Open Journal of Discrete Mathematics, 2016, 6, 99-104
Published Online April 2016 in SciRes. http://www.scirp.org/journal/ojdm
http://dx.doi.org/10.4236/ojdm.2016.62010

Double Derangement Permutations

Pooya Daneshmand[1], Kamyar Mirzavaziri[2], Madjid Mirzavaziri[3]

[1]Ferdowsi University of Mashhad, International Campus, Mashhad, Iran
[2]National Organization for Development of Exceptional Talents (NODET) I, Mashhad, Iran
[3]Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
Email: pooyadaneshmand@yahoo.com, mirzavaziri@gmail.com, mirzavaziri@um.ac.ir

Received 28 September 2015; accepted 9 April 2016; published 12 April 2016

Abstract

Let n be a positive integer. A permutation a of the symmetric group S_n of permutations of $[n]=\{1,2,\cdots,n\}$ is called a *derangement* if $a(i)\neq i$ for each $i\in[n]$. Suppose that x and y are two arbitrary permutations of S_n. We say that a permutation a is a *double derangement with respect to x and y* if $a(i)\neq x(i)$ and $a(i)\neq y(i)$ for each $i\in[n]$. In this paper, we give an explicit formula for $D_n(x,y)$, the number of double derangements with respect to x and y. Let $0\le k\le n$ and let $\{i_1,\cdots,i_k\}$ and $\{a_1,\cdots,a_k\}$ be two subsets of $[n]$ with $i_j\neq a_j$ and $\ell=\left|\{i_1,\cdots,i_k\}\cap\{a_1,\cdots,a_k\}\right|$. Suppose that $\Delta(n,k,\ell)$ denotes the number of derangements x such that $x(i_j)=a_j$. As the main result, we show that if $0\le m\le n$ and z is a permutation such that $z(i)\neq i$ for $i\le m$ and $z(i)=i$ for $i>m$, then $D_n(e,z)=\sum_{k=0}^{m}\sum_{1\le i_1<\cdots<i_k\le m}(-1)^k\Delta(n,k,\ell(i_1,\cdots,i_k))$, where $\ell(i_1,\cdots,i_k)=\left|\{i_1,\cdots,i_k\}\cap\{z(i_1),\cdots,z(i_k)\}\right|$.

Keywords

Symmetric Group of Permutations, Derangement, Double Derangement

1. Introduction

Let n be a positive integer. A *derangement* is a permutation of the symmetric group S_n of permutations of $[n]=\{1,2,\cdots,n\}$ such that none of the elements appear in their original position. The number of derangements of S_n is denoted by D_n or n¡. A simple recursive argument shows that

How to cite this paper: Daneshmand, P., Mirzavaziri, K. and Mirzavaziri, M. (2016) Double Derangement Permutations. *Open Journal of Discrete Mathematics*, **6**, 99-104. http://dx.doi.org/10.4236/ojdm.2016.62010

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

The number of derangements also satisfies the relation $D_n = nD_{n-1} + (-1)^n$. It can be proved by the inclusion-exclusion principle that D_n is explicitly determined by $n!\sum_{i=0}^{n} \frac{(-1)^i}{i!}$. This implies that $\lim_{n\to\infty} \frac{D_n}{n!} = \frac{1}{e}$. These facts and some other results concerning derangements can be found in [1]. There are also some generalizations of this notion. The *problème des rencontres* asks how many permutations of the set $[n]$ have exactly *k* fixed points. The number of such permutations is denoted by $D_{n,k}$ and is given by $D_{n,k} = \binom{n}{k} D_{n-k}$. Thus, we can say that $\lim_{n\to\infty} \frac{D_{n,k}}{n!} = \frac{1}{k!e}$. Some probabilistic aspects of this concept and the related notions concerning the permutations of S_n is discussed in [2] and [3].

Let *e* be the identity element of the symmetric group S_n, which is defined by $e(i) = i$ for each $i \in [n]$. We can say that a permutation *a* of $[n]$ is a derangement if $a(i) \neq e(i)$ for each $i \in [n]$. We denote this by $a \perp e$. Thus, D_n is the number of *a* with $a \perp e$. If *c* is any fixed element of S_n then the number of $a \in S_n$ with $a \perp x$ is also D_n, since $a \perp x$ if and only if $ax^{-1} \perp e$. In the present paper, we extend the concept of a derangement to a *double derangement* with respect to two fixed elements *x* and *y* of S_n.

2. The Result

In the following, we assume that *n* is a positive integer and the identity permutation of the symmetric group S_n of permutations of $[n]$ is denoted by *e*. Moreover, for two permutations *a* and *b* of S_n, the notation $a \perp b$ means that $a(i) \neq b(i)$ for each $i \in [n]$. We also denote the number of elements of a set *A* by $|A|$.

Definition 1. *Suppose that x and y are two arbitrary permutations of* S_n. *We say that a permutation a is a double derangement with respect to x and y if* $a \perp x$ *and* $a \perp y$. *The number of double derangements with respect to x and y is denoted by* $D_n(x, y)$.

Proposition 1. *Let* $0 \leq k \leq n$ *and let* $\{i_1, \cdots, i_k\}$ *and* $\{a_1, \cdots, a_k\}$ *be two subsets of* $[n]$ *with* $i_j \neq a_j$ *and* $\ell = \left|\{i_1, \cdots, i_k\} \cap \{a_1, \cdots, a_k\}\right|$. *Then* $\Delta(n, k, \ell)$, *the number of derangements x such that* $x(i_j) = a_j$, *is determined by*

$$\Delta(n,k,\ell) = \begin{cases} \sum_{i=0}^{k-\ell-1} \binom{k-\ell-1}{i} \frac{D_{(n+1)-(k+i)}}{n-(k+i)} & \text{if } k \neq \ell \text{ and } 2k-\ell \leq n \\ D_{n-k} & \text{if } k = \ell \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $a_r \in \{i_1, \cdots, i_k\} \cap \{a_1, \cdots, a_k\}$. Thus $a_r = i_s$ for some $s \neq r$. Now there are two cases:

Case 1. $a_s \in \{i_1, \cdots, i_k\}$. Let $a_s = i_t$. In this case a derangement *x* satisfies the condition $x(i_j) = a_j$ if and only if the derangement x' of the set $[n] \setminus \{i_t\}$ satisfies the condition $x'(i_j) = a'_j$ for all $j \neq t$, where $a'_j = a_j$ for $j \neq s$ and $a'_s = a_t$. This provides a one to one correspondence between the derangements *x* of $[n]$ with $x(i_j) = a_j$ for the given sets $\{i_1, \cdots, i_k\}$ and $\{a_1, \cdots, a_k\}$ with ℓ elements in their intersections, and the derangements x' of $[n] \setminus \{i_t\}$ with $x_{i_j} = a'_j$ for the given sets $\{i_1, \cdots, i_k\} \setminus \{i_t\}$ and $\{a'_1, \cdots, a'_k\} \setminus \{a'_t\}$ with $\ell - 1$ elements in their intersections.

Case 2. $a_s \notin \{i_1, \cdots, i_k\}$. In this case a derangement *x* satisfies the condition $x(i_j) = a_j$ if and only if the derangement x' of the set $[n] \setminus \{a_s\}$ satisfies the condition $x'(i_j) = a_j$ for all $j \neq s$. This provides a one to one correspondence between the derangements *x* of $[n]$ with $x(i_j) = a_j$ for the given sets $\{i_1, \cdots, i_k\}$ and $\{a_1, \cdots, a_k\}$ with ℓ elements in their intersections, and the derangements x' of $[n] \setminus \{a_s\}$ with $x'(i_j) = a_j$ for the given sets $\{i_1, \cdots, i_k\} \setminus \{i_s\}$ and $\{a_1, \cdots, a_k\} \setminus \{a_s\}$ with $\ell - 1$ elements in their intersections.

These considerations show that $\Delta(n, k, \ell) = \Delta(n-1, k-1, \ell-1)$. Iterating this argument, we have

$$\Delta(n,k,\ell) = \Delta(n-1,k-1,\ell-1) = \Delta(n-2,k-2,\ell-2) = \cdots = \Delta(n-\ell,k-\ell,0).$$

We can therefore assume that $\ell = 0$. We thus evaluate $\Delta(n, k, 0)$, where $2k \leq n$. For $k = 0$, we obviously have $\Delta(n, 0, 0) = D_n$. For $k \geq 1$, we claim that

$$\Delta(n,k,0)=\Delta(n-1,k-1,0)+\Delta(n-2,k-1,0).$$

For a derangement *x* satisfying $x(i_j)=a_j$ there are two cases: $x(a_1)=i_1$ or $x(a_1)\neq i_1$.

If the first case occurs then we have to evaluate the number of derangements of the set $[n]\setminus\{i_1,a_1\}$ for the given sets $\{i_2,\cdots,i_k\}$ and $\{a_2,\cdots,a_k\}$ with 0 elements in their intersections. The number is equal to $\Delta(n-2,k-1,0)$.

If the second case occurs then we have to evaluate the number of derangements of the set $[n]\setminus\{a_1\}$ for the given sets $\{i_2,\cdots,i_k\}$ and $\{a_2,\cdots,a_k\}$ with 0 elements in their intersections. The number is equal to $\Delta(n-1,k-1,0)$.

We now use induction on *k* to show that

$$\Delta(n,k,0)=\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)},\quad 2\le 2k\le n.$$

For $k=1$ we have

$$\Delta(n,1,0)=\Delta(n-1,0,0)+\Delta(n-2,0,0)=D_{n-1}+D_{n-2}=\frac{D_n}{n-1}.$$

Now let the result be true for $k-1$. We can write

$$\begin{aligned}\Delta(n,k,0)&=\Delta(n-1,k-1,0)+\Delta(n-2,k-1,0)\\&=\sum_{i=0}^{k-2}\binom{k-2}{i}\frac{D_{n-(k-1+i)}}{(n-1)-(k-1+i)}+\sum_{i=0}^{k-2}\binom{k-2}{i}\frac{D_{(n-1)-(k-1+i)}}{(n-2)-(k-1+i)}\\&=\sum_{i=0}^{k-2}\binom{k-2}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}+\sum_{i=1}^{k-1}\binom{k-2}{i-1}\frac{D_{n-(k+i-1)}}{(n-1)-(k+i-1)}\\&=\frac{D_{(n+1)-k}}{n-k}+\sum_{i=1}^{k-2}\binom{k-2}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}+\frac{D_{(n+1)-(2k-1)}}{n-(2k-1)}+\sum_{i=1}^{k-2}\binom{k-2}{i-1}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}\\&=\frac{D_{(n+1)-k}}{n-k}+\sum_{i=1}^{k-2}\left[\binom{k-2}{i}+\binom{k-2}{i-1}\right]\frac{D_{(n+1)-(k+i)}}{n-(k+i)}+\frac{D_{(n+1)-(2k-1)}}{n-(2k-1)}\\&=\frac{D_{(n+1)-k}}{n-k}+\sum_{i=1}^{k-2}\binom{k-2}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}+\frac{D_{(n+1)-(2k-1)}}{n-(2k-1)}\\&=\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}.\end{aligned}$$

Corollary 1. *Let k be a positive integer. Then*

$$\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{D_{k+1-i}}{k-i}=k!.$$

Proof. Let $n=2k$, $i_j=j$ and $a_j=k+j$ for $j=1,\cdots,k$. Then a derangement *x* satisfies the condition $x(i_j)=a_j$ if and only if x' defined by $x'(i)=x(k+i)$ for $i\in[k]$ is a permutation of $[k]$. The number of such permutations x' is $k!$.

The following **Table 1** gives some small values of $\Delta(n,k,0)$.

The following lemma can be easily proved.

Lemma 1. *Let x and y be two arbitrary permutations and $m\ge 0$ be the number of i's for which $x(i)\neq y(i)$. Then there is a permutation z such that $z(i)\neq i$ for $i\le m$ and $z(i)=i$ for $i>m$ and $D_n(x,y)=D_n(e,z)$.*

Theorem 2. *Let $0\le m\le n$ and let z be a permutation such that $z(i)\neq i$ for $i\le m$ and $z(i)=i$ for $i>m$. Then*

$$D_n(e,z)=\sum_{k=0}^{m}\sum_{1\le i_1<\cdots<i_k\le m}(-1)^k\,\Delta(n,k,\ell(i_1,\cdots,i_k)),$$

Table 1. Values of $\Delta(n,k,0)$ for $1 \le n \le 10$ and $1 \le 2k \le n$.

$n\backslash k$	1	2	3	4	5
1	0	0	0	0	0
2	1	0	0	0	0
3	1	0	0	0	0
4	3	2	0	0	0
5	11	4	0	0	0
6	53	14	6	0	0
7	309	64	18	0	0
8	2119	362	78	24	0
9	16,687	2428	426	96	0
10	148,329	18,806	2790	504	120

where $\ell(i_1,\cdots,i_k) = \left|\{i_1,\cdots,i_k\} \cap \{z(i_1),\cdots,z(i_k)\}\right|$.

Proof. Let E_i be the set of all derangements *x* for which $x(i) = z(i)$, where $1 \le i \le m$. Then $D_n(e,z) = D_n - \left|\bigcup_{i=1}^m E_i\right|$. We use the inclusion-exclusion principle to determine $\left|\bigcup_{i=1}^m E_i\right|$. For each $0 \le k \le m$ and $1 \le i_1 < \cdots < i_k \le m$ we have

$$\left|E_{i_1} \cap \cdots \cap E_{i_k}\right| = \Delta\left(n,k,\ell(i_1,\cdots,i_k)\right),$$

where $\ell(i_1,\cdots,i_k) = \left|\{i_1,\cdots,i_k\} \cap \{z(i_1),\cdots,z(i_k)\}\right|$. This implies the result.

Our ultimate goal is to find an explicit formula for evaluating $D_n(e,c)$ for an arbitrary cycle *c*. Prior to that we need to state two elementary enumerative problems concerning subsets *A* of the set $[n]$ with *k* elements and exactly ℓ consecutive members.

Lemma 2. *Let* $S(n,k,\ell)$ *be the number of subsets* $A = \{a_1,\cdots,a_k\}$ *of* $[n]$ *for which the equation* $r = s+1$ *has exactly* ℓ *solutions for r and s in A. If* $0 \le \ell < k \le n$ *then*

$$S(n,k,\ell) = \binom{n-k+1}{k-\ell}\binom{k-1}{\ell}.$$

Moreover, $S(n,0,0) = 1$ and $S(n,k,\ell) = 0$ for other values of n,k,ℓ.

Proof. We can restate the problem as follows: We want to put *k* ones and $n-k$ zeros in a row in such a way that there are exactly ℓ appearance of one-one. To do this we put $n-k$ zeros and choose $k-\ell$ places of the $n-k+1$ possible places for putting $k-\ell$ blocks of ones in $\binom{n-k+1}{k-\ell}$ ways. Let the number of ones in the *i*-th block be $r_i \ge 1$. We then must have $r_1 + \cdots + r_{k-\ell} = k$. The number of solutions for the latter equation is $\binom{k-1}{\ell}$.

Now suppose that we write $1,2,\cdots,n$ around a circle. We thus assume that 1 is after *n* and so $n,1$ is also assumed to be consecutive. Under this assumption we have the following result.

Lemma 3. *Let* $C(n,k,\ell)$ *be the number of subsets* $A = \{a_1,\cdots,a_k\}$ *of* $[n]$ *for which the equation* $r \equiv s+1$ *(mod n) has exactly* ℓ *solutions for r and s in A. If* $0 \le \ell < k < n$ *then*

$$C(n,k,\ell) = \frac{n}{k}\cdot\binom{n-k-1}{k-\ell-1}\binom{k}{\ell}.$$

Moreover, $C(n,0,0) = C(n,n,n) = 1$ and $C(n,k,\ell) = 0$ for other values of n,k,ℓ.

Proof. Similar to the above argument, we want to put k ones and $n-k$ zeros around a circle in such a way that there are exactly ℓ appearances of one-one. At first, we put them in a row. There are four cases:

Case 1. There is no block of ones before the first zero and after the last zero. In this case we put $n-k$ zeros and choose $k-\ell$ places of the $n-k-1$ possible places for putting $k-\ell$ blocks of ones in $\binom{n-k-1}{k-\ell}$ ways. Let the number of ones in the *i*-th block be $r_i \geq 1$. We then must have $r_1+\cdots+r_{k-\ell}=k$. The number of solutions for the latter equation is $\binom{k-1}{\ell}$.

Case 2. There is no block of ones before the first zero but there is a block after the last zero. In this case we put $n-k$ zeros and choose $k-\ell-1$ places of the $n-k-1$ possible places for putting $k-\ell-1$ blocks of ones in $\binom{n-k-1}{k-\ell-1}$ ways. Let the number of ones in the *i*-th block be $r_i \geq 1$. We then must have

$r_1+\cdots+r_{k-\ell}=k$. The number of solutions for the latter equation is $\binom{k-1}{\ell}$.

Case 3. There is a block of ones before the first zero but there is no block after the last zero. This is similar to the above case.

Case 4. There is a block of ones before the first zero and a block of ones after the last zero. In this case we must have $\ell-1$ appearance of one-one in the row format, since we want to achieve ℓ appearance of one-one in the circular format. Thus we put $n-k$ zeros and choose $k-(\ell-1)-2$ places of the $n-k-1$ possible places for putting $k-(\ell-1)-2$ blocks of ones in $\binom{n-k-1}{k-\ell-1}$ ways. Let the number of ones in the *i*-th block be $r_i \geq 1$. We then must have $r_1+\cdots+r_{k-(\ell-1)}=k$. The number of solutions for the latter equation is $\binom{k-1}{\ell-1}$.

These considerations prove that

$$C(n,k,\ell)=\binom{n-k-1}{k-\ell}\binom{k-1}{\ell}+2\binom{n-k-1}{k-\ell-1}\binom{k-1}{\ell}+\binom{n-k-1}{k-\ell-1}\binom{k-1}{\ell-1}.$$

A straightforward computation gives the result.

The following **Table 2** gives some small values of $C(10,k,\ell)$.

Table 2. Values of $C(10,k,\ell)$ for $1 \leq k \leq 10$ and $1 \leq \ell \leq k$.

$k\backslash\ell$	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	10	0	0	0	0	0	0	0	0	0	0
2	35	10	0	0	0	0	0	0	0	0	0
3	50	60	10	0	0	0	0	0	0	0	0
4	25	100	75	10	0	0	0	0	0	0	0
5	2	40	120	80	10	0	0	0	0	0	0
6	0	0	25	100	75	10	0	0	0	0	0
7	0	0	0	0	50	60	10	0	0	0	0
8	0	0	0	0	0	0	35	10	0	0	0
9	0	0	0	0	0	0	0	0	10	0	0
10	0	0	0	0	0	0	0	0	0	0	1

Theorem 3. *Let c be be a cycle of length* $m \le n$. *Then*

$$D_n(e,c) = \sum_{0 \le \ell \le 2k-\ell \le m} (-1)^k C(m,k,\ell)\Delta(n,k,\ell).$$

Proof. Let c_m be the cycle defined by $c_m(j) = j+1$ for $1 \le j \le m-1$, $c_m(m) = 1$ and $c_m(i) = i$ for $m+1 \le i \le n$. Then $D_n(e,c) = D_n(e,c_m)$.

Using the notations of Theorem 2, $\ell(i_1,\cdots,i_k) = \ell$ if and only if the subset $A = \{i_1,\cdots,i_k\}$ of $[m]$ has exactly ℓ solutions for the equation $r \equiv s+1 \pmod n$ for r,s in *A*. Thus the number of $\{i_1,\cdots,i_k\}$ with the property $\ell(i_1,\cdots,i_k) = \ell$ is $C(m,k,\ell)$. Applying Theorem 2, we have the result.

Example 1. *We evaluate* $D_5(e,c_5)$ *and* $D_5(e,c_3)$. *Applying Theorem* 3 *with* $m = 5$ *we have*

$$\begin{aligned} D_5(e,c_5) &= C(5,0,0)\Delta(5,0,0) - C(5,1,0)\Delta(5,1,0) + C(5,2,0)\Delta(5,2,0) \\ &\quad + C(5,2,1)\Delta(5,2,1) - C(5,3,1)\Delta(5,3,1) - C(5,3,2)\Delta(5,3,2) \\ &\quad + C(5,4,3)\Delta(5,4,3) - C(5,5,5)\Delta(5,5,5) \\ &= C(5,0,0)\Delta(5,0,0) - C(5,1,0)\Delta(5,1,0) + C(5,2,0)\Delta(5,2,0) \\ &\quad + C(5,2,1)\Delta(4,1,0) - C(5,3,1)\Delta(4,2,0) - C(5,3,2)\Delta(3,1,0) \\ &\quad + C(5,4,3)\Delta(2,1,0) - C(5,5,5)\Delta(0,0,0) \\ &= 1\times 44 - 5\times 11 + 5\times 4 + 5\times 3 - 5\times 2 - 5\times 1 + 5\times 1 - 1\times 1 = 13, \end{aligned}$$

and $(x(1),x(2),x(3),x(4),x(5))$ for the 13 double derangements *x* with respect to *e* and c_5 are

$$\begin{aligned} &(3,1,5,2,4),(3,4,5,1,2),(3,5,1,2,4),(3,5,2,1,4),(4,1,5,2,3), \\ &(4,1,5,3,2),(4,5,1,2,3),(4,5,1,3,2),(4,5,2,1,3),(5,1,2,3,4), \\ &(5,4,1,2,3),(5,4,1,3,2),(5,4,2,1,3). \end{aligned}$$

Applying Theorem 3 with $m = 3$ we have

$$\begin{aligned} D_5(e,c_3) &= C(3,0,0)\Delta(5,0,0) - C(3,1,0)\Delta(5,1,0) \\ &\quad + C(3,2,1)\Delta(5,2,1) - C(3,3,3)\Delta(5,3,3) \\ &= 1\times 44 - 3\times 11 + 3\times 3 - 1\times 1 = 19, \end{aligned}$$

and $(x(1),x(2),x(3),x(4),x(5))$ for the 19 double derangements with respect to *e* and c_3 are

$$\begin{aligned} &(3,4,5,1,2),(3,5,4,1,2),(3,4,5,2,1),(3,5,4,2,1),(4,5,2,1,3), \\ &(5,4,2,1,3),(4,5,2,3,1),(5,4,2,3,1),(4,1,5,2,3),(5,1,4,2,3), \\ &(4,1,5,3,2),(5,1,4,3,2),(3,5,2,1,4),(3,4,2,5,1),(3,1,5,2,4), \\ &(3,1,4,5,2),(5,1,2,3,4),(4,1,2,5,3),(3,1,2,5,4). \end{aligned}$$

The above example shows that how can we evaluate $D_n(e,c)$ for a cycle *c*. Moreover, Theorem 2 gives a formula for evaluating $D_n(e,z)$ for any permutation *z*. Applying Lemma 1, we can compute $D_n(x,y)$ for any permutations *x* and *y*.

References

[1] Graham, R.L., Knuth, D.E. and Patashnik, O. (1988) Concrete Mathematics. Addison-Wesley, Reading.

[2] Pitman, J. (1997) Some Probabilistic Aspects of Set Partitions. *American Mathematical Monthly*, **104**, 201-209. http://dx.doi.org/10.2307/2974785

[3] Knopfmacher, A., Mansour, T. and Wagner, S. (2010) Records in Set Partitions. *The Electronic Journal of Combinatorics*, **17**, R109.