a palindrome if  $k < 2n - 1$ .

The proof of Theorem 1 is done in Section 3.

The computations from **Table 1** illustrate the result from Theorem 1 if  $b = 10, n = 5$  and  $k \in \{5, 6, 7, 8, 9, 10, 11, 12\}$ .

**Theorem 2.** a) Let  $b \geq 3$  a numeration base. Let  $k \geq n \geq 4$ ,  $k, n$  be integers. Then the product

$$(b^n + b^{n-1} - 1)(b^k - 1) \tag{3}$$

is almost a palindrome.

The proof of Theorem 2 is done in Section 4.

The computations from **Table 2** illustrate the result from Theorem 2 if  $b = 10, n = 5$  and  $k \in \{5, 6, 7, 8, 9, 10, 11, 12\}$ .

**Table 1.** The product (2) if  $b = 10, n = 5$  and  $k \in \{5, 6, 7, 8, 9, 10, 11, 12\}$ .

$k$	$(b^{2n-1} + b^n - 1)(b^k - 1)$
5	$1000099999 \times 99999 = 100008999800001$
6	$1000099999 \times 999999 = 1000098998900001$
7	$1000099999 \times 9999999 = 10000998989900001$
8	$1000099999 \times 99999999 = 100009998899900001$
9	$1000099999 \times 999999999 = 1000099997999900001$
10	$1000099999 \times 9999999999 = 10000999988999900001$
11	$1000099999 \times 99999999999 = 100009999898999900001$
12	$1000099999 \times 999999999999 = 1000099998998999900001$

**Table 2.** The product (3) if  $b = 10, n = 5$  and  $k \in \{5, 6, 7, 8, 9, 10, 11, 12\}$ .

$k$	$(b^n + b^{n-1} - 1)(b^k - 1)$
5	$109999 \times 99999 = 10999790001$
6	$109999 \times 999999 = 109998890001$
7	$109999 \times 9999999 = 1099989890001$
8	$109999 \times 99999999 = 10999899890001$
9	$109999 \times 999999999 = 109998999890001$
10	$109999 \times 9999999999 = 1099989999890001$
11	$109999 \times 99999999999 = 10999899999890001$
12	$109999 \times 999999999999 = 109998999999890001$

### 3. Proof of Theorem 1

*Proof.* We first assume that  $b \geq 3$  and distinguish three cases:

*Case 1:*  $k > 2n - 1$  We show that the product (2) is equal to the palindrom

$$\begin{aligned}
 & \left[ 1(0)^{\wedge n-1} (b-1)^{\wedge n-1} (b-2)(b-1)^{\wedge k-2n} (b-2)(b-1)^{\wedge n-1} (0)^{\wedge n-1} 1 \right]_b. \\
 & \left[ 1(0)^{\wedge n-1} (b-1)^{\wedge n-1} (b-2)(b-1)^{\wedge k-2n} (b-2)(b-1)^{\wedge n-1} (0)^{\wedge n-1} 1 \right]_b \\
 & = b^{2n+k-1} + \left( \sum_{i=k+1}^{n+k-1} (b-1)b^i \right) + (b-2)b^k + \left( \sum_{i=2n}^{k-1} (b-1)b^i \right) \\
 & \quad + (b-2)b^{2n-1} + \left( \sum_{i=n}^{2n-2} (b-1)b^i \right) + 1 \\
 & = b^{2n+k-1} + (b^{n+k} - b^{k+1}) + (b-2)b^k + (b^k - b^{2n}) \\
 & \quad + (b-2)b^{2n-1} + (b^{2n-1} - b^n) + 1 \\
 & = b^{2n+k-1} + b^{n+k} - b^k - b^{2n-1} - b^n + 1 \\
 & = (b^{2n-1} + b^n - 1)(b^k - 1).
 \end{aligned}$$

*Case 2:*  $k = 2n - 1$  We will use repeatedly that  $k - n = n - 1$ . We show that the product (2) is equal to the palindrom

$$\begin{aligned}
 & \left[ 1(0)^{\wedge n-1} (b-1)^{\wedge n-1} (b-3)(b-1)^{\wedge n-1} (0)^{\wedge n-1} 1 \right]_b. \\
 & \left[ 1(0)^{\wedge n-1} (b-1)^{\wedge n-1} (b-3)(b-1)^{\wedge n-1} (0)^{\wedge n-1} 1 \right]_b \\
 & = b^{4n-2} + \left( \sum_{i=2n}^{3n-2} (b-1)b^i \right) + (b-3)b^{2n-1} + \left( \sum_{i=n}^{2n-2} (b-1)b^i \right) + 1 \\
 & = b^{4n-2} + (b^{k+n} - b^{2n}) + (b-3)b^{2n-1} + (b^{2n-1} - b^n) + 1 \\
 & = b^{2k} + b^{n+k} - b^k - b^{2n-1} - b^n + 1 \\
 & = (b^{2n-1} + b^n - 1)(b^k - 1).
 \end{aligned}$$

*Case 3:*  $k < 2n - 1$  We show that the product (2) is equal to the palindrom

$$\left[ 1(0)^{\wedge n-1} (b-1)^{\wedge k-n} (b-2)(b-1)^{\wedge 2n-k-2} (b-2)(b-1)^{\wedge k-n} (0)^{\wedge n-1} 1 \right]_b.$$

$$\begin{aligned}
 & \left[ 1(0)^{\wedge n-1} (b-1)^{\wedge k-n} (b-2)(b-1)^{\wedge 2n-k-2} (b-2)(b-1)^{\wedge k-n} (0)^{\wedge n-1} 1 \right]_b \\
 &= b^{2n+k-1} + \left( \sum_{i=2n}^{n+k-1} (b-1)b^i \right) + (b-2)b^{2n-1} + \left( \sum_{i=k+1}^{2n-2} (b-1)b^i \right) \\
 & \quad + (b-2)b^k + \left( \sum_{i=n}^{k-1} (b-1)b^i \right) + 1 \\
 &= b^{2n+k-1} + (b^{k+n} - b^{2n}) + (b-2)b^{2n-1} + (b^{2n-1} - b^{k+1}) \\
 & \quad + (b-2)b^k + (b^n - b^k) + 1 \\
 &= b^{2n+k-1} + b^{n+k} - b^{2n-1} - b^k - b^n + 1 \\
 &= (b^{2n-1} + b^n - 1)(b^k - 1).
 \end{aligned}$$

Assume now that  $b = 2$  the Cases 1 and 3 follow verbatim. In Case 2 the product 2 becomes the almost palindrome

$$\left[ 1(0)^{\wedge n-1} (1)^{\wedge n-2} (01)(1)^{\wedge n-1} (0)^{\wedge n-1} 1 \right]_2. \quad \square$$

### 4. Proof of Theorem 2

*Proof.* We first assume that  $b \geq 3$  and distinguish three cases:

*Case 1:*  $k > n + 1$  We show that the product (3) is equal to the almost palindrome

$$\begin{aligned}
 & \left[ 10((b-1)^{\wedge n-3})(b-1)(b-2)(b-1)^{\wedge k-n-1} (b-2)(b-1)(0)^{\wedge n-3} 01 \right]_b. \quad (4) \\
 & \left[ 10((b-1)^{\wedge n-3})(b-1)(b-2)(b-1)^{\wedge k-n-1} (b-2)(b-1)(0)^{\wedge n-3} 01 \right]_b \\
 &= b^{n+k} + \left( \sum_{i=k+2}^{k+n-2} (b-1)b^i \right) + (b-1)b^{k+1} + (b-2)b^k \\
 & \quad + \left( \sum_{i=n+1}^{k-1} (b-1)b^i \right) + (b-2)b^n + (b-1)b^{n-1} + 1 \\
 &= b^{n+k} + (b^{n+k-1} - b^{k+2}) + b^{k+2} - b^{k+1} + b^{k+1} - 2b^k \\
 & \quad + (b^k - b^{n+1}) + b^{n+1} - 2b^n + b^n - b^{n-1} + 1 \\
 &= b^{n+k} + b^{n+k-1} - b^k - b^{n-1} - b^n + 1 \\
 &= (b^n + b^{n-1} - 1)(b^k - 1). \quad (5)
 \end{aligned}$$

*Case 2:*  $k = n + 1$  We show that the product (3) is equal to the almost palindrome:

$$\begin{aligned}
 & 10(b-1)^{\wedge n-3} (b-1)(b-2)(b-2)(b-1)(0)^{\wedge n-2} 1. \\
 & 10(b-1)^{\wedge n-3} (b-1)(b-2)(b-2)(b-1)(0)^{\wedge n-2} 1 \\
 &= b^{2n+1} + \left( \sum_{i=n+3}^{2n-1} (b-1)b^i \right) + (b-1)b^{n+2} + (b-2)b^{n+1} + (b-2)b^n + (b-1)b^{n-1} + 1 \\
 &= b^{2n+1} + ((b^{n-3} - 1)b^{n+3}) + (b-1)b^{n+2} + (b-2)b^{n+1} + (b-2)b^n + (b-1)b^{n-1} + 1 \\
 &= b^{2n+1} - b^n + b^{2n} - b^{n-1} - b^{n+1} + 1 \\
 &= (b^n + b^{n-1} - 1)(b^{n+1} - 1).
 \end{aligned}$$

*Case 3:  $k = n$*  We show that the product (3) is equal to the almost palindrome:

$$\begin{aligned}
 & 10(b-1)^{\wedge n-3} (b-1)(b-3)(b-1)(0)^{\wedge n-2} 1. \\
 & 10(b-1)^{\wedge n-3} (b-1)(b-3)(b-1)(0)^{\wedge n-2} 1 \\
 & = b^{2n} + \left( \sum_{i=n+2}^{2n-2} (b-1)b^i \right) + (b-1)b^{n+1} + (b-3)b^n + (b-1)b^{n-1} + 1 \\
 & = b^{2n} + \left( b^{n+2} (b^{n-3} - 1) \right) + (b-1)b^{n+1} + (b-3)b^n + (b-1)b^{n-1} + 1 \\
 & = b^{2n} + b^{2n-1} - 2b^n - b^{n-1} + 1 \\
 & = (b^n + b^{n-1} - 1)(b^n - 1).
 \end{aligned}$$

Assume now that  $b = 3$ . The Cases 1, 2 and 3 follow verbatim. If  $b = 2$  the Cases 1 and 2 follow verbatim. In Case 3 the product 3 becomes the almost palindrome

$$10(1)^{\wedge n-3} 0110^{\wedge n-3} 01. \quad \square$$

## 5. Conclusion

Motivated by possible applications to the study of palindromes and other sequences of integers we find integer solutions and almost solutions of the equation  $N \cdot M = \text{reversal}(N \cdot M)$ . The results are valid in a general numeration base. Our results support the following conjecture: for integers  $\ell, k, n$ , such that  $1 \leq n \leq k, 1 \leq \ell \leq n-1$ , the product  $(10^{2n-\ell} + 10^n - 1)(10^k - 1)$  is a palindrome or almost a palindrome. Theorem 1 covers the case  $\ell = 1$  and Theorem 2 covers the case  $\ell = n-1$ .

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

## References

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