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Further Results on Acyclic Chromatic Number*

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ABSTRACT

An acyclic coloring of a graph is a proper vertex coloring such that the union of any two color classes induces a disjoint collection of trees.The purpose of this paper is to derive exact values of acyclic chromatic number of some graphs.

Keywords: Acyclic Coloring; Acyclic Chromatic Number; Central Graph; Middle Graph; Total Graph

1. Introduction

Graph coloring is a branch of graph theory which deals with such partitioning problems. For example, suppose that we have world map and we would like to color the countries so that if two countries share a boundary line, then they need to get different colors. We can translate the map to graph by letting countries be represented by vertices and two vertices are made adjacent if and only if the corresponding countries share a boundary line. Then the problem of map coloring is equivalent to vertex coloring of the corresponding graph. Hence the original map coloring now reduces to vertex coloring of the associated graph.

Coloring of a graph is an assignment of colors to the elements like vertices or edges or faces (regions) of a graph. It is said to be a proper coloring, if no two adjacent elements are assigned the same color. The most common types of graph colorings are vertex coloring, edge coloring and face coloring.

A vertex coloring of a graph G is an assignment of colors to its vertices so that no two vertices have the same color. The chromatic number $\chi(G)$ of a graph G is the minimum number of colors needed to label the vertices, so that adjacent vertices receive different colors.

A proper vertex coloring of a graph is acyclic if every cycle uses at least three colors [1]. The acyclic chromatic number of G, denoted by $a(G)$, is the minimum colors required for its acyclic coloring.

2. Acyclic Coloring of Central Graph of C_n

2.1. Central Graph [2]

Let G be a finite undirected graph with no loops and multiple edges. The central graph of a graph G, $C(G)$ is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G.

2.2. Structural Properties of Central Graphs

Let $G=(p,q)$ be any undirected simple graph, then by the definition of $C(G)$ of a graph.

- The number of vertices in the central graph of G is $p(C(G))=p+q$.
- For any (p,q) graph there exists exactly p vertices of degree $p-1$ and q vertices of degree 2 in $C(G)$.
- The central graph of two isomorphic graphs is also isomorphic.
- The maximum degree in $C(G)$ is $\Delta=p-1$.
- Central graph of any graph is connected.
- If G is any graph with odd p then $C(G)$ is Eulerian.

2.3. Theorem

The acyclic coloring of central graph of cycle, $a(C(C_n))=n-2$, for $n>4$.

Proof

Consider the graph C_n with vertex set $V=\{v_i/1\le i\le n\}$. Let $G=C(C_n)$ be the central graph of C_n, which is obtained by sub dividing each edge of C_n exactly once and joining non adjacent vertices of C_n. Let the newly introduced vertices be $v_{h,k}$ h, $k=1,2,3,\cdots,n$ with $h<k$. Consider the color class $C=\{c_i/1\le i\le n-2\}$. Now assign a proper coloring to the vertices as follows. The coloring is in such a way that the sub graph induced by any two color is a forest containing at most the path P_4. The vertices v_i are as-

*Middle, total graphs of cycle.

signed the cololur c_i for $i = 1, 2$; c_{i-1} for $i = 3, 4$; c_{i-2} for $5 \le i \le n$.

Case 1: When $n > 5$.

The newly $v_{2,3}, v_{4,5}$ are assigned the colors c_1 and c_4 respectively and all others are colored properly.

Case 2: When $n = 5$.

$v_{2,3} = v_{4,5} = c_1$, all others are assigned so that the coloring is proper. Now the coloring is obviously acyclic, by the very arrangement of the colors. It is also minimum, because if we replace any color by an already used color, it will become either improper or cyclic (**Figures 1** and **2**).

2.4. Note

$a\left(C\left(C_n\right)\right) = 3$, for $n = 3, 4$.

3. Acyclic Coloring of Line Graph of Central Graph of K_n

3.1. Definition

Let G be a finite undirected graph with no loops and multiple edges, the line graph of G, denoted by $L(G)$, is the intersection graph $\Omega(X)$. Thus the points of $L(G)$ are the lines of G, with two points of $L(G)$ are adjacent whenever the corresponding lines of G are.

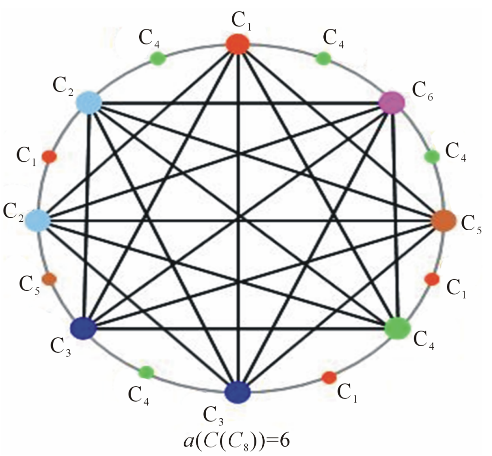

Figure 1. Acyclic coloring of central graph of C_8.

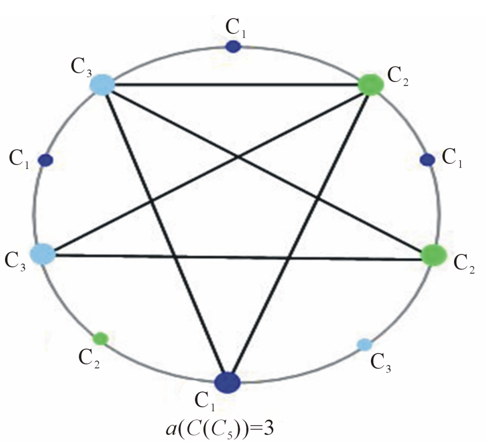

Figure 2. Acyclic coloring of central graph of C_5.

3.2. Structural Properties of Line Graph of Central Graph of K_n

Line graph of central graph of K_n is denoted by $L\left(C\left(K_n\right)\right)$

- Number of vertices in $L\left(C\left(K_n\right)\right) = n(n-1)$.
- Maximum Degree of vertices = Minimum Degree of vertices $= n-1$.
- $L\left(C\left(K_n\right)\right)$ contains n copies of vertex disjoint K_{n-1}.
- There is a cycle C' of length $2n$ with alternate edges from each of the complete graph K_{n-1}.

3.3. Theorem

For any complete graph K_n, $a\left\{L\left(C\left(K_n\right)\right)\right\} = n-1$.

Proof:

Let K_n be the complete graph on n vertices. Consider its line graph of central graph $G = L\left(C\left(K_n\right)\right)$, it contains n copies of vertex disjoint sub graphs K_{n-1}^j, $j = 1, 2, \cdots, n$ and which are marked in anti-clockwise direction. Let

$$V\left[L\left(C\left(K_n\right)\right)\right] = \left\{u_1^j, u_2^j, u_3^j, \cdots, u_{n-1}^j\right\}$$

where $j = 1, 2, 3, \cdots, n$; so that the total number of vertices in G is $n(n-1)$. Here there exist a unique bridge between each pair of sub graphs K_{n-1}^j. The bridge in the consecutive pairs of sub graph $\left(K_{n-1}^j, K_{n-1}^{j+1}\right)$, is given by for $2i < n$ it is $\left(u_{2i-1}^j, u_{2i}^{j+1}\right)$ and for $2i \ge n$ it is $\left(u_{x-1}^j, u_x^{j+1}\right)$, $2i \equiv x(\mathrm{mod}\, n-1)$ only for $x = 1, 2, \cdots, n-1$, form a bridge in the sub graph $\left(K_{n-1}^i, K_{n-1}^{i+1}\right)$. In a similar manner bridges are formed in non consecutive pairs also. Consider the color class $C = \left\{c_1, c_2, c_3, \cdots, c_{n-1}\right\}$. Assign the color c_i to the vertex u_i^j for $j = 1, 2, 3, \cdots, n$. Next we prove that the coloring is acyclic. That is the coloring does not induce a bi-chromatic cycle. Clearly for each complete sub graph K_{n-1}^j the coloring is acyclic (it never induce a bi-chromatic cycle). Now exactly two pairs of sub graphs K_{n-1}^j, $j = 1, 2, 3, \cdots, n$, never allow to induce a bi-chromatic cycle for any pair c_i, as there is only a unique bridge between each pair of sub graphs K_{n-1}^j. Note that bi-chromatic cycle is possible only for even cycles. The coloring is in such a way that more than three sub graphs K_{n-1}^j. never allow to induce a bi-chromatic cycle for any pair c_i. The maximum number of times a color will occur in any bi-chromatic path in this coloring is three. So the above said coloring acyclic. Also the coloring is minimum, as $G = L\left(C\left(K_n\right)\right)$, contains the subgraph K_{n-1}^j, minimum $n-1$ colors are required for its proper coloring (**Figure 3**).

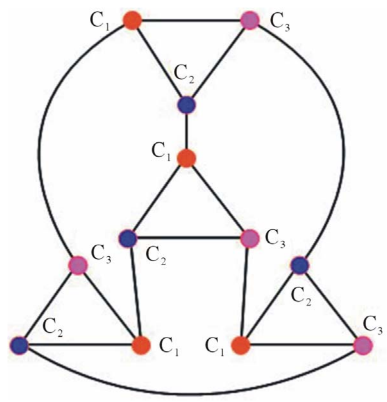

Figure 3. Acyclic coloring of $L(C(K_4))$.

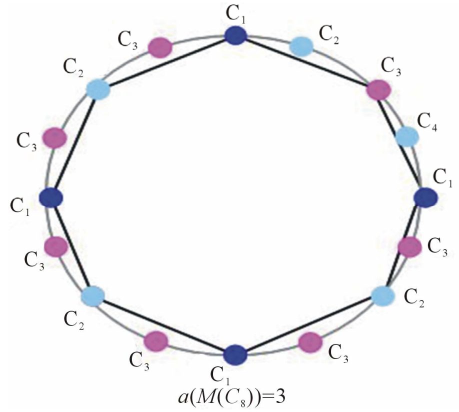

Figure 4. Acyclic coloring of middle graph of C_8.

4. Acyclic Coloring of Middle Graph of C_n

4.1. Middle Graph [3]

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G)\cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of following holds:

1) x, y are in $E(G)$ and x, y are adjacent in G; 2) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

4.2. Theorem

The acyclic chromatic number of the middle graph of C_n is $a(M(C_n))=3$, for $n \geq 3$.

Proof

$V(C_n)=\{v_1, v_2, \cdots, v_n\}$ and $E(C_n)=\{e_1, e_2, \cdots, e_n\}$ in which $e_i = v_i v_{i+1}$ with $v_{n+1}=v_1$. Let $G=M(C_n)$ be the middle graph of the n-cycle. By the definition of middle graph

$$V(M(C_n))=\{v_1, v_2, \cdots, v_n\}\cup\{e_1, e_2, \cdots, e_n\},$$

and

$$E(M(C_n))=\{e_i e_{i+1}/1 \leq i \leq n-1\}\cup e_n e_1$$
$$\cup\{e_i v_{i+1}/1 \leq i \leq n-1\}\cup e_n v_1 \cup\{v_i e_i/1 \leq i \leq n\}.$$

Then in the middle graph, there are n-vertices of degree 2 and another n-vertices of degree 4. Let C_n^* be the cycle of length n in G with degree of each vertex 4 and C_n^{**} be the cycle of length n in G with degree of vertices alternately 2 and 4. The cycle C_n^* are assigned the colors c_1 and c_2 alternately with last vertex preceding to v_n by c_3. All other vertices except vertices adjacent to v_n^* (which are colored as c_2) are colored as c_3. The coloring is minimum, as for any cycle minimum 3 colors needed for its acyclic coloring. The coloring is acyclic (**Figure 4**).

5. Acyclic Coloring of Total Graph of C_n

5.1. Total Graph [3]

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph of G, denoted by $T(G)$ is defined as follows. The vertex set of $T(G)$ is $V(G)\cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following holds:

1) x, y are in $V(G)$ and x is adjacent to y in G; 2) x, y are in $E(G)$ and x, y are adjacent in G; 3) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

5.2. Some Structural Properties of Total Graph of C_n

- Every cycle has a 4-regular total graph.
- The number of vertices in the total graph of C_n is 2 times the number of vertices in the cycle C_n.
- The number of edges in the total graph of C_n is 4 times the number of edges in the cycle C_n.
- The total graph of C_n is Eulerian.
- The total graph of C_n is Hamiltonian.

5.3. Theorem

The acyclic chromatic number of the total graph of C_n is $a(T(C_n))=4$, for $n \geq 4$.

Proof

Let $V(C_n)=\{v_1, v_2, \cdots, v_n\}$ and $E(C_n)=\{e_1, e_2, \cdots, e_n\}$ in which $e_i = v_i v_{i+1}$ with $v_{n+1}=v_1$. Let $G=T(C_n)$ be the total graph of the n-cycle. By the definition of total graph $V(T(C_n))=\{v_1, v_2, \cdots, v_n\}\cup\{e_1, e_2, \cdots, e_n\}$, and

$$E(T(C_n))=\{e_i e_{i+1}/1 \leq i \leq n-1\}\cup\{v_i v_{i+1}/1 \leq i \leq n-1\}$$
$$\cup e_n e_1 \cup\{e_i v_{i+1}/1 \leq i \leq n-1\}\cup e_n v_1$$
$$\cup\{v_i e_i/1 \leq i \leq n\}\cup v_n v_1.$$

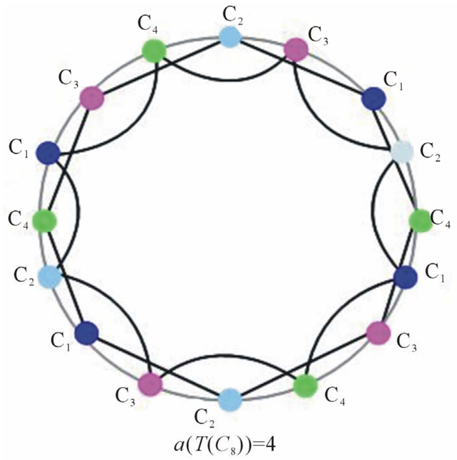

Figure 5. Acyclic coloring of middle graph of C_8.

By Menger's theorem as, there are four pair wise vertex-independent paths between any two non adjacent vertices, the total graph of C_n is 4-connected. To prove that $a(T(C_n))=4,$ if possible consider the color class $C=\{c_1,c_2,\cdots,c_m\}$ with $m<4,$ such that the coloring is acyclic. Then there exist no pair (v_h,v_k) such that they induce a bi-chromatic cycle. *i.e.*, there exist a three vertex cut in $T(C_n)$. This is a contradiction to the fact that $T(C_n)$ is 4-connected. Also acyclic chromatic number is can't be 5, as in this case we can replace a color by an already used color.

Therefore $a(T(C_n))=4,$ **for** $n\geq 4$ (**Figure 5**).

5.4. Note

$a(T(C_3))=5.$

6. Acknowledgements

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