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The Rupture Degree of Graphs with *k*-Tree

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Abstract

A *k*-tree of a connected graph *G* is a spanning tree with maximum degree at most *k*. The rupture degree for a connected graph *G* is defined by $r(G)=\max\{\omega(G-X)-|X|-m(G-X): X\subset V(G), \omega(G-X)>1\}$, where $m(G-X)$ and $\omega(G-X)$, respectively, denote the order of the largest component and number of components in $G-X$. In this paper, we show that for a connected graph *G*, if $r(G)\le(k-3)|X|-m(G-X)+2$ for any cut-set $X\subset V(G)$, then *G* has a *k*-tree.

Keywords

The Rupture Degree, *k*-Tree, Induced Graph

1. Introduction

Throughout this paper, We consider only finite undirected graphs without loops and multiple edges. A graph $G=(V,E)$ always means a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Let Δ denote the maximum degree of *G*, and $G[S]$ denote the subgraph of *G* induced by a subset *S* of $V(G)$. We by $d_G(v)$ denote the degree of a vertex *v* in a graph *G* and $N_G(v)$ the neighbor vertex set of *v*. Further for a nonempty subset *S* of $V(G)$, we put $d_G(S)=\sum_{u\in S}d_G(u)$ and $N_i(S)=\{v\in V(G)\,|\,|N_G(v)\cap S|=i\}$.

A *k*-tree of a connected graph *G* is a spanning tree with maximum degree *k*. Clearly, if $k=2$, it reduces to that of a Hamiltonian path in *G*. Since every tree with maximum degree Δ has a Δ-tree, thus here we doesn't consider trees.

A nonempty set *S* of independent vertices of *G* is called a frame of *G*, if $G-S'$ is connected for any $S'\subseteq S$. If $|S|=k$ then *S* is called a *k*-frame.

In [1] and [2], Win, Aung and Kyaw gave the ore-type condition and Fan-type condition for *k* tree as fellows (Theorem A and B).

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Theorem A. If $d_G(S) \geq n-1$ for every independent set *S* of *k* vertices of graph *G*, then *G* has a *k* tree.

Theorem B. Let $V^{\star} = \left\{ v \in V(G) \mid d_G(v) < \frac{n-1}{k} \right\}$ and suppose, either $V^{\star} = \varnothing$ or $G[V^{\star}]$ is a complete graph, then *G* has a *k* tree.

Further, Kyaw in [3] gave a stronger result for *k* tree as theorem C.

Theorem C. Let *G* be a connected graph and $k(\geq 2)$ an integer. If $d_G(S) + \sum_{i=2}^{k+1}(k-i)N_i(S) \geq n-1$ for every $k+1$-frame *S* in *G*, then *G* has a *k* tree.

The rupture degree of a graph *G* is introduced in [4], which is an important parameter for measuring the structure characteristics of the connected graph *G* and defined as

$$r(G) = \max\{\omega(G-X) - |X| - m(G-X) : X \subset V(G), \omega(G-X) > 1\}$$

where $m(G-X)$ and $\omega(G-X)$, respectively, denote the order of the largest component and the number of components in $G-X$.

In this paper, we consider the rupture degree and existence of *k*-tree in a connected graph *G* and thus give a new sufficient condition for a graph to have *k* tree.

Any undefined terms can be found in the standard references on graph theory, including Bondy and Murty [5].

2. Main Result

Let *G* be a connected graph and *k* an integrity with $2 \leq k \leq \Delta$. Now, we by proving the following theorem to discuss the relationship between the rupture degree and existence of *k*-tree in graph *G*.

Theorem 1. *Let G be a connected graph but not a tree. If* $r(G) \leq (k-3)|X| - m(G-X) + 2$ *for any cut-set* $X \subset V(G)$, *then G has a k-tree.*

Let *H* be an induced subgraph of *G* and with maximal order among all subgraphs containing *k*-tree, and let *A* be a set adjacent to some vertices in *G* but not in $V(H)$. Clearly, if $A = \varnothing$, then *G* has *k*-tree. Now, we suppose that $r(G) \leq (k-3)|X| - m(G-X) + 2$ for any cut-set $X \subset V(G)$ and *A* is nonempty for connected graph *G*. We by finding a contradiction to prove the above theorem. Firstly, we prove the following claims.

Claim 1. *Let T be a k-tree of H. Then* $d_T(x) = k$ *for* $x \in A$.

Proof. Let *T* be a *k*-tree of *H*, which has maximal order among all the induced subgraphs of *G* having a *k*-tree. On the contrary, suppose that if there exist some vertex $x \in A$ such that $d_T(x) < k$, then *H* could be expanded by joining *xy* for a neighbor *y* of *x* which is not in *H*. This is contradictive to the maximality of *H*. Thus $d_T(x) = k$ for any $x \in A$.

Let *T* be a *k*-tree of *H* and $x \in A$. Since $d_T(x) = k$, we suppose that $C_1, C_2, \cdots, C_k$ are all components of $T - x$.

Claim 2. *If there exist an edge* $u_m u_n \in E(H)$ *for* $u_m \in C_m, u_n \in C_n$ $(1 \leq m \neq n \leq k)$, *then* $d_T(u_m) = k$ *and* $d_T(u_n) = k$.

Proof. Suppose that $d_T(u_m) < k$ (or $d_T(u_n) < k$) for $u_m \in C_m$ (or $u_n \in C_n$). Since $u_m u_n \in E(H)$, we may obtain a new *k*-tree T^* from *T* of *H* by deleting one of the edges joining *x* to components C_n (or C_m) from *T* and adding the edge $u_m u_n$ in *T*. Clearly, we obtain another *k*-tree of *H* and in the latter *k*-tree, *x* has degree less than *k*. Then *H* could be expanded and this is contradictive to the maximality of *H*. So the conclusion holds.

Claim 3. *Let T be a k-tree of H and M is subset of* $V(T)$ *with degree k. Then M is non-empty and satisfies the following property*: *Let* $\mathbb{C}_i$, $1 \leq i \leq r$, *be the components of* $T - M$. *If for some i and j with* $i \neq j$, *the vertex* x_i *of* $\mathbb{C}_i$ *is adjacent in H to the vertex* x_j *of* $\mathbb{C}_j$, *then* x_i *and* x_j *has degree k and* $x_i, x_j \in N \subset V(T)$.

Proof. Since *A* is nonempty and for every $x \in A$ with $d_T(x) = k$, *M* is non-empty. By Claim 2, the property holds with $r = k$ because the way in which we have picked the vertices u_n or u_m which go into *N*. Clearly, $M \cup N$ be a subset of the set of vertices of *T* of degree *k* in *T*. At the same time, we find that a vertex *v* of *T* adjacent to a vertex in *G* but not in $V(H)$ is in $M \cup N$.

Let *T* be a *k*-tree of *H* with as many *k* degree vertices as possible and M, N be subsets of $V(T)$ with degree k as above. Suppose that $\mathbb{C}_i$ is one component of $T-M$. If $N\cap V(\mathbb{C}_i)$ is nonempty, we select $y\in N\cap V(\mathbb{C}_i)$ and suppose that $\mathbb{C}_{ij}$, $1\le i\le s$, be the components of $T-M-\{y\}$.

Claim 4. *For m and n, with* $1\le m\ne n\le s$, *if there exists a vertex* y_m *of* $\mathbb{C}_{im}$ *adjacent to a vertex* y_n *of* $\mathbb{C}_{in}$ *in H. Then* $d_T(y_m)=k$ *and* $d_T(y_n)=k$.

Proof. Suppose that $d_T(y_m)<k$ (or $d_T(y_n)<k$) for $y_m\in\mathbb{C}_{im}$ (or $y_n\in\mathbb{C}_{in}$). Since $y_my_n\in E(H)$, we may obtain a new *k*-tree T^{**} from *T* of *H* by deleting one of the edges joining $y\in N\cap V(\mathbb{C}_i)$ to components $\mathbb{C}_{in}$ (or $\mathbb{C}_{im}$) from *T* and adding the edge $y_my_n\in E(H)$ in *T*. Clearly, *y* has degree less than *k* in *k*-tree T^{**} of *H*. Then combine $y\in N\cap V(\mathbb{C}_i)$ we know this is contradictive to Claim 3. The conclusion holds.

By taking T,M,N as the claims and let $|M\cup N|$ be maximal, then we obtain the follows claim as a straightforward consequence.

Claim 5. *Let T be a k-tree of H with maximal number of k degree vertices. Then, there is no edge of H joining any components of* $T-M\cup N$.

Proof. Given our choice of *T*, *M* and *N* as above, we derive a contradiction by assuming that there is an edge *yz* of *H* with endpoints *y* and *z* joining two components of $T-M\cup N$. If the path in *T* joining *y* and *z* contains a vertex of *M*, then by claim 3, either *y* or *z* is in *N* which is absurd. Then this path contains no vertex of *M* and *y* and *z* are therefore in the same component $\mathbb{C}_i$ of $T-M$ for some *i* with $1\le i\le r$. Now let *w* be a vertex of *N* on the path in *T* joining *y* and *z* and let $\mathbb{C}_{ij}$, $1\le j\le s$, be the components of $\mathbb{C}_i-\{w\}$. Now let N_0 be the set of all vertices of $\mathbb{C}_i-\{w\}$ having property of claim 4. Further, let $M^*=M\cup\{w\}$ and $N^*=N\cup N_0-\{w\}$. Since N_0 contains *y* or *z*, claim 2 holds with M^* and N^* replacing *M* and *N*, respectively. Moreover, $|M^*\cup N^*|$ is greater than $|M\cup N|$. But this contradicts to our choice of T,M and *N*. This shows that there is no edge in *H* joining any components of $T-M\cup N$.

Now we are ready for the proof of theorem 2.1.

The proof of theorem 2.1. Since *A* is nonempty and thus *H* is an induced proper subgraph of *G*, we have $\omega(G-M\cup N)\ge\omega(H-M\cup N)+1$. By claim 5 we know that $\omega(H-M\cup N)=\omega(T-M\cup N)$. At the same time, we know that $\omega(T-M\cup N)$ reaches minimum when $T[M\cup N]$ is itself a tree. Thus we have

$$\begin{aligned}\omega(G-M\cup N)&\ge\omega(H-M\cup N)+1=\omega(T-M\cup N)+1\\&\ge k|M\cup N|-2(|M\cup N|-1)+1=(k-2)|M\cup N|+3\end{aligned}$$

On the other hand, since $r(G)\le(k-3)|X|-m(G-X)+2$ for any cut-set $X\subset V(G)$ we have

$$\omega(G-X)-|X|-m(G-X)\le(k-3)|X|-m(G-X)+2$$

and,

$$\omega(G-X)\le(k-2)|X|+2$$

This is a contradiction. Therefore *A* is empty and the proof is completed.

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