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Some Edge Product Cordial Graphs in the Context of Duplication of Some Graph Elements

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Abstract

For a graph $G=(V(G),E(G))$, a function $f:E(G)\to\{0,1\}$ is called an edge product cordial labeling of *G*, if the induced vertex labeling function is defined by the product of the labels of the incident edges as such that the number of edges with label 1 and the number of edges with label 0 differ by at most 1 and the number of vertices with label 1 and the number of vertices with label 0 differ by at most 1. In this paper, we show that the graphs obtained by duplication of a vertex, duplication of a vertex by an edge or duplication of an edge by a vertex in a crown graph are edge product cordial. Moreover, we show that the graph obtained by duplication of each of the vertices of degree three by an edge in a gear graph is edge product cordial. We also show that the graph obtained by duplication of each of the pendent vertices by a new vertex in a helm graph is edge product cordial.

Keywords

Graph Labeling, Edge Product Cordial Labeling, Duplication of a Vertex

1. Introduction

We begin with a simple, finite, undirected graph $G=(V(G),E(G))$ where $V(G)$ and $E(G)$ denote the vertex set and the edge set respectively. For all other terminology, we follow Gross [1]. We will provide a brief summary of definitions and other information which are necessary for the present investigations.

Definition 1. *A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition. If the domain of the mapping is the set of vertices, edges or both then the labeling is called a vertex labeling, an edge labeling or a total labeling.*

Definition 2. *For a graph G, an edge labeling function is defined as* $f: E(G) \to \{0,1\}$ *and the induced vertex labeling function* $f^*: V(G) \to \{0,1\}$ *is given by* $f^*(v) = f(e_1) f(e_2) \cdots f(e_k)$ *if* $e_1, e_2, \cdots, e_k$ *are the edges incident with the vertex v.*

We denote the number of vertices of G having label i under f^* *by* $v_f(i)$ *and the number of edges of G having label i under f by* $e_f(i)$ *for* $i = 0,1$.

The function f is called an edge product cordial labeling of G if $|v_f(0) - v_f(1)| \le 1$ *and* $|e_f(0) - e_f(1)| \le 1$. *A graph G is called edge product cordial if it admits edge product cordial labeling.*

The concept of edge product cordial labeling was introduce by Vaidya and Barasara [2] *in which they proved that* C_n *for n odd, trees of order greater than* 2, *unicyclic graphs of odd order, crowns, armed crowns, helms, closed helms, webs, flowers graph are edge product cordial. They also proved that wheel and gear for even are not edge product cordial. They also* [3] *proved that* T_n, DT_n *for odd,* Q_n *for odd,* DQ_n *for odd are edge product cordial labeling. They also proved that* DT_n *for even,* Q_n *for even,* DQ_n *for even,* DF_n *are not edge product cordial labeling.*

Definition 3. *The graph* $W_n = C_n + K_1$ *is called wheel graph, the vertex corresponding to* K_1 *is called apex vertex and vertices corresponding to* C_n *are called rim vertices.*

Definition 4. *The helm* H_n *is the graph obtained from a wheel* W_n *by attaching a pendent edge at each vertex of the n-cycle.*

Definition 5. *A gear graph is obtained from the wheel graph* W_n *by adding a vertex between every pair of adjacent vertices of the n-cycle.*

Definition 6. *The crown* $C_n \odot K_1$ *is obtained by joining a single pendent edge to each vertex of* C_n.

Definition 7. *The neighborhood of a vertex v of a graph is the set of all vertices adjacent to v. It is denoted by* $N(v)$.

Definition 8. *Duplication of a vertex of the graph G is the graph* G' *obtained from G by adding a new vertex* v' *to G such that* $N(v') = N(v)$.

Definition 9. *Duplication of a vertex* v_k *by a new edge* $e = v_k' v_k''$ *in a graph G produces a new graph* G' *such that* $N(v_k') = \{v_k, v_k''\}$ *and* $N(v_k'') = \{v_k, v_k'\}$.

The concept of duplication of vertex by edge was introduce by Vaidya and Barasara [4].

Definition 10. *Duplication of an edge* $e = uv$ *by a new vertex w in a graph G produces a new graph* G' *such that* $N(w) = \{u, v\}$.

The concept of duplication of edge by vertex was introduce by Vaidya and Dani [5].

2. Main Results

Theorem 1. *The graph obtained by duplication of an arbitrary vertex of the cycle in a crown graph is an edge product cordial graph.*

Proof. Let C_n be a cycle with consecutive vertices $v_1, v_2, \cdots, v_n$ and edges $e_i = v_i v_{i+1}$ for each $i \in \{1,2,3,\cdots,n\}$. Let u_i be a new vertex adjacent to v_i with $e_i' = u_i v_i$ for each $i \in \{1,2,3,\cdots,n\}$. Resulting graph is a crown graph G_1.

Let G be the graph obtained by duplication of the vertex v_n by a new vertex v' of G_1 such that $e_1'' = v'v_1$, $e_2'' = v'v_{n-1}$ and $e_3'' = v'u_n$.

Thus $|V(G)| = 2n+1$ and $|E(G)| = 2n+3$.

Now for $n=3$ and $n=4$ **Figure 1** shows that the graphs are edge product cordial as follows:

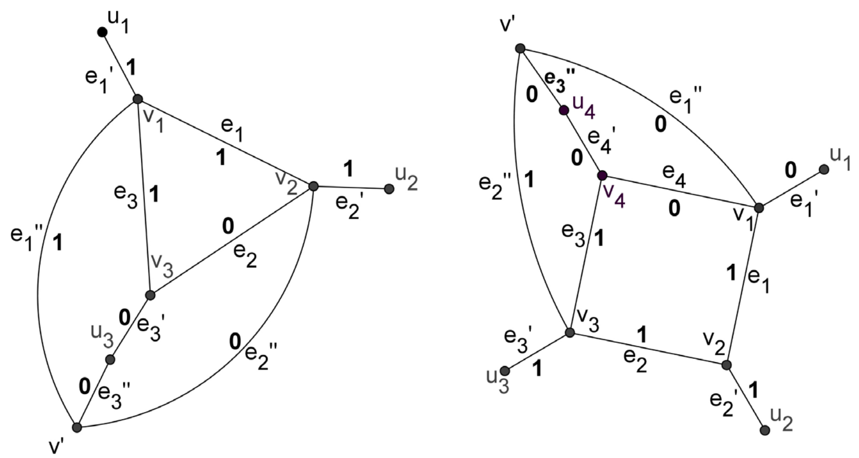

Figure 1. $n=3$ and $n=4$.

Case 1: When n is odd. For $n \ge 5$, define $f: E(G) \to \{0,1\}$ as follows:

$$f(e) = \begin{cases} 1, \text{ if } e = e_i, \text{ for } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil; \\ 0, \text{ if } e = e_i, \text{ for } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n; \\ 1, \text{ if } e = e_i', \text{ for } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil; \\ 0, \text{ if } e = e_i', \text{ for } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n; \\ 0, \text{ if } e = e_i'', \text{ for } i = 1,2,3. \end{cases}$$

In the view of above labeling pattern we have,

$$v_f(1) = v_f(0) - 1 = n \text{ and } e_f(1) = e_f(0) - 1 = n+1.$$

Case 2: When n is even. For $n \ge 6$, define $f: E(G) \to \{0,1\}$ as follows:

$$f(e) = \begin{cases} 1, \text{ if } e = e_i, \text{ for } 1 \le i \le \frac{n}{2} + 1; \\ 0, \text{ if } e = e_i, \text{ for } \frac{n}{2} + 2 \le i \le n; \\ 1, \text{ if } e = e_i', \text{ for } 1 \le i \le \frac{n}{2} + 1; \\ 0, \text{ if } e = e_i', \text{ for } \frac{n}{2} + 2 \le i \le n; \\ 0, \text{ if } e = e_i'', \text{ for } i = 1,2,3. \end{cases}$$

In the view of the above labeling pattern we have,

$$v_f(0) = v_f(1) - 1 = n \text{ and } e_f(0) = e_f(1) - 1 = n + 1.$$

Thus, from both the cases we have $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.

Hence, graph G admits edge product cordial labeling. Thus, G is an edge product cordial graph. □

Illustration 1. *The graph obtained by duplication of an arbitrary vertex of the cycle C_n in a crown graph is an edge product cordial graph as shown in* Figure 2 *as follows*:

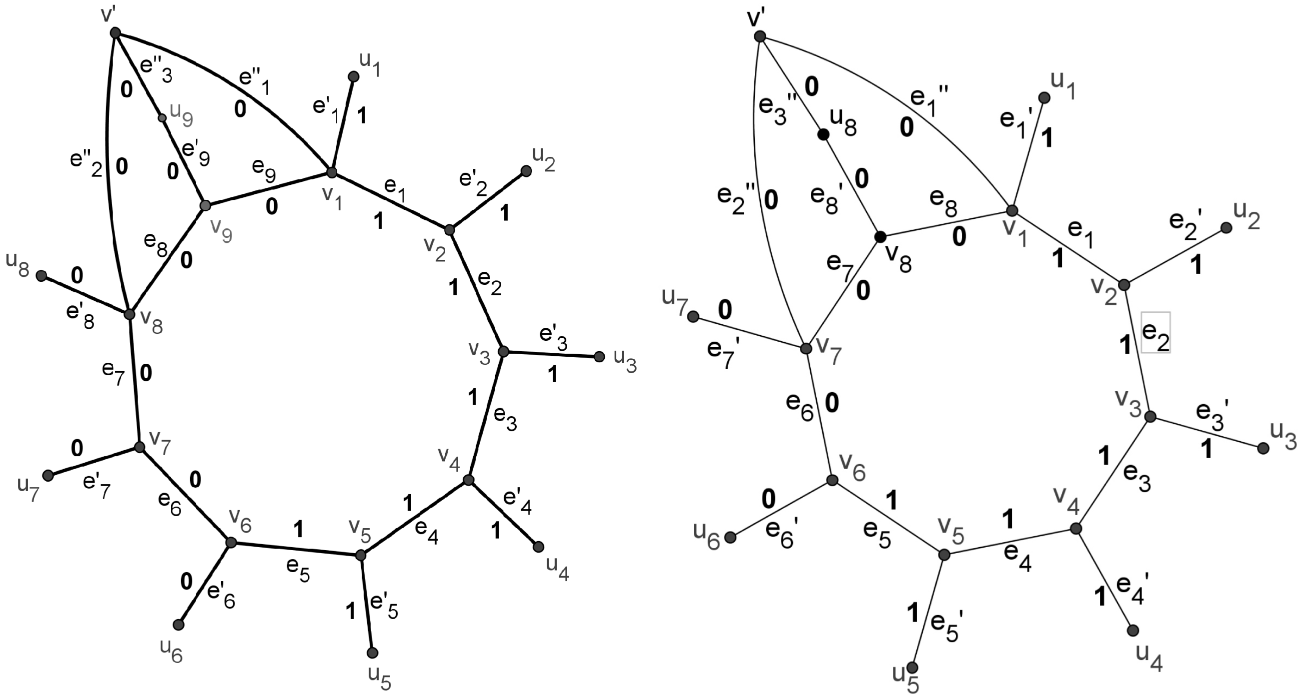

Figure 2. $n = 9$ and $n = 8$.

Theorem 2. *The graph obtained by duplication of an arbitrary vertex of the cycle by a new edge in a crown graph is edge product cordial graph.*

Proof. Let C_n be a cycle with consecutive vertices $v_1, v_2, \cdots, v_n$ and edges $e_i = v_i v_{i+1}$ for each $i \in \{1, 2, \cdots, n\}$. Let u_i be a new vertex adjacent to v_i with $e_i = u_i v_i$ for each $i \in \{1, 2, \cdots, n\}$. Resulting graph is a crown graph G_1.

Let G be the graph obtained by duplication of the vertex v_n by an edge $e_1'' = v'v''$ of G_1 such that $e_2'' = v'v_n$ and $e_3'' = v''v_n$. Thus $|V(G)| = 2n + 2$ and $|E(G)| = 2n + 3$. Define $f : E(G) \to \{0, 1\}$ as follows:

$$f(e) = \begin{cases} 1, \text{ if } e = e_i, \text{ for } i = 1, 2; \\ 0, \text{ if } e = e_i, \text{ for } 3 \le i \le n; \\ 1, \text{ if } e = e_i', \text{ for } i \in \{1, 2, \cdots, n\}; \\ 0, \text{ if } e = e_i'', \text{ for } i = 1, 2, 3. \end{cases}$$

In the view of the above labeling pattern we have, $v_f(0) = v_f(1) = n + 1$ and

$e_f(0) = e_f(1) - 1 = n + 1$. Thus, we have $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.

Hence, graph G admits edge product cordial labeling. Thus, G is an edge product cordial graph. □

Illustration 2. *The graph obtained by duplication of an arbitrary vertex of the cycle* C_n *by a new edge in a crown graph is edge product cordial graph as shown in* Figure 3.

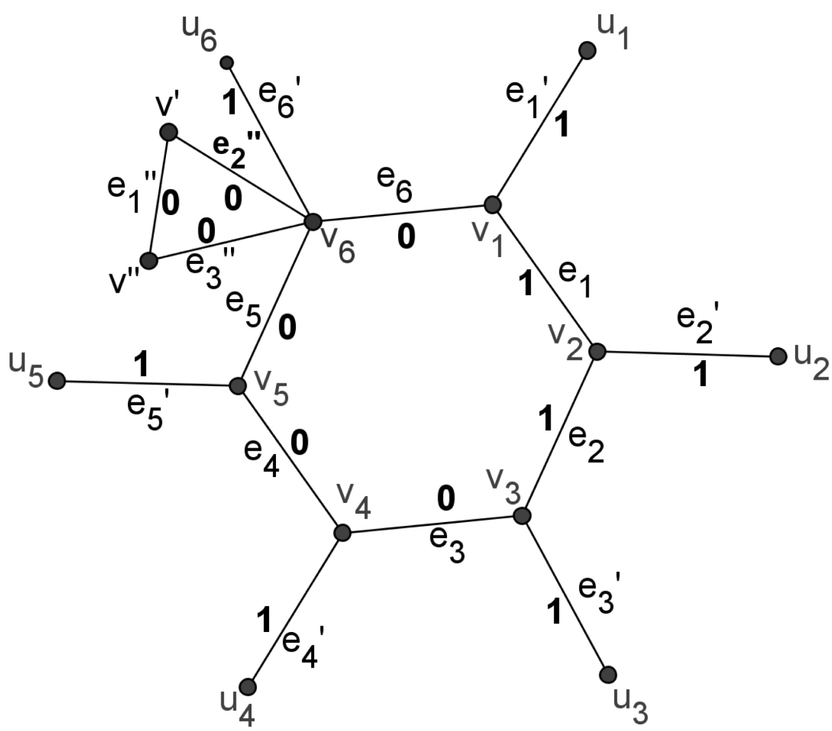

Figure 3. $n = 6$.

Theorem 3. *The graph obtained by duplication of an arbitrary edge of the cycle* C_n *by a new vertex in a crown graph is edge product cordial.*

Proof. Let C_n be a cycle with the consecutive vertices $v_1, v_2, \cdots, v_n$ and edges $e_i = v_i v_{i+1}$ for each $i \in \{1, 2, \cdots, n\}$. Let u_i be a new vertex adjacent to v_i with $e_i' = u_i v_i$ for each $i \in \{1, 2, \cdots, n\}$. Resulting graph is a crown graph G_1.

Let G be the graph obtained by duplication of an edge $e_1 = v_1 v_2$ by a vertex v' in G_1 such that $e_1'' = v' v_1$ and $e_2'' = v' v_2$. Thus $|V(G)| = 2n + 1$ and $|E(G)| = 2n + 2$. Define $f : E(G) \to \{0,1\}$ as follows:

$$f(e) = \begin{cases} 1, \text{ if } e = e_i, \text{ for } i = 1; \\ 0, \text{ if } e = e_i, \text{ for } 2 \le i \le n; \\ 1, \text{ if } e = e_i', \text{ for } i \in \{1, 2, \cdots, n\}; \\ 0, \text{ if } e = e_i'', \text{ for } i = 1, 2. \end{cases}$$

In the view of above labeling pattern we have, $v_f(0) - 1 = v_f(1) = n$ and $e_f(0) = e_f(1) = n + 1$. Thus, we have $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.

Hence, graph G admits edge product cordial labeling. Thus, G is an edge product cordial graph. □

Illustration 3. *The graph obtained by duplication of an arbitrary edge of the cycle* C_n *by a new vertex in a crown graph is edge product cordial graph as shown in* Figure 4.

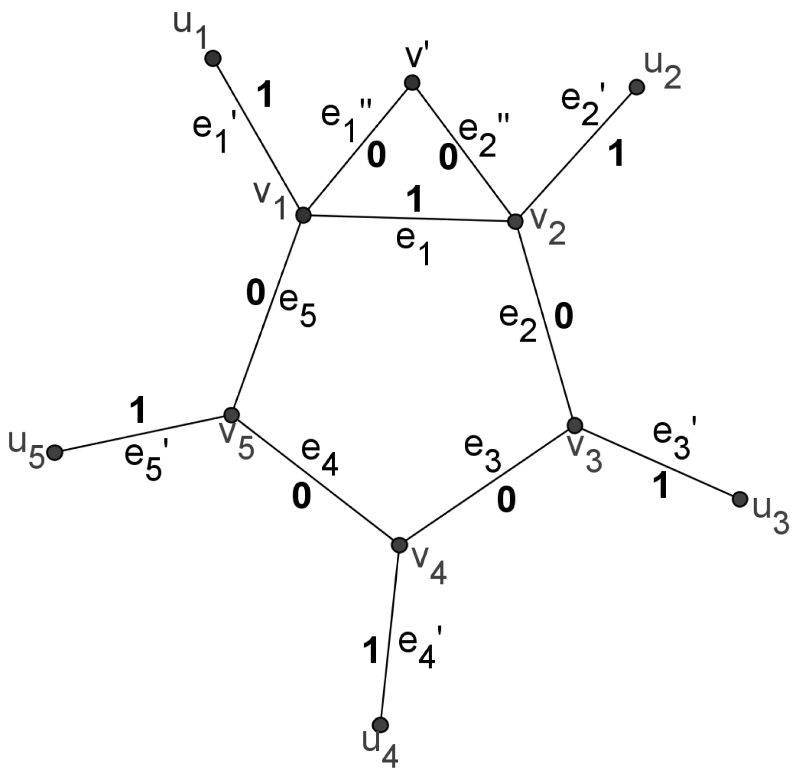

Figure 4. $n=5$.

Theorem 4. *The graph obtained by duplication of each pendent vertex by a new vertex in a crown graph is edge product cordial graph.*

Proof. Let C_n be a cycle with consecutive vertices $v_1, v_2, \cdots, v_n$ and edges $e_i = v_i v_{i+1}$ for each $i \in \{1,2,\cdots,n\}$. Let u_i be a new vertex adjacent to v_i with $e_i' = u_i v_i$ for each $i \in \{1,2,\cdots,n\}$. Resulting graph is a crown graph G_1.

Let G be the graph obtained by duplication of each pendent vertex u_i by a new vertex v_i' of G_1 such that $e_i'' = v_i' v_i$ for $i \in \{1,2,\cdots,n\}$. Thus $|V(G)| = 3n$ and $|E(G)| = 3n$.

Case 1: When n is odd, define $f : E(G) \to \{0,1\}$ as follows:

$$f(e) = \begin{cases} 0, \text{ if } e = e_i, \text{ for } i \in \{1,2,\cdots,n\}; \\ 1, \text{ if } e = e_i', \text{ for } i \in \{1,2,\cdots,n\}; \\ 1, \text{ if } e = e_i'', 1 \le i \le \left\lceil \frac{n}{2} \right\rceil; \\ 0, \text{ if } e = e_i'', \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n. \end{cases}$$

In the view of above labeling pattern we have,

$$v_f(0)=v_f(1)-1=\left\lfloor\frac{3n}{2}\right\rfloor \text{ and } e_f(0)=e_f(1)-1=\left\lfloor\frac{3n}{2}\right\rfloor.$$

Case 2: When n is even, define $f:E(G)\to\{0,1\}$ as follows:

$$f(e)=\begin{cases}0, \text{ if } e=e_i, \text{ for } i\in\{1,2,\cdots,n\};\\ 1, \text{ if } e=e_i', \text{ for } i\in\{1,2,\cdots,n\};\\ 1, \text{ if } e=e_i'', 1\le i\le\frac{n}{2};\\ 0, \text{ if } e=e_i'', \frac{n}{2}+1\le i\le n.\end{cases}$$

In the view of above labeling pattern we have, $v_f(0)=v_f(1)=\frac{3n}{2}$ and $e_f(0)=e_f(1)=\frac{3n}{2}$.

Thus, from both the cases we have $|v_f(0)-v_f(1)|\le 1$ and $|e_f(0)-e_f(1)|\le 1$. Hence, graph G admits edge product cordial labeling. Thus, G is an edge product cordial graph.

□

Illustration 4. *The graph obtained by duplication of each pendent vertex by a new vertex in a crown graph is edge product cordial graph as shown in Figure 5 as follows:*

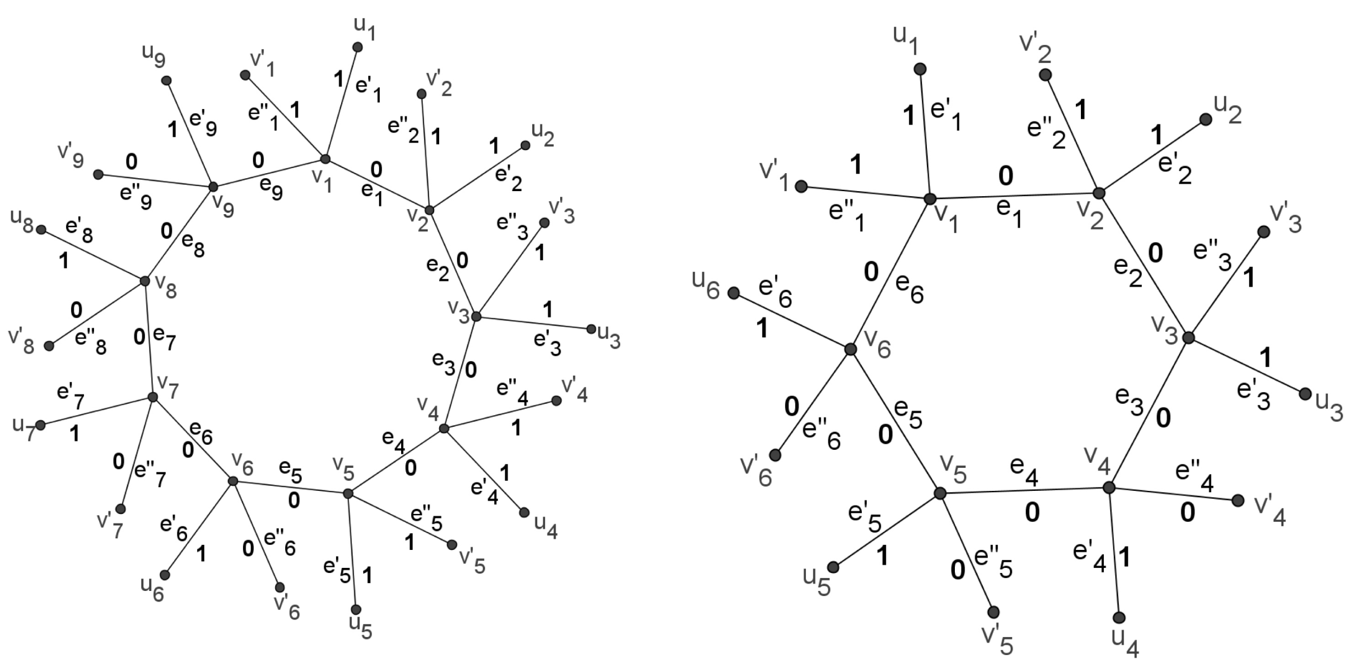

Figure 5. $n=9$ and $n=6$.

Theorem 5. *The graph obtained by duplication of each of the vertices of degree three by an edge in a gear graph is an edge product cordial graph.*

Proof. Let W_n be the wheel graph with apex vertex v and consecutive rim vertices $v_1,v_2,\cdots,v_n$. To obtained the gear graph G_n subdivide each of the rim edges

$v_1v_2, v_2v_3, \cdots, v_{n-1}v_n, v_nv_1$ of the wheel graph by the vertices $u_1, u_2, \cdots, u_n$ respectively such that $e_i = vv_i$, $e_{l_i} = v_iu_i$ and $e_{r_i} = u_iv_{i+1}$ for $i \in \{1, 2, \cdots, n\}$.

Let G be the graph obtained from G_n by duplication of each vertex v_i by an edge $f_i = v_i'v_i''$ such that $f_i' = v_i'v_i$ and $f_i'' = v_i''v_i$ for $i \in \{1, 2, \cdots, n\}$. Thus $|V(G)| = 4n+1$ and $|E(G)| = 6n$.

Define $f : E(G) \to \{0,1\}$ as follows:

$$f(e) = \begin{cases} 0, \text{ if } e = e_i, \text{ for } i = \{1,2,3,\cdots,n\}; \\ 0, \text{ if } e = e_{l_i}, \text{ for } i = \{1,2,3,\cdots,n\}; \\ 0, \text{ if } e = e_{r_i}, \text{ for } i \in \{1,2,3,\cdots,n\}; \\ 1, \text{ if } e = f_i, \text{ for } i = \{1,2,3,\cdots,n\}; \\ 1, \text{ if } e = f_i', \text{ for } i = \{1,2,3,\cdots,n\}; \\ 1, \text{ if } e = f_i'', \text{ for } i = \{1,2,3,\cdots,n\}. \end{cases}$$

In the view of the above labeling pattern we have, $v_f(0) - 1 = v_f(1) = 2n$ and $e_f(0) = e_f(1) = 3n$. Thus, we have $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. Hence, graph G admits edge product cordial labeling. Thus, G is edge product cordial graph.

□

Illustration 5. *The graph obtained by duplication of each vertex of degree three by an edge in a gear graph is an edge product cordial graph as shown in* Figure 6.

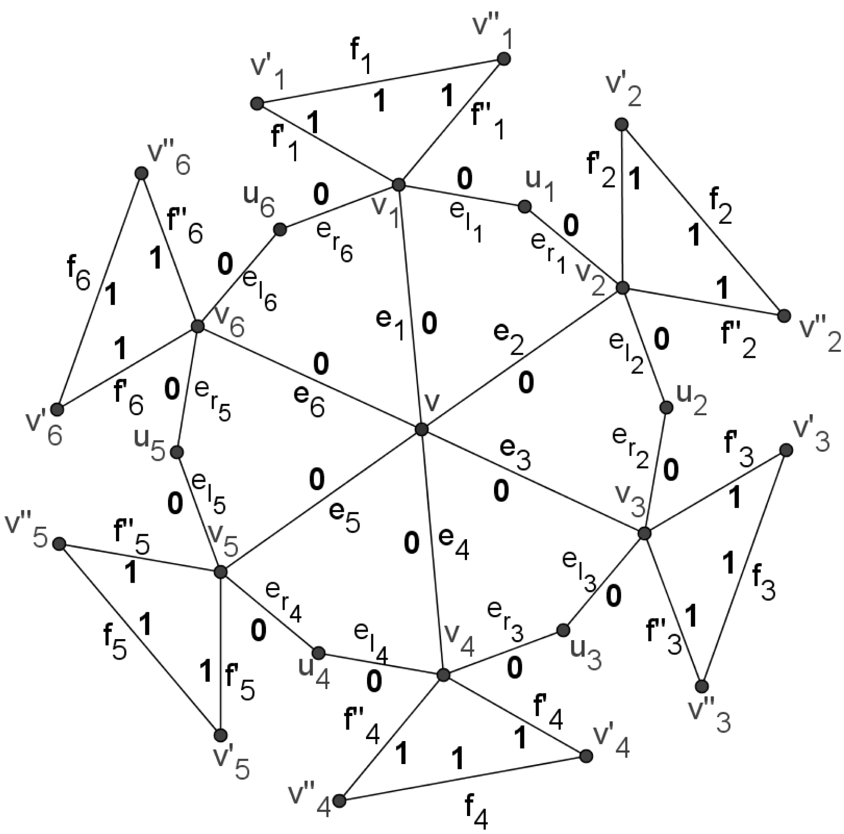

Figure 6. $n = 6$.

Theorem 6. *The graph obtained by duplication of each of the pendent vertices by a new vertex in a helm graph is edge product cordial graph.*

Proof. Let v be the apex vertex and $v_1, v_2, \cdots, v_n$ be the consecutive rim vertices of the wheel W_n with edges $e_i = v_i v_{i+1}$ and $e_i' = v v_i$ for $i \in \{1, 2, \cdots, n\}$. Let u_i be a new vertex adjacent to v_i with edges $e_i'' = u_i v_i$ for $i \in \{1, 2, \cdots, n\}$. Resulting graph is helm graph H_n.

Let G be the graph obtained from H_n by duplication of each pendent vertex u_i by a new vertex v_i' such that $f_i = v_i' v_i$ for $i \in \{1, 2, \cdots, n\}$. Thus $|V(G)| = 3n+1$ and $|E(G)| = 4n$.

Case 1: When n is odd, define $f : E(G) \to \{0,1\}$ as follows:

$$f(e) = \begin{cases} 1, \text{ if } e = e_i, \ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor; \\ 0, \text{ if } e = e_i, \ \left\lfloor \frac{n}{2} \right\rfloor + 1 \le i \le n; \\ 0, \text{ if } e = e_i', \text{ for } i \in \{1, 2, \cdots, n\}; \\ 1, \text{ if } e = e_i'', \ 1 \le i \le n-1; \\ 0, \text{ if } e = e_i'', \text{ for } i = n; \\ 1, \text{ if } e = f_i, \ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 2; \\ 0, \text{ if } e = f_i, \ \left\lfloor \frac{n}{2} \right\rfloor + 3 \le i \le n. \end{cases}$$

In the view of above labeling pattern we have, $v_f(0) = v_f(1) = \left\lceil \frac{3n}{2} \right\rceil$ and $e_f(0) = e_f(1) = 2n$.

Case 2: When n is even, define $f : E(G) \to \{0,1\}$ as follows:

$$f(e) = \begin{cases} 1, \text{ if } e = e_i, \ 1 \le i \le \frac{n}{2} - 1; \\ 0, \text{ if } e = e_i, \ \frac{n}{2} \le i \le n; \\ 0, \text{ if } e = e_i', \text{ for } i \in \{1, 2, \cdots, n\}; \\ 1, \text{ if } e = e_i'', \ 1 \le i \le \frac{n}{2} + 1; \\ 0, \text{ if } e = e_i'', \ \frac{n}{2} + 2 \le i \le n; \\ 1, \text{ if } e = f_i, \text{ for } i \in \{1, 2, \cdots, n\}. \end{cases}$$

In the view of above labeling pattern we have, $v_f(0) = v_f(1) - 1 = \frac{3n}{2}$ and $e_f(0) = e_f(1) = 2n$.

Thus, from both the cases we have $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. Hence, graph G admits edge product cordial labeling. Thus, G is an edge product cordial graph.

□

Illustration 6. *The graph obtained by duplication of each pendent vertex by a new vertex in a helm graph is edge product cordial graph as shown in* Figure 7.

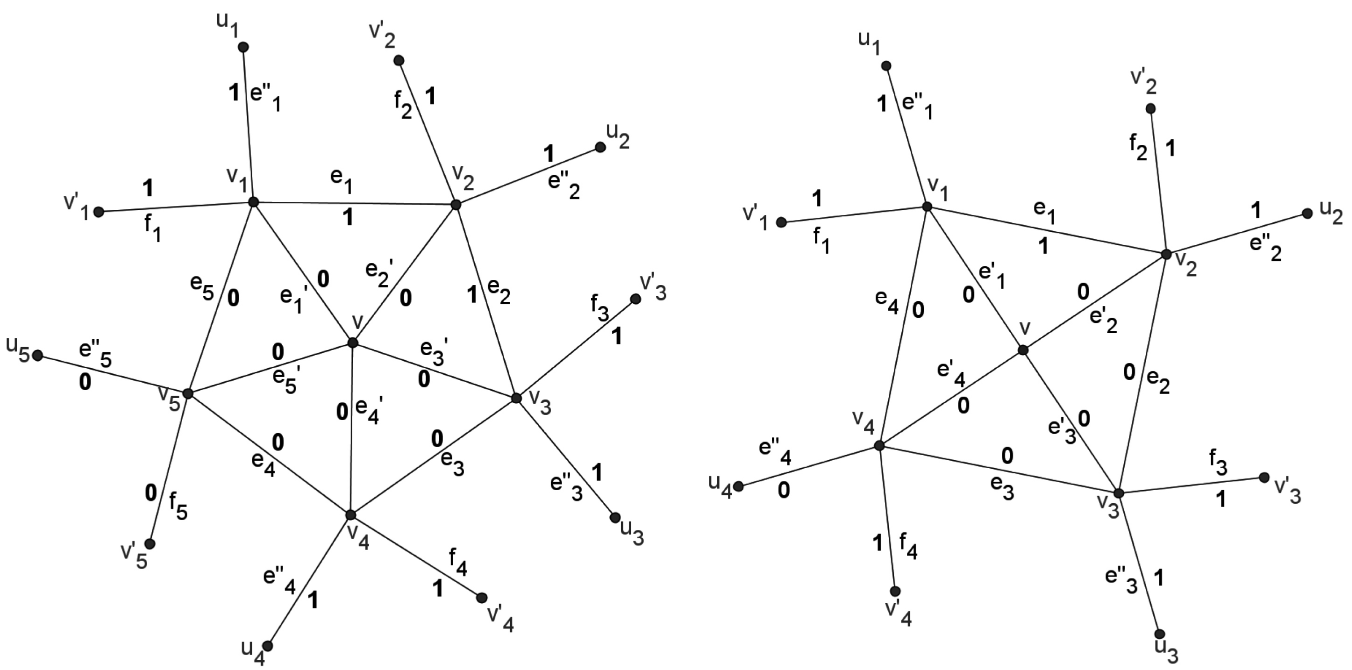

Figure 7. $n = 5$ and $n = 4$.

3. Conclusion

We have derived six results for edge product cordial related to crown graph, gear graph and helm graph in the context of duplication of various graph elements. Similar problem can be discussed for other graph family for edge product cordial labeling.

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