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On the 2-Domination Number of Complete Grid Graphs

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Abstract

A set D of vertices of a graph $G = (V, E)$ is called k-dominating if every vertex $v \in V - D$ is adjacent to some k vertices of D. The k-domination number of a graph G, $\gamma_k(G)$, is the order of a smallest k-dominating set of G. In this paper we calculate the k-domination number (for $k = 2$) of the product of two paths $P_m \times P_n$ for $m = 1, 2, 3, 4, 5$ and arbitrary n. These results were shown an error in the paper [1].

Keywords

k-Dominating Set, k-Domination Number, 2-Dominating Set, 2-Domination Number, Cartesian Product Graphs, Paths

1. Introduction

Let $G = (V, E)$ be a graph. A subset of vertices $D \subseteq V$ is called a 2-dominating set of G if for every $v \in V$, either $v \in D$ or v is adjacent to at least two vertices of D. The 2-domination number $\gamma_2(G)$ is equal to $\min\{|D| : D \text{ is a } 2-\text{dominating set of } G\}$.

The Cartesian product $G \times H$ of two graphs G and H is the graph with vertex set $V(G \times H) = V(G) \times V(H)$, where two vertices (v_1, v_2), $(u_1, u_2) \in G \times H$ are adjacent if and only if either $v_1u_1 \in E(G)$ and $v_2 = u_2$ or $v_2u_2 \in E(H)$ and $v_1 = u_1$.

Let G be a path of order n with vertex set $V(G) = \{1, 2, \cdots, n\}$. Then for two paths of order m and n respectively, we have $P_m \times P_n = \{(i, j) : 1 \le i \le m, 1 \le j \le n\}$. The jth column of $P_m \times P_n$ is $K_j = \{(i, j) : i = 1, \cdots, m\}$. If D is a 2-dominating set for $P_m \times P_n$, then we put $W_j = D \cap K_j$. Let $s_j = |W_j|$. The sequence $(s_1, s_2, \cdots, s_n)$ is called a 2-dominating sequence corresponding to D. For a graph G, we refer to minimum and maximum degrees by $\delta(G)$ and $\Delta(G)$, and for simplicity denoted those by δ and Δ, respectively. Also, we denote by $|V|$ and $|E|$ to order and size of graph G, respectively.

2. Notation and Terminology

Fink and Jacobson [2] [3] in 1985 to introduced the concept of multiple domination. A subset $D \subseteq V$ is k-dominating in G if every vertex of $V - D$ has at least k neighbors in D. The cardinality of a minimum k-dominating set is called the k-domination number $\gamma_k(G)$ of G. Clearly, $g_1(G) = g(G)$. Naturally, every k-dominating set of a graph G contains all vertices of degree less than k. Of course, every $(k+1)$-dominating set is also a k-dominating set and so $\gamma_k(G) \le \gamma_{k+1}(G)$. Moreover, the vertex set V is the only $(\Delta+1)$-dominating set but evidently it is not a minimum Δ-dominating set. Thus every graph G satisfies

$$\gamma_k(G) \le \gamma_{k+1}(G) \le \cdots \le \gamma_\Delta(G) < \gamma_{\Delta+1}(G) = |V|.$$

For a comprehensive treatment of domination in graphs, see the monographs by Haynes *et al.* [4]. Also, for more information see [5] [6]. Fink and Jacobson [2], introduced the following theorems:

Theorem 2.1 [2]. If $k \ge 2$, is an integer and G is a graph with $k \le \Delta(G)$, then $\gamma_k(G) \ge \gamma(G) + k - 2$.

Theorem 2.2 [2]. If T is a tree, then $\gamma_2(T) \ge \frac{|T|+1}{2}$.

In [6], Hansberg and Volkmann, proved the following theorem.

Theorem 2.4 [6]. Let $G = (V, E)$ be a graph of order n and minimum degree δ and let $k \in N$. If $\frac{\delta+1}{\ln(\delta+1)} \ge 2k$, then $\gamma_k(G) \le \frac{|V|}{\delta+1}\left(k\ln(\delta+1) + \sum_{i=0}^{k-1} \frac{\delta^i}{i!(\delta+1)^{k-1}}\right)$.

Cockayne, *et al.* [7], established an upper bound for the k-domination number of a graph G has minimum degree k, they gave the following result.

Theorem 2.3 [7]. Let G be a graph with minimum degree at least k, then $\gamma_k(G) \le \frac{k|V|}{(k+1)}$.

Blidia, *et al.* [8], studied the k-domination number. They introduced the following results.

Theorem 2.5 [8]. Let G be a bipartite graph and S is the set of all vertices of degree at most $k-1$, then $\gamma_k(G) \le \frac{|V|+|S|}{2}$.

Favaron, *et al.* [9], gave new upper bounds of $\gamma_k(G)$.

Corollary 2.6 [9]. Let G be a graph of order n and minimum degree δ. If $k \le \delta$ is an integer, then $\gamma_k(G) \le \frac{\delta}{2\delta+1-k}|V|$.

In [4], Haynes *et al.* showed that the 2-domination number is bounded from below by the total domination number for every nontrivial tree.

Theorem 2.7 [4]. For every nontrivial tree, $\gamma_2(T) \ge \gamma_t(T)$.

Also, Volkmann [10] gave the important following result.

Theorem 2.8 [10]. Let G be a graph with minimum degree $\delta \ge k+1$, then $\gamma_{k+1}(G) \le \frac{|V| + \gamma_k(G)}{2}$.

Shaheen [11] considered the 2-domination number of Toroidal grid graphs and gave an upper and lower bounds. Also, in [12], he introduced the following results.

Theorem 2.9 [12].

1) $\gamma_2(C_n) = \lceil n/2 \rceil$.

2) $\gamma_2(C_3 \times C_n) = n : n \equiv 0 (\bmod 3)$,

$\gamma_2(C_3 \times C_n) = n+1 : n \equiv 1,2 (\bmod 3)$.

3) $\gamma_2(C_4 \times C_n) = n + \lceil n/2 \rceil : n \equiv 0,3,5 (\bmod 8)$,

$\gamma_2(C_4 \times C_n) = n + \lceil n/2 \rceil + 1 : n \equiv 1,2,4,6,7 (\bmod 14)$.

4) $\gamma_2(C_5 \times C_n) = 2n$.

5) $\gamma_2(C_6 \times C_n) = 2n : n \equiv 0 (\bmod 3)$,

$\gamma_2(C_6 \times C_n) = 2n + 2 : n \equiv 1,2 (\bmod 3)$.

6) $\gamma_2(C_7 \times C_n) = \lceil 5n/2 \rceil : n \equiv 0,3,11 (\bmod 14)$,

$\gamma_2(C_7 \times C_n) = \lceil 5n/2 \rceil + 1 : n \equiv 5,6,7,8,9,10 (\bmod 14)$,

$\gamma_2(C_7 \times C_n) = \lceil 5n/2 \rceil + 2 : n \equiv 1,2,4,12,13 (\bmod 14)$.

In this paper we calculate the *k*-domination number (for *k* = 2) of the product of two paths $P_m \times P_n$ for *m* = 1, 2, 3, 4, 5 and arbitrary *n*. These results were shown an error in the paper [1]. We believe that these results were wrong. In our paper we will provide improved and corrected her, especially for *m* = 3, 4, 5.

The following formulas appeared in [1],

$$\gamma_2(P_n) = \lceil (n+1)/2 \rceil \cdot \gamma_2(P_2 \times P_n) = n \cdot \gamma_2(P_3 \times P_n) = 2n - \lceil n/2 \rceil \cdot \gamma_2(P_4 \times P_n) = 2n.$$

$$\gamma_2(P_5 \times P_n) = 3n - \lceil n/2 \rceil \cdot \gamma_2(P_{2k+1} \times P_n) = (k+1)n - \lceil n/2 \rceil.$$

$$\gamma_2(P_m \times P_n) = \lceil m/2 \rceil n - \lceil n/2 \rceil : m \equiv 1 (\bmod 2),$$
$$\gamma_2(P_m \times P_n) = \lceil m/2 \rceil n : m \equiv 0 (\bmod 2).$$

In this paper, we correct the results in [1] and proves the following:

$$\gamma_2(P_n) = \lceil (n+1)/2 \rceil \cdot \gamma_2(P_2 \times P_n) = n \cdot \gamma_2(P_3 \times P_n) = n + \lceil n/3 \rceil.$$

$$\gamma_2(P_4 \times P_n) = 2n - \lfloor n/4 \rfloor : n \equiv 3,7 (\bmod 8),$$
$$\gamma_2(P_4 \times P_n) = 2n - \lfloor n/4 \rfloor + 1 : n \equiv 0,1,2,4,5,6 (\bmod 8).$$

$$\gamma_2(P_5 \times P_n) = 2n + \lceil n/7 \rceil : n \equiv 1,2,3,5 (\bmod 7),$$
$$\gamma_2(P_5 \times P_n) = 2n + \lceil n/7 \rceil + 1 : n \equiv 0,4,6 (\bmod 7).$$

3. Main Results

Our main results here are to establish the domination number of Cartesian product of two paths P_m and P_n for *m* = 1, 2, 3, 4, 5 and arbitrary *n*. We study 2-dominating sets in complete grid graphs using one technique: by given a minimum of upper 2-dominating set *D* of $P_m \times P_n$ and then we establish that *D* is a minimum 2-dominating set of $P_m \times P_n$ for several values of *m* and arbitrary *n*. Definitely we have $\gamma_2(P_m \times P_n) = |D|$.

Let *G* be a path of order *n* with vertex set $V(G) = \{1, 2, \cdots, n\}$. For two paths of order *m* and *n* respectively is:

$P_m \times P_n = \{(i, j) : 1 \le i \le m, 1 \le j \le n\}$. The *j*th column $P_m \times P_n$ is $K_j = \{(i, j) : i = 1, \cdots, m\}$.

If *D* is a 2-dominating set for $P_m \times P_n$ then we put $W_j = D \cap K_j$. Let $s_j = |W_j|$. The sequence $(s_1, s_2, \cdots, s_n)$ is called a 2-dominating sequence corresponding to *D*. Always we have $s_1, s_n \geq \lceil m/3 \rceil$. Suppose that $s_j = 0$ for some *j* (where $j \neq 1$ or *n*). The vertices of the *j*th column can only be 2-dominated by vertices of the $(j-1)$st columns and $(j+1)$st columns. Thus we have $s_{j-1} + s_{j+1} = 2m$, then $s_{j-1} = s_{j+1} = m$. In general $s_{j-1} + 4s_j + s_{j+1} \geq 2m$.

Notice 3.1.

1) The study of 2-dominating sequence $(s_1, s_2, \cdots, s_n)$ is the same as the study of the 2-dominating sequence $(s_n, s_{n-1}, \cdots, s_1)$.

2) If subsequence $(s_j, s_{j+1}, \cdots, s_{j+k})$ is not possible, then its reverse $(s_{j+k}, \cdots, s_{j+1}, s_j)$ is not possible.

3) We say that two subsequences $(s_j, \cdots, s_{j+q}), (s_{j+q+1}, \cdots, s_{j+r})$ are equivalent, if the sequence $(s_j, \cdots, s_{j+q}, s_{j+q+1}, \cdots, s_{j+r})$ is possible.

We need the useful following lemma.

Lemma 3.1. There is a minimum 2-dominating set for $P_m \times P_n$ with 2-dominating sequence $(s_1, s_2 -, \cdots, s_n)$ such that, for all $j = 1, 2, \cdots, n$, is $\lfloor m/4 \rfloor \leq s_j \leq \lceil 3m/4 \rceil$.

Proof. Let D be a minimum 2-dominating set for $P_m \times P_n$ with 2-dominating sequence $(s_1, s_2 -, \cdots, s_n)$. Assume that for some *j*, s_j is large. Then we modify D by moving two vertices from column *j*, one to column $j-1$ and another one to column $j+1$, such that the resulting set is still 2-dominating set for $P_m \times P_n$. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let $W = D \cap \{(i,j),(i+1,j),(i+2,j),(i+3,j)\}$. If $|W| = 4$, then we define $D_1 = (D - W) \cup \{(i,j),(i+1,j-1),(i+2,j+1),(i+3,j)\}$, see Figure 1. We repeat this process if necessary eventually leads to a 2-dominating set with required properties. Also, we get D_1 is a 2-dominating set for $P_m \times P_n$ with $|D| = |D_1|$. Thus, we can assume that every four consecutive vertices of the *j*th column include at most three vertices of *D*. This implies that $s_j \leq \lceil 3m/4 \rceil$, for all $1 \leq j \leq n$.

To prove the lower bound, we suppose that $|K_j \cap D|$ is be a maximum, *i.e.*, $s_j = \lceil 3m/4 \rceil$. Then for each $(i,j) \notin D$, we have $|\{(i-1,j+1),(i,j+1),(i+1,j+1)\} \cap D| \geq 1$. When $s_j = \lceil 3m/4 \rceil$, there at must $m - \lceil 3m/4 \rceil = \lfloor m/4 \rfloor$ vertices does not in $K_j \cap D$. This implies that $s_{j+1} \geq \lfloor m/4 \rfloor$. So, the same as for $s_{j-1} \geq \lfloor m/4 \rfloor$. □

By Lemma 3.1, always we have a minimum 2-dominating set D with 2-dominating sequence $(s_1, s_2, \cdots, s_n)$, such that $\lfloor m/4 \rfloor \leq s_j \leq \lceil 3m/4 \rceil$, for all $j = 1, 2, \cdots, n$.

***Lemma* 3.2.** $\gamma_2(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

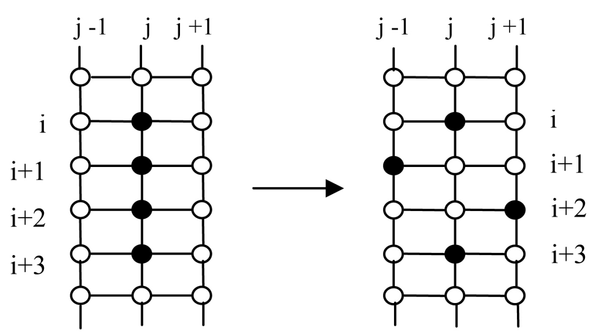

Figure 1. Modify D.

Proof. Let $D=\left\{(2k-1);1\le k\le\left\lceil\frac{n}{2}\right\rceil\right\}$.

We have D is a 2-dominating set of P_n for $n\equiv 1(\bmod 2)$ with $|D|=\left\lceil\frac{n+1}{2}\right\rceil$, also $D\cup\{(n)\}$ is a 2-dominating set of P_n for $n\equiv 0(\bmod 2)$ with $|D\cup\{(n)\}|=\left\lceil\frac{n+1}{2}\right\rceil$.

Let D_1 be a minimum 2-dominating set for P_n with $V(P_n)=\{x_1,x_2,\cdots,x_n\}$. Since $x_1x_n\notin E(P_n)$, we need to $x_1,x_n\in D_1$, also if $x_j\notin D_1$ then x_{j-1},x_{j+1} are belong to D_1, this implies that $x_{2j-1}\in D_1$ for $2\le j\le\left\lfloor\frac{n}{2}\right\rfloor$. Thus implies that

$|D_1|\ge 2+\left\lfloor\frac{n}{2}\right\rfloor-1=\left\lceil\frac{n+1}{2}\right\rceil$. We result that $\gamma_2(P_n)=\left\lceil\frac{n+1}{2}\right\rceil$. □

Theorem 3.1. $\gamma_2(P_2\times P_n)=n$.

Proof. Let a set $D=\left\{(1,2k-1):1\le k\le\left\lceil\frac{n}{2}\right\rceil\right\}\cup\left\{(2,2k):1\le k\le\left\lfloor\frac{n}{2}\right\rfloor\right\}$.

It is clear that $|D|=n$. (1)

We can check that D is 2-dominating set for $P_2\times P_n$, see Figure 2. Let D_1 be a minimum 2-dominating set for $P_2\times P_n$ with dominating sequence $(s_1,\cdots,s_n)$. If $s_i\ge 1$ for all

$$j=1,\cdots,n\text{, then }|D_1|=\sum_{j=1}^{n}s_j\ge n. \quad (2)$$

Let $s_j=0$ for some *j*, then $s_{j-1}=s_{j+1}=2$, also we have $s_1\ge 1$ and $s_n\ge 1$. Now we define a new sequence $(s_1',\cdots,s_n')$, (not necessarily a 2-dominating sequence) as follows:

For $s_j=2$, if $j=1$ or *n*, we put $s_j'=s_j-1$, $s_2'=s_2+1/2$ and $s_{n-1}'=s_{n-1}+1/2$.

If $j\ne 1$ or *n*, we put $s_j'=s_j-1$, $s_{j-1}'=s_{j-1}+1/2$ and $s_{j+1}'=s_{j+1}+1/2$.

Otherwise $s_j'=s_j$.

We get a sequence $(s_1',\cdots,s_n')$ have property that each $s_j'\ge 1$ with

$$|D|=\sum_{j=1}^{n}s_j=\sum_{j=1}^{n}s_j'\ge n. \quad (3)$$

By (1), (2) and (3) is $\gamma_2(P_2\times P_n)=n$. This completes the proof of the theorem. □

Theorem **3.2.** $\gamma_2(P_3\times P_n)=n+\left\lceil\frac{n}{3}\right\rceil$.

Proof. Let
$$D=\left\{(2,3k-2):1\le k\le\left\lceil\frac{n}{3}\right\rceil\right\}\cup\left\{(2,3k):1\le k\le\left\lfloor\frac{n}{3}\right\rfloor\right\}\cup\left\{(1,3k-1),(3,3k-1):1\le k\le\left\lceil\frac{n-1}{3}\right\rceil\right\}.$$

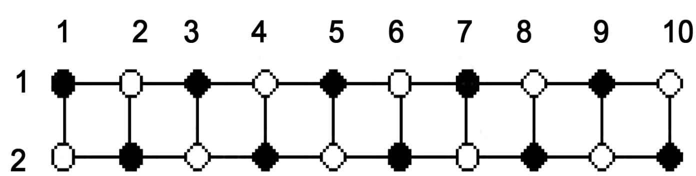

Figure 2. A 2-dominating set for $P_2\times P_{10}$.

$$D' = \left\{(1,3k-2),(3,3k-2):1 \le k \le \left\lceil \frac{n}{3} \right\rceil\right\} \cup \left\{(2,3k-1):1 \le k \le \left\lceil \frac{n-1}{3} \right\rceil\right\}$$
$$\cup \left\{(2,3k):1 \le k \le \left\lfloor \frac{n}{3} \right\rfloor\right\}.$$

We have $|D| = n + \left\lceil \frac{n}{3} \right\rceil$ and $|D'| = n + \left\lceil \frac{n}{3} \right\rceil$. (4)

By definition D and D' we note that

D is 2-dominating set for $P_3 \times P_n$ when $n = 0,2 \pmod 3$, (see **Figure 3**, for $P_3 \times P_{14}$).

D' is 2-dominating set for $P_3 \times P_n$ when $n = 1 \pmod 3$, (see **Figure 4**, for $P_3 \times P_{10}$).

Let D_1 be a minimum 2-dominating set for $P_3 \times P_n$ with 2-dominating sequence $(s_1, \cdots, s_n)$ we have $s_1, s_n \ge 1$ and

if $s_1, s_n = 1$ then $s_2, s_{n-1} \ge 2$,

if $s_1, s_n = 2$ then $s_2, s_{n-1} \ge 1$.

Also for $1 < j < n$, if $s_j = 0$ then $s_{j-1} = s_{j+1} = 3$,

$s_j = 1$ then $s_{j-1} + s_{j+1} \ge 3$,

$s_j = 2$ then $s_{j-1} + s_{j+1} \ge 2$,

If no one of $s_j = 0$ for all *j*, then $|D_1| = \sum_{j=1}^{n} s_j \ge n + \left\lceil \frac{n}{3} \right\rceil$. (5)

Let $s_j = 0$ ($j \ne 1$ or *n*) for some *j*, we define a sequence $(s'_1, \cdots, s'_n)$, (not necessarily a 2-dominating sequence) as follows:

If $s_j = 3$, then we put $s'_j = s_j - 1$, $s'_{j-1} = s_{j-1} + 1/2$ and $s'_{j+1} = s_{j+1} + 1/2$, otherwise $s'_j = s_j$. We have $|D_1| = \sum_{j=1}^{n} s_j = \sum_{j=1}^{n} s'_j$. We note that the sequence $(s'_1, \cdots, s'_n)$ have the property if $s'_j = 1$ then $s'_{j-1} + s'_{j+1} \ge 3$. Thus implies that

$$|D_1| = \sum_{j=1}^{n} s'_j \ge n + \left\lceil \frac{n}{3} \right\rceil. \quad (6)$$

From (4), (5) and (6) we get the required result. □

***Theorem* 3.3.** $\gamma_2(P_4 \times P_n) = \begin{cases} 2n - \left\lfloor \frac{n}{4} \right\rfloor : n \equiv 3,7 \pmod 8, \\ 2n - \left\lfloor \frac{n}{4} \right\rfloor + 1 : n \equiv 0,1,2,4,5,6 \pmod 8. \end{cases}$

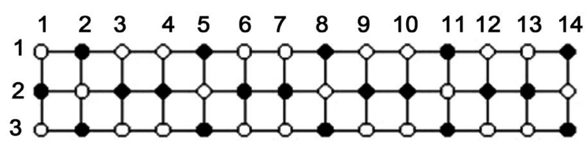

Figure 3. A 2-dominating set for $P_3 \times P_{14}$.

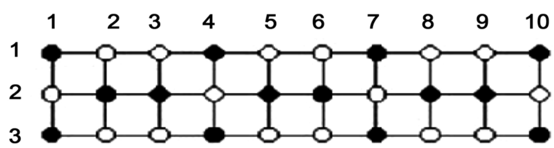

Figure 4. A 2-dominating set for $P_3 \times P_{10}$.

Proof. Let a set D defined as follows:

$$D=\left\{\{(2,1),(3,1)\}\cup\left\{(1,4k-2),(4,4k-2);1\le k\le\left\lceil\frac{n-1}{4}\right\rceil\right\}\right.$$
$$\cup\left\{(2,8k-5);1\le k\le\left\lceil\frac{n-2}{8}\right\rceil\right\}$$
$$\cup\left\{(1,8k-4),(3,8k-4),(4,8k-4);1\le k\le\left\lceil\frac{n-3}{8}\right\rceil\right\}$$
$$\cup\left\{(2,8k-3);1\le k\le\left\lceil\frac{n-4}{8}\right\rceil\right\}\cup\left\{(3,8k-1);1\le k\le\left\lceil\frac{n-6}{8}\right\rceil\right\}$$
$$\left.\cup\left\{(1,8k),(2,8k),(4,8k);1\le k\le\left\lceil\frac{n-7}{8}\right\rceil\right\}\cup\left\{(3,8k+1);1\le k\le\left\lfloor\frac{n-1}{8}\right\rfloor\right\}\right\}$$

$$D'=\{(2,n)\},\quad D''=\{(3,n)\}.$$

We can check that the following sets are 2-dominating set for $P_4\times P_n$ (see **Figure 5**, for $P_4\times P_{11}$) as indicated:

D is 2-dominating set for $P_4\times P_n$ when $n\equiv 0,4(\bmod 8)$.

$D\cup D'$ is 2-dominating set for $P_4\times P_n$ when $n\equiv 1,2,7(\bmod 8)$.

$D\cup D''$ is 2-dominating set for $P_4\times P_n$ when $n\equiv 3,5,6(\bmod 8)$.

We have

$$|D|=\begin{cases}2n-\left\lfloor\frac{n}{4}\right\rfloor-1: n\equiv 3,7(\bmod 8),\\ 2n-\left\lfloor\frac{n}{4}\right\rfloor: n\equiv 1,2,5,6(\bmod 8),\\ 2n-\left\lfloor\frac{n}{4}\right\rfloor+1: n\equiv 0,4(\bmod 8).\end{cases}$$

Let D_1 be a minimum 2-dominating set for $P_4\times P_n$ with 2-dominating sequence $(s_1,\cdots,s_n)$ we shall show that

$$|D_1|=\begin{cases}2n-\left\lfloor\frac{n}{4}\right\rfloor: n\equiv 3,7(\bmod 8),\\ 2n-\left\lfloor\frac{n}{4}\right\rfloor+1: n\equiv 0,1,2,4,5,6(\bmod 8).\end{cases}$$

By Lemma 3.1, we have $1\le s_j\le 3$. Thus

If $s_j=1$ then $s_{j-1}+s_{j+1}\ge 5$.

If $s_j=2$ then $s_{j-1}+s_{j+1}\ge 2$.

If $s_j=3$ then $s_{j-1}+s_{j+1}\ge 2$.

Also, we have $s_1,s_n\ge 2$. If $s_1,s_n=2$ then $s_2,s_{n-1}\ge 2$, and if $s_1,s_n=3$ then

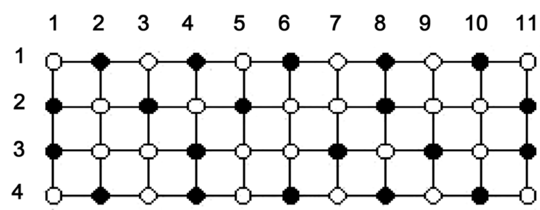

Figure 5. A 2-dominating set for $P_4\times P_{11}$.

$s_2, s_{n-1} \geq 1$.

We define a new set D_1' with sequence $(s_1', \cdots, s_n')$, (not necessarily a 2-dominating sequence) as follows: if $s_j \geq 2$, let $M_j = s_j - \frac{7}{4}$. Now, for $j = 2$ to $j = n-1$, if $s_j \geq 2$, then we put

$$s_j' = s_j - M_j,\quad s_{j-1}' = s_{j-1} + \frac{M_j}{2} \quad \text{and} \quad s_{j+1}' = s_{j+1} + \frac{M_j}{2}$$

Thus, for $3 \leq j \leq n-2$, we have $s_j \geq \frac{7}{4}$. Since if $s_j \geq 2$ then $s_j' \geq \frac{7}{4}$ and if $s_j = 1$, then $s_{j-1} + s_{j+1} = 5$ this implies that $M_{j-1} + M_{j+1} = 5 - \frac{14}{4} = \frac{6}{4}$, which implies that $s_j' = s_j + \frac{M_{j-1}}{2} + \frac{M_{j+1}}{2} = 1 + \frac{3}{4} = \frac{7}{4}$.

We have three cases:

Case 1: $s_1, s_n \geq 2$, then $s_2, s_{n-1} \geq 2$, these implies that $s_1' \geq s_1 + \frac{1}{8}$ and $s_n' \geq s_n + \frac{1}{8}$ also

$$|D_1| = \sum_{j=1}^{n} s_j = \sum_{j=1}^{n} s_j' = s_1' + s_n' + \sum_{j=2}^{n-1} s_j' \geq 2 + \frac{1}{8} + 2 + \frac{1}{8} + \frac{7(n-2)}{4} = \frac{7n}{4} + \frac{3}{4}.$$

Case 2: $s_1, s_n = 3$ then $s_2, s_{n-1} \geq 2$. Thus implies that $s_1', s_n' = 3$ and $s_2', s_{n-1}' \geq 1 + \frac{1}{8}$. Then

$$|D_1| = \sum_{j=1}^{n} s_j = \sum_{j=1}^{n} s_j' = s_1' + s_2' + s_{n-1}' + s_n' + \sum_{j=2}^{n-2} s_j' \geq 3 + 1 + \frac{1}{8} + 3 + 1 + \frac{1}{8} + \frac{7}{4} = \frac{7n}{4} + \frac{5}{4}$$

Case 3: $s_1 = 2,\ s_n = 3$ and $s_2 \geq 2,\ s_{n-1} \geq 1$ or $s_1 = 3,\ s_n = 2$ and $s_2 \geq 1,\ s_{n-1} \geq 2$. Two cases are similar by symmetry. We consider the first case:

$s_1 = 2,\ s_2 \geq 2$ and $s_n = 3,\ s_{n-1} \geq 1$, this implies that

$s_1' = 2 + \frac{1}{8},\ s_2' = \frac{7}{4},\ s_n' = 3,\ s_{n-1}' = 1 + \frac{1}{8}$ and

$$|D_1| = \sum_{j=1}^{n} s_j = \sum_{j=1}^{n} s_j' = s_1' + s_2' + s_{n-1}' + s_n' + \sum_{j=3}^{n-2} s_j' \geq 2 + \frac{1}{8} + \frac{7}{4} + 3 + 1 + \frac{1}{8} + \frac{7}{4}(n-4) = \frac{7n}{4} + 1$$

But, we have the 2-domination number is positive integer number, also we have

$$2n - \left\lfloor \frac{n}{4} \right\rfloor = \frac{7n}{4} + \frac{3}{4} \quad \text{for} \quad n \equiv 3,7 \pmod 8,$$

$$2n - \left\lfloor \frac{n}{4} \right\rfloor + 1 = \begin{cases} \frac{7n}{4} + 1 & \text{For} \quad n \equiv 0,4 \pmod 8, \\ \frac{7n}{4} + \frac{5}{4} & \text{For} \quad n \equiv 1,5 \pmod 8, \\ \frac{7n}{4} + \frac{6}{4} & \text{For} \quad n \equiv 2,6 \pmod 8, \end{cases}$$

Thus implies that

$$|D_1| \geq \left\{ \begin{aligned} &2n - \left\lfloor \frac{n}{4} \right\rfloor; n \equiv 3,7 \pmod 8, \\ &2n - \left\lfloor \frac{n}{4} \right\rfloor + 1; n \equiv 0,1,2,4,5,6 \pmod 8, \end{aligned} \right\}$$

Finally, we get

$$\gamma_2(P_4 \times P_n) = 2n - \left\lfloor \frac{n}{4} \right\rfloor : n \equiv 3,7 \pmod 8,$$

$$\gamma_2(P_4 \times P_n) = 2n - \left\lfloor \frac{n}{4} \right\rfloor + 1 : n \equiv 0,1,2,4,5,6 \pmod 8),$$

This complete the proof of the theorem. □

Theorem 3.4.

$$\gamma_2(P_5 \times P_n) = \begin{cases} 2n + \left\lceil \frac{n}{7} \right\rceil : n \equiv 1,2,3,5 \pmod 7, \\ 2n + \left\lceil \frac{n}{7} \right\rceil + 1 : n \equiv 0,4,6 \pmod 7. \end{cases}$$

Proof. Let a set D defined as follows:

$$\begin{aligned} D = \{ & \{(2,1),(4,1)\} \cup \{(1,j),(2,j),(5,j) : j \equiv 2 \pmod 7\} \\ & \cup \{(3,j) : j \equiv 3 \pmod 7\} \cup \{(1,j),(4,j),(5,j) : j \equiv 4 \pmod 7\} \\ & \cup \{(2,j),(3,j) : j \equiv 5 \pmod 7\} \cup \{(2,j),(5,j) : j \equiv 6 \pmod 7\} \\ & \cup \{(1,j),(4,j) : j \equiv 0 \pmod 7\} \cup \{(3,j),(4,j) : j \equiv 1 \pmod 7\} \text{ and } j \neq 1 \} \end{aligned}$$

We can check that the following sets are 2-dominating set for $P_5 \times P_n$ (see **Figure 6**, for $P_5 \times P_{23}$) as indicated:

$$\{D - \{K_n \cap D\}\} \cup \{(2,n),(3,n),(5,n)\} : n \equiv 1 \pmod 7.$$
$$D \cup \{(2,n)\} : n \equiv 0,4 \pmod 7.$$
$$D : n \equiv 2 \pmod 7.$$
$$\{D - \{K_n \cap D\}\} \cup \{(2,n),(4,n)\} : n \equiv 3,5 \pmod 7.$$
$$\{D - \{K_n \cap D\}\} \cup \{(1,n),(3,n),(5,n)\} : n \equiv 6 \pmod 7.$$

We have $D \leq 2n + \left\lceil \frac{n}{7} \right\rceil$ and

$$\gamma_2(P_5 \times P_n) \leq \begin{cases} 2n + \left\lceil \frac{n}{7} \right\rceil : n \equiv 1,2,3,5 \pmod 7, \\ 2n + \left\lceil \frac{n}{7} \right\rceil + 1 : n \equiv 0,4,6 \pmod 7. \end{cases}$$

This complete the proof of the theorem. □

Lemma 3.3. The following cases are not possible:

1) (1, 2, 3, 1).

2) (1, 2, 1).

3) (1, 4, 1, 1).

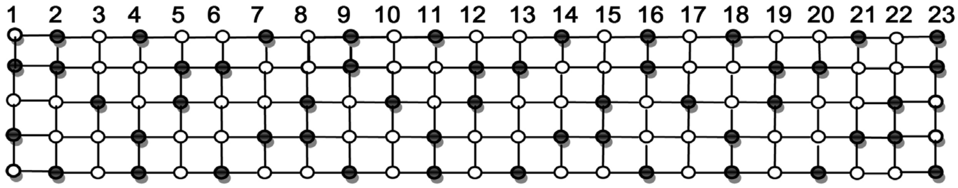

Figure 6. A 2-dominating set for $P_5 \times P_{23}$.

4) (1, 3, 1, 3, 1, 3).

5) (2, 1, 3).

6) (2, 2, 2, 2, 2, 2).

Proof. It follows directly from the drawing.

Lemma 3.4.

1) There is one case for subsequence $(s_j, s_{j+1}, s_{j+2}, s_{j+3}, s_{j+4}) = (2,2,2,2,2)$.

2) There is one case for subsequence $(s_j, s_{j+1}, s_{j+2}, s_{j+3}) = (1,3,1,3)$.

3) There is one case for subsequence $(s_j, s_{j+1}, s_{j+2}, s_{j+3}, s_{j+4}) = (1,3,1,3,1)$.

4) There is one case for subsequence $(s_j, s_{j+1}, s_{j+2}, s_{j+3}, s_{j+4}) = (1,2,3,2,1)$.

Proof. It follows directly from the drawing (see Figure 7).

Lemma 3.5.

1) $\sum_j^{j+3} s_j \geq 8$.

2) $\sum_j^{j+5} s_j \geq 12$.

3) $\sum_j^{j+6} s_j \geq 14$.

4) If $s_j = 3$ then $\sum_j^{j+6} s_j \geq 15$.

5) If $s_j = 4$ then $\sum_j^{j+6} s_j \geq 16$.

Proof. 1) By Lemma 3.3, imply that $\sum_j^{j+3} s_j \geq 8$.

2) By 1, we have $\sum_j^{j+3} s_j \geq 8$. If $\sum_j^{j+3} s_j = 8$, then we have the cases

$$(s_j, s_{j+1}, s_{j+2}, s_{j+3}) = (1,2,3,2),(1,3,1,3),(1,3,2,2),(1,4,1,2),(2,2,2,2).$$

From Lemma 3.3, we have $s_{j+4} + s_{j+5} \geq 4$, this implies that $\sum_j^{j+5} s_j \geq 12$.

If $\sum_j^{j+4} s_j \geq 9$ then $s_{j+4} + s_{j+5} \geq 3$. This implies that $\sum_j^{j+6} s_j \geq 12$.

3) We have $\sum_j^{j+2} s_j \geq 5$ and $\sum_{j+4}^{j+6} s_j \geq 5$. If $\sum_{j+4}^{j+6} s_j = 5$, then there is one case $(s_{j+4}, s_{j+5}, s_{j+6}) = (1,3,1)$ (where the cases $(1,2,1),(1,2,2)$ are not possible). But the case $(1,3,1)$ is not compatible with any of the cases when $\sum_j^{j+3} s_j = 8$, this implies that

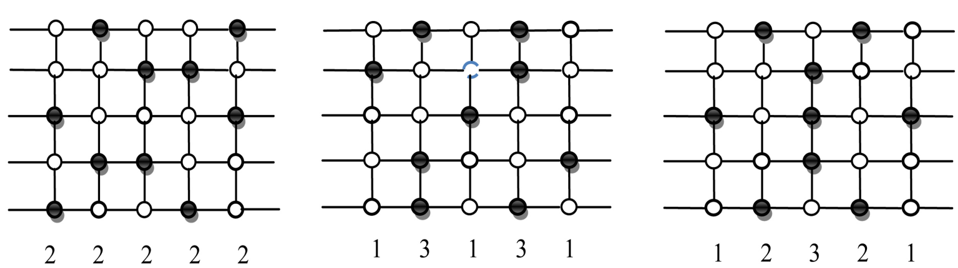

Figure 7. Cases 1, 2, 3 and 4 of Lemma 3.4.

$\sum_{j}^{j+3} s_j \geq 9$. Then $\sum_{j}^{j+6} s_j \geq 14$ (where the case $(1,3,1,3,1,3)$ is not possible). If $\sum_{j+4}^{j+6} s_j \geq 6$ then $\sum_{j}^{j+6} s_j = \sum_{j}^{j+3} s_j + \sum_{j+4}^{j+6} s_j \geq 8+6=14$.

4) We have $s_j \geq 3$, then from 2 is $\sum_{j}^{j+6} s_j \geq 15$.

5) We have $s_j \geq 4$, then from 2 is $\sum_{j}^{j+6} s_j \geq 16$. This complete the proof of the Lemma. □

Lemma 3.6. If $\sum_{j}^{j+6} s_j = 14$, then $s_j = 1$ or $s_{j+6} = 1$.

Proof. We suppose the contrary $s_j, s_{j+6} \geq 2$. From Lemma 3.5, $s_j, s_{j+6} < 3$, else $\sum_{j}^{j+6} s_j \geq 15$. Now, we must study the case $s_j = s_{j+6} = 2$. We have $\sum_{j+2}^{j+5} s_j = 10$, by Lemma 3.3, the case $(2,2,2,2,2,2)$ is not possible, this implies that not all elements of the subsequence $(s_{j+1}, \cdots, s_{j+5})$ are equal to the value 2. If $s_{j+1}, s_{j+2}, s_{j+3}, s_{j+4}, s_{j+5} \geq 2$ where at least one of them is equal or greater than 3, then $\sum_{j}^{j+6} s_j \geq 15$, this is a contradiction with $\sum_{j}^{j+6} s_j = 14$. Now, we have $\sum_{j}^{j+5} s_j = 10$, where one of the subsequence element $(s_{j+1}, \cdots, s_{j+5})$ is at most equal the value 1 (where $1 \leq s_j \leq 4$). We consider the cases $s_j = 1$ for $j+1 \leq j \leq j+5$:

1) $s_{j+1} = 1$ or $s_{j+5} = 1$ (where two cases are similar), we study the case $s_{j+1} = 1$ then $s_{j+2} = 4$, these implies that $s_j + s_{j+1} + s_{j+2} = 7$. By Lemma 3.5, we have $\sum_{j+3}^{j+6} s_j \geq 8$ then $\sum_{j}^{j+6} s_j \geq 15$, this is a contradiction.

2) $s_{j+2} = 1$ or $s_{j+4} = 1$ (where two cases are similar), we study the case $s_{j+2} = 1$ then $s_{j+1} \geq 3$, (because the case $(2,2,1)$ is not possible). If $s_{j+1} = 3$ then $s_{j+3} \geq 3$ and we have $s_{j+6} = 2$ then $\sum_{j+4}^{j+5} s_j \geq 4$ (because two cases $(1,2,2),(2,1,2)$ are not possible). Thus implies that $\sum_{j}^{j+6} s_j \geq 2+3+1+3+4+2=15$, this is a contradiction.

3) $s_{j+3} = 1$, then we have two subcases results from $s_{j+2} + s_{j+4} \geq 6$:

Subcase 1: $s_{j+2} = s_{j+4} = 3$ then $s_{j+1}, s_{j+5} \geq 2$ (because two cases $(s_j, s_{j+1}, s_{j+2}) = (2,1,3)$ and $(s_{j+4}, s_{j+5}, s_{j+6}) = (3,1,2)$ are not possible). Thus implies that $\sum_{j}^{j+6} s_j \geq 15$, this is a contradiction.

Subcase 2: If $s_{j+2} = 2,\ s_{j+4} = 4$ or conversely (two cases are similar in studying), so we will study case $s_{j+2} = 2,\ s_{j+4} = 4$ then $s_{j+5} \geq 1$, if $s_{j+5} \geq 2$, then $\sum_{j}^{j+6} s_j \geq 15$, because $s_{j+4} + s_{j+5} + s_{j+6} \geq 8$, we have $\sum_{j}^{j+3} s_j \geq 8$. Then $\sum_{j}^{j+6} s_j \geq 15$, this is a contradiction).

If $s_{j+5} = 1$, then $s_{j+4} + s_{j+5} + s_{j+6} = 7$. We have $\sum_{j}^{j+3} s_j \geq 8$. This implies that

$\sum_{j}^{j+6} s_j \geq 15$ this is a contradiction.

Finally, we get if $\sum_{j}^{j+6} s_j = 14$, then $s_j = 1$ or $s_{j+6} = 1$. This completely the proof. □

Result 3.1. If $\sum_{j}^{j+6} s_j = 14$, then from Lemma 3.6, we have the cases for subsequence $(s_j, s_{j+1}, s_{j+2}, s_{j+3}, s_{j+4}, s_{j+5}, s_{j+6})$:

$$a_1 : (1,2,3,2,1,4,1), \quad a_2 : (1,2,3,2,2,2,2), \quad a_3 : (1,2,3,2,2,3,1),$$
$$a_4 : (1,2,3,3,1,3,1), \quad a_5 : (1,3,1,3,1,4,1), \quad a_6 : (1,3,1,3,2,2,2),$$
$$a_7 : (1,3,1,3,2,3,1), \quad a_8 : (1,3,1,3,3,2,1), \quad a_9 : (1,3,1,4,1,3,1),$$
$$a_{10} : (1,3,2,2,2,2,2), \quad a_{11} : (1,3,2,2,2,3,1), \quad a_{12} : (1,3,2,2,3,2,1),$$
$$a_{13} : (1,3,2,3,1,3,1), \quad a_{14} : (1,4,1,2,3,2,1), \quad a_{15} : (1,4,1,3,1,3,1).$$

It is 15 cases (where $s_j = 1$ with $\sum_{j}^{j+6} s_j = 14$). We have three cases with $s_{j+1} = 2$, $s_{j+1} = 3$ and $s_{j+1} = 4$.

Case 1: $s_{j+1} = 2$ (including the cases $s_j = 1$ and $s_{j+1} = 2$ or $s_{j+6} = 1$ and $s_{j+5} = 2$). We have these cases are $a_1, a_2, a_3, a_4, a_8, a_{12}, a_{14}$ and comes before these cases, $s_{j-1} = 4$ or comes after these cases $s_{j+7} = 4$, *i.e.*, if $s_j = 1,\ s_{j+1} = 2$ then $s_{j-1} = 4$ and if $s_{j+6} = 1,\ s_{j+5} = 2$ then $s_{j+7} = 4$.

Case 2: $s_{j+1} = 3,\ s_{j+1} = 4$ and these are the 8 remaining cases. We will study these cases after rejecting isomorphism cases when there is two cases or more, where $(s_j, \cdots, s_{j+6}) = (s_{j+6}, \cdots, s_j)$, then we will study only one case. We have 8 cases as follows:

$$a_5 : (1,3,1,3,1,4,1),\ a_6 : (1,3,1,3,2,2,2),\ a_7 : (1,3,1,3,2,3,1),\ a_9 : (1,3,1,4,1,3,1),$$
$$a_{10} : (1,3,2,2,2,2,2),\ a_{11} : (1,3,2,2,2,3,1),\ a_{13} : (1,3,2,3,1,3,1),\ a_{15} : (1,4,1,3,1,3,1).$$

We note that two cases a_5, a_{15} are similar where one of them is contrary to the other one, so we study the case a_5. Also, two cases a_7, a_{13} are similar, so we study the case a_7. Then we study these cases: $a_5, a_6, a_7, a_9, a_{10}, a_{11}$. □

Notice 3.2. We note that all the possible cases in Result 3.1, do not begin or end with 3 or 4 and it do not begin or end with $s_j + s_{j+1} \geq 5$ or $s_{j+5} + s_{j+6} \geq 5$ such that $s_j = 2$ or $s_{j+6} = 2$, and $s_{j+1} = 3$ or $s_{j+5} = 3$. Thus implies that if $s_j = 2,\ s_{j+1} = 3$, then $\sum_{j}^{j+6} s_j \geq 15$. Also, we note cases a_5, a_6, a_7 are beginning with (1, 3, 1, 3), but from Lemma 3.4, we get $s_{j-1} = 4$. Now, remains our three cases for studying by the following lemma are:

$$a_9 : (1,3,1,4,1,3,1),\ a_{10} : (1,3,2,2,2,2,2),\ a_{11} : (1,3,2,2,2,3,1). \ \square$$

Result 3.2. If $s_{j+1} = 3,\ s_j = 1$ where $k_j \cap s = \{(1, j)\}$ or $k_j \cap s = \{(2, j)\}$ then $s_{j-1} = 4$, also for $k_j \cap s = \{(4, j)\}$ or $k_j \cap s = \{(5, j)\}$ because it are similar to two cases $k_j \cap s = \{(2, j)\}$ or $k_j \cap s = \{(1, j)\}$, respectively. □

Lemma 3.7. If $\sum_{j}^{j+6} s_j = 14$, such that $s_{j+5} = 3,\ s_{j+6} = 1$, then $\sum_{j+7}^{j+13} s_j \geq 15$. Furthermore, if $\sum_{j+7}^{j+13} s_j = 15$ then $\sum_{j+14}^{j+20} s_j \ge 15$.

Proof. By Result 3.2, if $k_{j+6} \cap s = \{(1, j+6)\}$, $k_{j+6} \cap s = \{(2, j+6)\}$, $k_{j+6} \cap s = \{(4, j+6)\}$ or $k_{j+6} \cap s = \{(5, j+6)\}$ then $s_{j+7} = 4$. From Lemma 3.5, we get $\sum_{j+7}^{j+13} s_j \ge 16$. Assume $k_{j+6} \cap s = \{(3, j+6)\}$ then we have two cases for $k_{j+5} \cap s$:

Case 1. $k_{j+5} \cap s = \{(1, j+5), (3, j+5), (5, j+5)\}$. Then $s_{j+7} = 4$, by lemma 3.5, $\sum_{j+7}^{j+13} s_j \ge 16$.

Case 2. $k_{j+5} \cap s = \{(1, j+5), (2, j+5), (5, j+5)\}$ or $k_{j+5} \cap s = \{(1, j+5), (4, j+5), (5, j+5)\}$ and both cases are similar, so we will consider the first case. We have $3 \le s_{j+7} \le 4$ then by Lemma 3.5, $\sum_{j+7}^{j+13} s_j \ge 15$. If $s_{j+7} = 4$ then $\sum_{j+7}^{j+13} s_j \ge 16$. Assume $s_{j+7} = 3$, if $\sum_{j+7}^{j+13} s_j \ge 16$ the proof is finish. Assume $\sum_{j+7}^{j+13} s_j = 15$ then we have cases $s_{j+8} = 1, 2, 3 \text{ or } 4$.

Subcase 2.1. If $s_{j+8} = 4$ then $s_{j+9} \ge 1$. This implies that

$$\sum_{j+7}^{j+13} s_j \ge 3 + 4 + 1 + \sum_{j+10}^{j+13} s_j = 8 + 8 = 16$$

{By Lemma 3.5, $\sum_{j}^{j+3} s_j \ge 8$}.

Subcase 2.2. If $s_{j+8} = 3$ then $\sum_{j+9}^{j+13} s_j \ge 9$. If $\sum_{j+9}^{j+13} s_j > 9$ then $\sum_{j+7}^{j+13} s_j \ge 16$. Assume that $\sum_{j+9}^{j+13} s_j = 9$ then we have only one case $(s_{j+9}, \cdots, s_{j+13}) = (1,3,1,3,1)$ or $(s_{j+9}, \cdots, s_{j+13}) = (1,2,3,2,1)$. For any case we have $s_{j+8} = 4$. So, we get $\sum_{j+9}^{j+13} s_j > 9$. Which implies that $\sum_{j+7}^{j+13} s_j \ge 16$.

Subcase 2.3. If $s_{j+8} = 1$ then $s_{j+9} = 4$ {because the case $(s_{j+5}, s_{j+6}, s_{j+7}, s_{j+8}, s_{j+9}) = (3,1,3,1,3)$ is not possible, by Lemma 3.3}. Then

$$\sum_{j+7}^{j+13} s_j \ge 3 + 1 + 4 + \sum_{j+10}^{j+13} s_j \ge 8 + 8 = 16.$$

Subcase 2.4. If $s_{j+8} = 2$ then $s_{j+7} = 3,\ s_{j+8} = 2$, we have the following cases:

2.4.1. $s_{j+9} \ge 3$ then $\sum_{j+7}^{j+13} s_j \ge 3 + 2 + 3 + \sum_{j+10}^{j+13} s_j \ge 8 + 8 = 16$.

2.4.2. $s_{j+9} \ne 1$ {because there is only one case for $(s_{j+7}, s_{j+8}, s_{j+9}) = (3,2,1)$ such that

$$\{K_{j+7} \cup K_{j+8} \cup K_{j+9}\} \cap S = \{(2, j+7), (3, j+7), (4, j+7), (1, j+8), (5, j+8), (3, j+9)\}$$

But according to distribution vertices $k_{j+5} \cap S$ and $k_{j+6} \cap S$ we have

$$k_{j+5} \cap 7 \ne \{(2, j+7), (3, j+7), (4, j+7)\}.$$

2.4.3. $s_{j+9} = 2$ then $s_{j+7} + s_{j+8} + s_{j+9} = 7$. This implies that

$(s_{j+7}, s_{j+8}, s_{j+9}) = (3,2,2)$. We will study the cases that leads to $\sum_{j+7}^{j+13} s_j = 15$, *i.e.*, $\sum_{j+10}^{j+13} s_j = 8$, {because the cases which leads to $\sum_{j+7}^{j+13} s_j \geq 16$ the proof will be done}. Now, we have the fixed case $(s_{j+7}, s_{j+8}, s_{j+9}) = (3,2,2)$ We will consider the vertices $k_{j+10} \cap S$ which imply the following:

2.4.3.1. If $s_{j+10} = 4$ then $(3,2,2,4,s_{j+11},s_{j+12},s_{j+13})$, this implies that $\sum_{j+11}^{j+13} s_j = 4$ and $(s_{j+11},s_{j+12},s_{j+13}) = (1,2,1)$ is not possible.

2.4.3.2. If $s_{j+10} = 3$ then $(3,2,2,3,s_{j+11},s_{j+12},s_{j+13})$ and $\sum_{j+11}^{j+13} s_j = 5$ which imply that $(s_{j+11},s_{j+12},s_{j+13}) = (2,1,2),(2,2,1),(1,2,2)$ or (1, 3, 1), and the only possible case is (1, 3, 1). Thus implies that $(s_{j+7},\cdots,s_{j+13}) = (3,2,2,3,1,3,1)$. By Lemma 3.4 and Lemma 3.5 is $s_{j+14} = 4$, these implies that $\sum_{j+14}^{j+20} s_j \geq 16$.

2.4.3.3. If $s_{j+10} = 2$ then $(3,2,2,2,s_{j+11},s_{j+12},s_{j+13})$, *i.e.*, $\sum_{j+11}^{j+13} s_j = 6$. We have $s_{j+11} \neq 1$ {because the case $(2,2,1)$ is not possible}. Then we have the following cases for $s_{j+11}, s_{j+12}, s_{j+13}$:

1). If $s_{j+11} = 4$ then $s_{j+12} = 1$ and $s_{j+13} = 1$, but the case $(4,1,1)$ is not possible.

2). If $s_{j+11} = 3$ and $s_{j+12} = 1$ then $s_{j+13} = 2$, also the case $(3,1,2)$ is not possible.

3). If $s_{j+11} = 3$, $s_{j+12} = 2$ and $s_{j+13} = 1$ then $(s_j,\cdots,s_{j+6}) = (3,2,2,2,3,2,1)$ which gets $s_{j+7} = 4$ and $\sum_{j+7}^{j+13} s_j \geq 16$.

4). If $s_{j+11} = 2$ and $s_{j+12} = 2$ then $s_{j+13} = 2$, but the case $(3,2,2,2,2,2,2)$ is not possible. If $s_{j+11} = 2$, $s_{j+12} = 3$ and $s_{j+13} = 1$ then we gets $(s_j,\cdots,s_{j+6}) = (3,2,2,2,2,3,1)$ During the proof of Lemma, we notice that if $s_j = 3$ and $s_{j+1} = 1$, then $\sum_{j+2}^{j+8} s_j \geq 15$. This complete the proof. □

Result 3.3. Based on the Lemma 3.6, and the other Lemmas and results precede it. We see that when we have case of $\sum_{j}^{j+6} s_j = 14$, then the only case that comes after it, is $\sum_{j+7}^{j+13} s_j = 15$ such that $(s_{j+7},\cdots,s_{j+13}) = (3,2,2,2,2,3,1)$ which continues in the same way or it is followed by 7 columns contain 16 vertices from S {by Lemma 3.6, $\sum_{j+14}^{j+20} s_j \geq 15$, because $s_{j+12} = 3$, $s_{j+13} = 1$}. When this case is repeated then $\sum_{j=n-6}^{n} s_j \geq 15$ and then when the case $\sum_{j}^{j+6} s_j = 14$ it is necessary, the case $\sum_{j+6+q}^{j+6+q-1+7r} s_j \geq 16$ exists as well {where $j+6+q-1+7r \leq n$} these implies that $\sum_{j=1}^{n} s_j \geq \left\lceil \frac{15n}{7} \right\rceil$ then

$\gamma_2(P_5 \times P_n) = \sum_{j=1}^{n} s_j \geq 2n + \left\lceil \frac{n}{7} \right\rceil$. □

Lemma 3.8. Let S be 2-dominating set for $P_5 \times P_n$ then:

1) $s_1 \ge 2$ and $s_1 + s_2 \ge 4\left(s_{n-1} + s_n \ge 4, s_n \ge 2\right)$.

2) If $s_1 + s_2 = 4$ then $s_1 + s_2 + s_3 = 8$ ($s_{n-1} + s_n = 4$ then $s_{n-2} + s_{n-1} + s_n = 8$).

3) $s_1 + s_2 + s_3 \ge 6\left(s_{n-2} + s_{n-1} + s_n \ge 6\right)$.

4) $\sum_{j=1}^{4} s_j \ge 9\left(\sum_{j=n-3}^{n} s_j \ge 9\right)$.

5) $\sum_{j=1}^{5} s_j \ge 10\left(\sum_{j=n-4}^{n} s_j \ge 10\right)$ and if $\sum_{j=1}^{5} s_j = 10$ then $\sum_{j=1}^{6} s_j \ge 14$, also if $\sum_{j=n-4}^{n} s_j = 10$ then $\sum_{j=n-5}^{n} s_j \ge 14$

6) $\sum_{j=1}^{6} s_j \ge 13\left(\sum_{j=n-5}^{n} s_j \ge 13\right)$.

7) $\sum_{j=1}^{7} s_j \ge 15\left(\sum_{j=n-6}^{n} s_j \ge 15\right)$.

8) If $s_1 + s_2 = 5$ then either $\sum_{j=1}^{5} s_j \ge 11$ or $\sum_{j=1}^{6} s_j \ge 14$, also if $s_{n-1} + s_n = 5$ then either $\sum_{j=n-4}^{n} s_j \ge 11$ or $\sum_{j=n-5}^{n} s_j \ge 14$.

Proof. The study of dominating sequence $(s_1, s_2, \cdots, s_n)$ is the same as the study of the dominating sequence $(s_n, s_{n-1}, \cdots, s_1)$, so we study one case $(s_1, s_2, \cdots, s_n)$. Also, the study of $\sum_{j=1}^{r} s_j$ is the same as the study of $\sum_{j=n-r+1}^{n} s_j$.

1) We have $s_1 \ge 2$, if $s_1 = 2$ then $s_2 \ge 3$ thus, $s_1 + s_2 \ge 5$ if $s_1 \ge 3$ then $s_2 \ge 1\left(1 \le s_j \le 4\right)$ these implies that $s_1 + s_2 \ge 4$.

2) If $s_1 + s_2 = 4$, then we have only one the case $k_1 \cap s = \{(1,1),(3,1),(5,1)\}$ these implies that $k_2 \cap s = \{(3,2)\}$ and $s_3 = 4$ then $s_1 + s_2 + s_3 = 8$.

3) If $s_1 + s_2 \ge 5$, then $\sum_{j=1}^{3} s_j \ge 6$ {because $1 \le s_j \le 4$} and if $s_1 + s_2 = 4$ then by 2, is $\sum_{j=1}^{3} s_j = 8$.

4) If $s_1 + s_2 = 4$ then $\sum_{j=1}^{4} s_j = 8$ these implies that $\sum_{j=1}^{4} s_j \ge 9$ and if $s_1 + s_2 \ge 6$ then $\sum_{j=1}^{4} s_j \ge 9$ {because $s_3 + s_4 \ge 3$}. Assume that $s_1 + s_2 = 5$, then we have three cases:

4.1) $s_1 = 2,\ s_2 = 3$ then $s_3 + s_4 \ge 4$, because the case $(s_2, s_3, s_4) = (3,1,2)$ is not possible. Also the case $(s_2, s_3, s_4) = (3,2,1)$ is not possible, else when $k_2 \cap s = \{(2,2),(3,2),(4,2)\}$ and this is not possible.

4.2) $s_1 = 3,\ s_2 = 2$ then $s_3 + s_4 \ge 4$ because the cases $(s_2, s_3, s_4) = (2,2,1)$, $(s_2, s_3, s_4) = (2,1,2)$ are not possible.

4.3) $s_1 = 4,\ s_2 = 1$ then $s_3 + s_4 \ge 4$, because the cases $(s_1, s_2, s_3, s_4) = (4,1,2,1)$, $(s_1, s_2, s_3, s_4) = (4,1,2,2)$ are not possible. Thus implies that we have $\sum_{j=1}^{4} s_j \ge 9$.

5) By Lemma 3.4, we have two cases for $\sum_{j=1}^{4} s_j = 9$ and these two cases are $(1,2,3,2,1),(1,3,1,3,1)$, furthermore these cannot be shown here because $s_1 \geq 2$. Thus implies that we $\sum_{j=1}^{5} s_j \geq 10$.

6). If $s_1 + s_2 \geq 5$ then $\sum_{j=1}^{6} s_j = s_1 + s_2 + \sum_{j=3}^{6} s_j \geq 5 + 8 = 13$.

(where by Lemma 3.5, we have $\sum_{j}^{j+3} s_j \geq 8$). Let $s_1 + s_2 = 4$ then $\sum_{j=1}^{3} s_j = 8$ these implies that $\sum_{j=1}^{6} s_j \geq 8 + \sum_{j=4}^{6} s_j$. Thus implies that $\sum_{j=1}^{6} s_j \geq 8 + 5 = 13$ {because $\sum_{j}^{j+2} s_j \geq 5$}.

7) If $s_1 \geq 3$ then from Lemma 3.5, $\sum_{j=1}^{7} s_j \geq 15$. Let $s_1 = 2$ {because $s_1 > 1$} then $s_2 \geq 3$. This implies that $\sum_{j=1}^{7} s_j \geq 15$ {by Notice 3.2}.

8) If $s_1 + s_2 = 5$ then either $\sum_{j=1}^{5} s_j \geq 11$ or $\sum_{j=1}^{6} s_j \geq 14$. We have $s_1 + s_2 = 5$, then we have three cases:

8.1) $s_1 = 4,\ s_2 = 1$, then $s_3 + s_4 + s_5 \geq 7$ because the cases $(s_1,s_2,s_3,s_4,s_5) = (4,1,2,2,2),(4,1,3,2,1),(4,1,2,3,1)$ or $(4,1,3,1,2)$ are not possible. Thus implies that $\sum_{j=1}^{5} s_j \geq 11$.

8.2) $s_1 = 2,\ s_2 = 3$, then $\sum_{j=1}^{5} s_j \geq 10$ and if $\sum_{j=1}^{5} s_j = 10$ then $(s_1,s_2,s_3,s_4,s_5) = (2,3,1,3,1)$. By Lemma 3.4, $s_6 = 4$. Thus implies that $\sum_{j=1}^{6} s_j \geq 14$.

8.3) $s_1 = 3,\ s_2 = 2$, then (s_1,s_2,s_3,s_4,s_5) it has minimal numerals in the following cases $(s_1,s_2,s_3,s_4,s_5) = (3,2,2,2,2),(3,2,1,4,1)$ or $(3,2,3,1,3)$ and for the case $(s_3,s_4,s_5) = (1,3,1)$ is not compatible with the case $(s_1,s_2) = (3,2)$. Thus implies that $\sum_{j=1}^{5} s_j \geq 11$. This completes the proof. □

Theorem 3.5.

$$\gamma_2(P_5 \times P_n) = \begin{cases} 2n + \left\lceil \frac{n}{7} \right\rceil : n \equiv 1,2,3,5 \pmod 7, \\ 2n + \left\lceil \frac{n}{7} \right\rceil + 1 : n \equiv 0,4,6 \pmod 7. \end{cases}$$

Proof. By Result 3.3, we have $\gamma_2(p_5 \times p_n) = \sum_{j=1}^{n} s_j \geq \left\lceil \frac{15n}{7} \right\rceil$. By Theorem 3.4, we get $\gamma_2(p_5 \times p_n) = 2n + \left\lceil \frac{n}{7} \right\rceil : n \equiv 1,2,3,5 \pmod 7$.

Now, for $n \equiv 0,4,6 \pmod 7$, by Theorem 3.4, we have $\gamma_2(p_5 \times p_n) \leq 2n + \left\lceil \frac{n}{7} \right\rceil + 1$.

From Result 3.3, we have $\gamma_2(p_5 \times p_n) \ge 2n + \left\lceil \frac{n}{7} \right\rceil$. We will study the cases:

1) $n \equiv 0 \pmod 7$. We have $\gamma_2(p_5 \times p_n) = \sum_{j=1}^{n} s_j$. So, we consider the following:

a) $s_1 + s_2 = 4$ then $s_1 + s_2 + s_3 = 8$ and by Lemma 3.8,

$$\gamma_2(p_5 \times p_n) = \sum_{j=1}^{n} s_j = s_1 + s_2 + s_3 + \sum_{j=4}^{n-4} s_j + \sum_{j=n-3}^{n} s_j \ge 8 + 2(n-2) + \frac{n-7}{7} + 9,$$

$$\gamma_2(p_5 \times p_n) \ge 17 + 2n - 14 + \frac{n-7}{7} = 2n + \frac{n+14}{7} = 2n + \left\lceil \frac{n}{7} \right\rceil + 2 \ge 2n + \left\lceil \frac{n}{7} \right\rceil + 1.$$

b) $s_1 + s_2 \ge 5$ if $s_1 + s_2 \ge 6$ then

$$\gamma_2(p_5 \times p_n) = \sum_{j=1}^{n} s_j = s_1 + s_2 + \sum_{j=3}^{n-5} s_j + \sum_{j=n-4}^{n} s_j \ge 6 + 2(n-7) + \frac{n-7}{7} + 10$$

$$= 2n + \frac{n-7+14}{7} = 2n + \left\lceil \frac{n}{7} \right\rceil + 1.$$

Let $s_1 + s_2 = 5$ then by Lemma 3.8, $\sum_{j=1}^{5} s_j \ge 11$ or $\sum_{j=1}^{6} s_j \ge 14$. If $\sum_{j=1}^{5} s_j \ge 11$ then

$$\gamma_2(p_5 \times p_n) = \sum_{j=1}^{n} s_j = \sum_{j=1}^{5} s_j + \sum_{j=6}^{n-2} s_j + s_{n-1} + s_n \ge 11 + 2(n-7) + \frac{n-7}{7} + 5$$

$$= 2n + \frac{n}{7} + 1 = 2n + \left\lceil \frac{n}{7} \right\rceil + 1.$$

{where the case $s_{n-1} + s_n = 4$ is the same as $s_1 + s_2 = 4$}. If $\sum_{j=1}^{5} s_j < 11$ then by Lemma 3.8, we have $\sum_{j=1}^{6} s_j \ge 14$

$$\gamma_2(p_5 \times p_n) = \sum_{j=1}^{n} s_j = s_1 + s_2 + \sum_{j=3}^{n-5} s_j + \sum_{j=n-4}^{n} s_j \ge 6 + 2(n-7) + \frac{n-7}{7} + 10$$

$$= 2n + \frac{n-7+14}{7} = 2n + \left\lceil \frac{n}{7} \right\rceil + 1.$$

And with Theorem 3.4, we get $\gamma_2(p_5 \times p_n) = 2n + \left\lceil \frac{n}{7} \right\rceil + 1 : n \equiv 0 \pmod 7$.

2) When $n \equiv 4 \pmod 7$ we have two cases:

a) $s_1 + s_2 = 4$. Thus implies that $s_1 + s_2 + s_3 = 8$ then

$$\gamma_2(p_5 \times p_n) = \sum_{j=1}^{n} s_j = \sum_{j=1}^{3} s_j + \sum_{j=4}^{n-1} s_j + s_n \ge 8 + \frac{15(n-4)}{7} + 2$$

$$= 2n + \frac{n+10}{7} = 2n + \left\lceil \frac{n}{7} \right\rceil + 1.$$

b) $s_1 + s_2 \ge 5$ {where $s_{n-1} + s_n \ge 5$} then

$$\gamma_2(p_5 \times p_n) = \sum_{j=1}^{n} s_j = s_1 + s_2 + \sum_{j=3}^{n-2} s_j + s_{n-1} + s_n \ge 5 + 2(n-4) + \frac{n-4}{7} + 5$$

$$= 2n + \frac{n+10}{7} = 2n + \left\lceil \frac{n}{7} \right\rceil + 1.$$

Then by Theorem 3.4, we get $\gamma_2(p_5 \times p_n) = 2n + \left\lceil \frac{n}{7} \right\rceil + 1 : n \equiv 4 \pmod 7$.

3) $n \equiv 6 \pmod 7$. We have two cases:

a) If $s_1 + s_2 = 4$ then $s_1 + s_2 + s_3 = 8$. Thus implies that

$$\gamma_2(p_5 \times p_n) = \sum_{j=1}^{n} s_j = s_1 + s_2 + s_3 + \sum_{j=4}^{n-3} s_j + s_{n-2} + s_{n-1} + s_n \ge 8 + 2(n-6) + \frac{n-6}{7} + 6$$
$$= 2n + \left\lceil \frac{n}{7} \right\rceil + 1.$$

b) If $s_1 + s_2 \ge 5$ then $s_{n-1} + s_n \ge 5$. Thus implies that

$$\gamma_2(p_5 \times p_n) = \sum_{j=1}^{n} s_j = \sum_{j=1}^{4} s_j + \sum_{j=5}^{n-2} s_j + s_{n-1} + s_n \ge 9 + 2(n-6) + \frac{n-6}{7} + 5$$
$$= 2n + \frac{n+8}{7} = 2n + \left\lceil \frac{n}{7} \right\rceil + 1.$$

By Theorem 3.4, we get $\gamma_2(p_5 \times p_n) = 2n + \left\lceil \frac{n}{7} \right\rceil + 1 : n \equiv 6 \pmod 7$. Finally, we get

$$\gamma_2(p_5 \times p_n) = \begin{cases} 2n + \left\lceil \frac{n}{7} \right\rceil : n \equiv 1,2,3,5 \pmod 7, \\ 2n + \left\lceil \frac{n}{7} \right\rceil + 1 : n \equiv 0,4,6 \pmod 7. \end{cases}$$

This completes the proof. □

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