

A Note on a Natural Correspondence of a Determinant and Pfaffian

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Abstract

A familiar and natural decomposition of square matrices leads to the construction of a Pfaffian with the same value as the determinant of the square matrix.

Keywords

Determinant, Pfaffian

A square matrix is the sum of its symmetric and skew symmetric parts, S and C respectively:

$$M = S + C .$$

For a 2 by 2 matrix it happens that the determinant is $|M| = |S| + |C|$.

But this is not true in general. However, in block matrix form, of any size, we do have

$$\begin{vmatrix} C & S \\ -S & -C \end{vmatrix} = |M|^2 .$$

This follows from the properties of determinants according to the sequence of equalities:

$$\begin{vmatrix} C & S \\ -S & -C \end{vmatrix} = \begin{vmatrix} S+C & S \\ -S-C & -C \end{vmatrix} = \begin{vmatrix} S+C & S \\ 0 & S-C \end{vmatrix} = |M|^2$$

Since $|S-C| = |(S-C)^T| = |S^T - C^T| = |S+C| = |M|$.

Next, define $B = SJ$. The matrix J has zero entries except for all ones on the secondary diagonal (NE to SW). The matrix B is just S with the order of columns reversed.

Form the skew matrix in block form $\begin{pmatrix} C & B \\ -B^T & {}^T C \end{pmatrix}$.

We have indicated the transpose across the secondary diagonal by ${}^T C$. (In general, ${}^T A = JA^T J$)

Then, we claim that the Pfaffian formed by the triangular array above the main diagonal in this matrix has the same value as the determinant of the original matrix M .

For example, $\begin{pmatrix} 3 & -3 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix}$ and the determinant is 33.

We calculate the equivalent Pfaffian as

$$\begin{vmatrix} -5 & 2 & 3 \\ & 4 & 2 \\ & & -5 \end{vmatrix} = 25 - 4 + 12 = 33$$

using the “cofactor” expansion of a Pfaffian. For the cofactor expansion see [1].

First, we show that $\begin{vmatrix} C & B \\ -B^T & {}^T C \end{vmatrix} = |M|^2$.

The calculation follows.

$$\begin{vmatrix} C & B \\ -B^T & {}^T C \end{vmatrix} = \begin{vmatrix} C & SJ \\ -JS & -JCJ \end{vmatrix} = \begin{vmatrix} I & 0 \\ 0 & J \end{vmatrix} \begin{vmatrix} C & S \\ -S & -C \end{vmatrix} \begin{vmatrix} I & 0 \\ 0 & J \end{vmatrix} = \begin{vmatrix} C & S \\ -S & -C \end{vmatrix}.$$

The last equation follows by taking the determinant of the three factors. And as mentioned above,

$$\begin{vmatrix} C & S \\ -S & -C \end{vmatrix} = |M|^2.$$

So the construction of the Pfaffian array delivers the value of the original determinant, up to sign.

It is easy enough to verify by calculation that the correct sign is given in the case of a two by two matrix rewritten as a Pfaffian array. In general the expansion of the determinant will contain a term which is a product of the diagonal elements of the matrix regardless of the other entries. In the same way, the expansion of the Pfaffian will contain this same term, independent of all other entries. So the sign of the Pfaffian is correct.

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References

- [1] Ishikawa, M. and Wakayama, M. (2006) Applications of the Minor Summation Formula III: Plucker Relations, Lattice Paths and Pfaffians. *Journal of Combinatorial Theory, Series A*, **113**, 113-155.

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