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A Note on Hamiltonian Circulant Digraphs of Outdegree Three*

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ABSTRACT

We construct Hamilton cycles in connected loopless circulant digraphs of outdegree three with connection set of the form $\{a, ka, b\}$ for an integer k satisfying the condition $(b-a)\gcd(a,n) \equiv a\alpha t(k-1) \bmod n$ for some integer t such that $0 \le t \le \gcd(a,n)$, where $\alpha = \gcd(|a|, k)$. This extends work of Miklavič and Šparl, who previously determined the Hamiltonicity of these digraphs in the case where $k=-1$ and $k=2$, to other values of k which depend on the generators a and b.

Keywords: Hamilton Cycle; Circulant Digraph

1. Definitions and Notation

The group of integers under the operation of addition modulo n is denoted by $\mathbb{Z}_n$. A subset S of $\mathbb{Z}_n$ is a *generating set* for $\mathbb{Z}_n$ if every element of $\mathbb{Z}_n$ can be written as a linear combination of elements in S. For elements $a_1, a_2, \cdots, a_m$ of $\mathbb{Z}_n$, the symbol $\langle a_1, a_2, \cdots, a_m \rangle$ denotes the subgroup of $\mathbb{Z}_n$ generated by the elements $a_1, a_2, \cdots, a_m$, which is comprised of all linear combinations of the elements $a_1, a_2, \cdots, a_m$ in $\mathbb{Z}_n$. For an element $a \in \mathbb{Z}_n$, the set $\{b + x : x \in \langle a \rangle\}$ is called the *left coset of* $\langle a \rangle$ *in* $\mathbb{Z}_n$, and is denoted by $b + \langle a \rangle$. For two integers a, b, the *greatest common divisor* of a and b is the least positive integer which divides both a and b, and is denoted by $\gcd(a,b)$.

A *digraph* is a pair (V, A) in which V is a set of *vertices* and A is a set of ordered pairs of elements of V called arcs. A *directed path* of length m in a digraph $D = (V, A)$ is a sequence $v_0, a_1, v_1, a_2, v_2, \cdots, a_m, v_m$ in which $v_i \in V$ and $a_i = (v_{i-1}, v_i) \in A$ for $i = 1, 2, \cdots, m$, and no vertices or arcs in the sequence are repeated except possibly $v_0 = v_m$. If $v_0 = v_m$ then the sequence is called a *directed cycle*. A digraph is *connected* if there is a directed path from v to w for any two vertices v and w. Two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are *isomorphic* if there is a bijection $\sigma : V_1 \to V_2$ such that $(v, w) \in A_1$ if and only if $(\sigma(v), \sigma(w)) \in A_2$. Such a mapping σ is called an *isomorphism* from D_1 to D_2.

An *automorphism* of a digraph D is an isomorphism from D to itself. A digraph D is *vertex-transitive* if, for any two vertices v and w of D, there is an automorphism of D mapping v to w.

For a subset $S \subseteq \mathbb{Z}_n$, the *circulant digraph* $Circ(n;S)$ is the digraph with vertex set $\mathbb{Z}_n$ and arcs from v to $v+s$ for all $v \in \mathbb{Z}_n$ and all $s \in S$. The set S is called the *connection set* of the digraph $Circ(n;S)$, and the *outdegree* of $Circ(n;S)$ is the cardinality of the connection set S. Clearly, the circulant digraph $Circ(n;S)$ is connected if and only if S is a generating set for $\mathbb{Z}_n$. A *Hamilton cycle* in a digraph with n vertices is a directed cycle with n vertices. A digraph is said to be *Hamiltonian* if it has a Hamilton cycle.

Each arc in $Circ(n;S)$ of the form $(v, v+s)$ is labeled s. A Hamilton cycle in $Circ(n;S)$ can be specified by the sequence of vertices encountered or by the sequence of arcs traversed. In the latter case, it is often more convenient to list the labels of the arcs, rather than the arcs themselves, since for each vertex there is exactly one out-arc with label s for each $s \in S$. A *Hamiltonian* arc *sequence* is an ordered sequence $s_1 s_2 \cdots s_n$ of the arc labels encountered in a Hamilton cycle. Since circulant digraphs are vertex-transitive, any cyclic shift of a Hamiltonian arc sequence of $Circ(n;S)$ is also a Hamiltonian arc sequence of $Circ(n;S)$, and traversing a Hamiltonian arc sequence of $Circ(n;S)$ starting from any vertex will yield a Hamilton cycle in $Circ(n;S)$. For any arc sequence x, x^t denotes the concatenation

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of t copies of x.

2. History and Statement of the Main Result

One long-standing open problem is that of determining which circulant digraphs are Hamiltonian. Clearly, Hamiltonian circulant digraphs must be connected. In 1948, Rankin [1] determined which connected circulant digraphs of outdegree two are Hamiltonian, and so we need only consider connected circulant digraphs of outdegree at least three. There has been some recent work on the problem of determining when a circulant digraph of outdegree three is Hamiltonian. In 1999, Locke and Witte [2] constructed some infinite families of connected non-Hamiltonian circulant digraphs of outdegree three. In 2009, Witte Morris, Morris and Webb [3] proved that the circulant digraph $Circ(n;\{2,3,c\})$ is *not* Hamiltonian if and only if n is a multiple of 6, $c \in \{(n/2)+2,(n/2)+3\}$ and c is even. Also in 2009, Miklavič and Šparl proved the following result.

Proposition 2.1. (Miklavič and Šparl, [4]) *For $k=-1$ or $k=2$, the circulant digraph $Circ(n;\{a,ka,b\})$ is Hamiltonian if and only if it is connected, except in the special case where it is isomorphic to $Circ(12;\{3,6,4\})$.*

The result of Witte Morris *et al* in [3] shows that Miklavič and Šparl's result does not hold for all values of *k*. For example, if 12 divides *n* then $Circ(n;\{2,3,2k\})$ is not Hamiltonian for $k=(n/4)+1$, and if 6 divides n and $n/6$ is odd then $Circ(n;\{2,3,2k\})$ is not Hamiltonian for $k=(n/6)+1$. The following result shows that Miklavič and Šparl's result does hold for other values of k which depend on a and b.

Theorem 2.2. *Let $a,b,k \in \mathbb{Z}_n \setminus \{0\}$, let $\alpha = \gcd(k,|a|)$, and suppose that*

$$(b-a)\gcd(a,n) \equiv a\alpha t(k-1) \bmod n \tag{1}$$

for an integer t such that $0 \le t \le \gcd(a,n)$. The circulant digraph $Circ(n;\{a,ka,b\})$ is Hamiltonian if and only if it is connected.

We prove this theorem in the next section, and in Section 4 we obtain two corollaries to this theorem in the case where *a* divides *n*, which yield two infinite families of Hamiltonian circulant digraphs of the form $Circ(n;\{a,ka,b\})$.

3. Proof of Theorem 2.1

Proof: If $Circ(n;\{a,ka,b\})$ is Hamiltonian, then it is certainly connected.

Conversely, if $Circ(n;\{a,ka,b\})$ is connected, then $\mathbb{Z}_n = \langle a,ka,b\rangle = \langle a,b\rangle$, and so $\mathbb{Z}_n$ can be partitioned into the cosets

$\langle a\rangle,\ b+\langle a\rangle,\ 2b+\langle a\rangle,\ \cdots,\ (\gcd(a,n)-1)b+\langle a\rangle$

of $\langle a\rangle$ in $\langle a,b\rangle$. Let C_i denote the coset $ib+\langle a\rangle$ for $i=0,1,\cdots,\gcd(a,n)-1$. We will show that the arc sequence

$$\left(a^{|a|-1}b\right)^{\gcd(a,n)-t}\left(\left(ka\right)^{|ka|-1}\left(a\left(ka\right)^{|ka|-1}\right)^{\alpha-1}b\right)^{t} \tag{2}$$

is a Hamiltonian arc sequence of $Circ(n;\{a,ka,b\})$.

Starting from any vertex $v \in C_i$ and traversing the arc sequence $a^{|a|-1}$, we form a walk which visits every vertex of the coset C_i exactly once. Since $[\langle a\rangle:\langle ka\rangle] = \gcd(|a|,k) = \alpha$, the set of distinct cosets of $\langle ka\rangle$ in $\langle a\rangle$ are

$\langle ka\rangle,\ a+\langle ka\rangle,\ 2a+\langle ka\rangle,\ \cdots,\ (\alpha-1)a+\langle ka\rangle.$

This implies that every vertex of the coset C_i can be written uniquely in the form $v+q(ka)+r$, where $0 \le q < |ka|$ and $0 \le r < \alpha$, for any fixed vertex v in C_i. Thus, starting from any vertex v in C_i and traversing the arc sequence $(ka)^{|ka|-1}\left(a(ka)^{|ka|-1}\right)^{\alpha-1}$, we visit every vertex of the coset C_i exactly once. Since $b+C_i = C_{i+1}$, traversing arc b from any vertex in coset C_i leads to a vertex of the coset C_{i+1}.

Hence, starting from any vertex v of $Circ(n;S)$, say $v \in C_i$, and traversing arc sequence $(xb)^{\gcd(a,n)-t}(yb)^{t}$, where x denotes the arc sequences $a^{|a|-1}$ and y denotes the arc sequence $(ka)^{|ka|-1}\left(a(ka)^{|ka|-1}\right)^{\alpha-1}$, we form a walk which visits every vertex of this circulant digraph exactly once, and then finishes back on a vertex of C_i. This walk ends back on the starting vertex v, and hence is a Hamilton cycle, if and only if the sum of the arc labels in the arc sequence (2) is equal to 0. Hence it remains to show that

$$\begin{aligned}&(\gcd(a,n)-t)\left[(|a|-1)a+b\right]\\&+t\left[\alpha(|ka|-1)(ka)+(\alpha-1)a+b\right]\\&\equiv 0 \bmod n.\end{aligned}$$

A straightforward calculation shows that this is equivalent to

$$(b-a)\gcd(a,n) \equiv a\alpha t(k-1) \bmod n,$$

which holds by assumption.

The construction described in the proof of Theorem 2.2 is shown in **Figure 1** for the case where $n=105$, $a=7$, $ka=21$ and $b=25$. Here $k=3$, $|a|=15$, $|ka|=15$, $\alpha=\gcd(k,|a|)=3$ and $\gcd(a,n)=7$. In this case condition (1) holds for $t=3$. The Hamiltonian arc sequence in (2) for the circulant digraph $Circ(105;\{7,21,25\})$ is

$$\left(a^{14}b\right)^{4}\left(\left(ka\right)^{20}\left(a\left(ka\right)^{4}\right)^{2}(b)\right)^{3}.$$

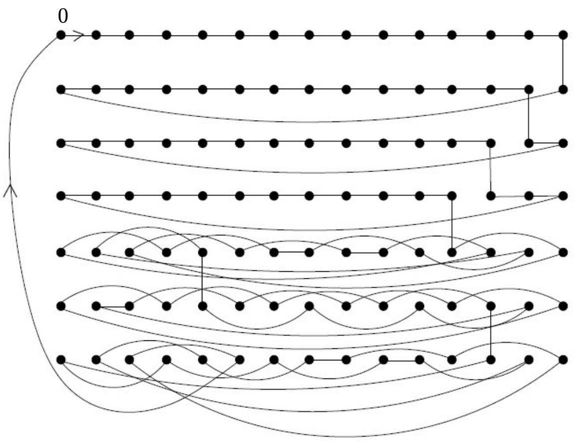

Figure 1. Hamilton cycle in $Circ(105;\{7,21,25\})$. The vertex in the ith row and the jth column is $25i+7j \bmod 105$, for $0 \le i < 7$ and $0 \le j < 15$.

The Hamilton cycle given by this arc sequence is shown in **Figure 1**, where each row of vertices represents a coset of $\langle 7 \rangle$ in $\mathbb{Z}_{105}$, and each column of vertices represents a coset of $\langle 25 \rangle$, so the vertex in the i th row and the j th column of the figure is $25i+7j \bmod 105$, for $0 \le i < 7$ and $0 \le j < 15$. The straight horizontal arcs have label $a=7$, the curved horizonal arcs have label $ka=21$, and the vertical arcs have label $b=25$.

4. Corollaries

Note that if $k=1$, then the assumption in (1) is that $a=b$, and in this case Theorem 2.2 simply states that $Circ(n;\{a\})$ is Hamiltonian if and only if $\langle a \rangle = \mathbb{Z}_n$. For other values of k satisfying condition 1, Theorem 2.2 implies that $Circ(n;\{a,ka,b\})$ is Hamiltonian if and only if $\gcd(a,ka,b,n)=1$ (*i.e.*, the digraph is connected). In the special case where a divides n, we obtain two corollaries.

Corollary 4.1. *If a divides n and $\gcd(n/a, b-a+1)=1$, then the circulant digraph $Circ(n;\{a,(b-a+1)a,b\})$ is Hamiltonian if and only if it is connected.*

Proof: If a divides n then $\gcd(a,n)=a$. For $k=b-a+1$, we have

$\alpha=\gcd(|a|,k)=\gcd(n/a, b-a+1)=1$, and so

$at\alpha(k-1)=at(b-a)=\gcd(a,n)t(b-a)$.

Thus condition (1) holds with $t=1$, and so Theorem 2.2 implies the result.

Example 4.1. If a is odd, a divides n and $\gcd(n/a,3)=1$, then Corollary 4.1 implies that $Circ(n;\{a,3a,a+2\})$ is Hamiltonian, since $\gcd(a,3a,a+2,n)=1$ and so this digraph is connected. For example, Corollary 4.1 guarantees that $Circ(35;\{5,15,7\})$ is Hamiltonian. We note that Rankin's result in [1] implies that neither $Circ(35;\{5,7\})$ nor $Circ(35;\{7,15\})$ is Hamiltonian, so all three arc labels must appear on any Hamilton cycle in $Circ(35;\{5,15,7\})$.

Corollary 4.2. *If a divides n, k divides n/a and $\gcd(a,k(k-1))=1$, then for any i such that $0 \le i \le n$, the circulant digraph $Circ(n;\{a,ka,(k(k-1)+1)a-k(k-1)i\})$ is Hamiltonian if and only if it is connected.*

Proof: If a divides n and k divides n/a, then $\gcd(a,n)=a$ and $\alpha=\gcd(n/a,k)=k$. Thus for $b=(k(k-1)+1)a-k(k-1)i$ and $t=a-i$, we have

$$
\begin{aligned}
(b-a)\gcd(a,n) &\equiv a(b-a) \pmod{n} \\
&\equiv a\big((k(k-1)+1)a-k(k-1)i-a\big) \pmod{n} \\
&\equiv a\big(k(k-1)a-k(k-1)i\big) \pmod{n} \\
&\equiv ak(k-1)(a-i) \pmod{n} \\
&\equiv a\alpha t(k-1) \pmod{n}.
\end{aligned}
$$

Hence condition (1) holds and so the result follows from Theorem 2.2.

Example 4.2. If a divides n, 3 divides n/a, $0<i<a$ and $\gcd(a,6i)=1$, then Corollary 4.2 implies that $Circ(n;\{a,3a,7a-6i\})$ is Hamiltonian, since $\gcd(a,3a,7a-6i,n)=1$ and so this digraph is connected. An example of a Hamilton cycle in $Circ(105;\{7,21,25\})$ is shown in **Figure 1**. Here $a=7$ and $i=4$, so $b=25$ and $t=a-i=3$.

In conclusion, in Theorem 2.2 we generalized Miklavič and Šparl's Proposition 2.1 to include all values of k that satisfy condition (1), showing that condition (1) is a sufficient condition on k to guarantee that the circulant digraph $Circ(n;\{a,ka,b\})$ is Hamiltonian if and only if it is connected. One interesting open problem is that of determining conditions on k which are both necessary and sufficient to guarantee this result. In Corollaries 4.1 and 4.2, for the special case where a divides n we obtained two infinite families of Hamiltonian circulant digraphs of the form $Circ(n;\{a,ka,b\})$.

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