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The Middle Equitable Dominating Graphs

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ABSTRACT

Let $G=(V,E)$ be a graph and $A(G)$ is the collection of all minimal equitable dominating set of G. The middle equitable dominating graph of G is the graph denoted by $M_{ed}(G)$ with vertex set the disjoint union of $V \cup A(G)$ and (u, v) is an edge if and only if $u \cap v \neq \phi$ whenever $u, v \in A(G)$ or $u \in v$ whenever $u \in v$ and $v \in A(G)$. In this paper, characterizations are given for graphs whose middle equitable dominating graph is connected and $K_p \subseteq M_{ed}(G)$. Other properties of middle equitable dominating graphs are also obtained.

Keywords: Eqitable Domination Number; Middle Equitable Dominating Graph; Intersection Graphs

1. Introduction

All the graphs considered here are finite and undirected with no loops and multiple edges. As usual $p=|V|$ and $q=|E|$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighbourhoods of a vertex v, respectively. A set D of vertices in a graph G is a dominating set if every vertex in $V-D$ is adjacent to some vertex in *D*. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. For terminology and notations not specifically defined here we refer reader to [1]. For more details about parameters of domination number, we refer to [2], and [3].

A subset D of $V(G)$ is called an equitable dominating set of a graph G if for every $u \in (V-D)$, there exists a vertex $v \in D$ such that $uv \in E(G)$ and $|\deg(u)-\deg(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by $\gamma_e(G)$ and is called equitable domination number of G. D is minimal if for any vertex $u \in D$, $D-\{u\}$ is not a equitable dominating set of G. A subset S of V is called a equitable independent set, if for any $u \in S$, $v \notin N_e(u)$, for all $v \in S-\{u\}$. If a vertex $u \in V$ be such that $|\deg(u)-\deg(v)| \geq 2$ for all $v \in N(u)$ then u is in each equitable dominating set. Such vertices are called equitable isolates. The equitable neighbourhood of u denoted by $N_e(u)$ is defined as $N_e(u)=\{v \in V / v \in N(u), |\deg(u)-\deg(v)| \leq 1\}$ and $u \in I_e \Leftrightarrow N_e(u)=\varnothing$. The cardinality of $N_e(u)$ is denoted by $\deg_e(u)$. The maximum and minimum equitable degree of a point in G are denoted respectively by $\Delta_e(G)$ and $\delta_e(G)$. That is $\Delta_e(G)=\max a_{u\in V(G)}|N_e(u)|$, $\delta_e(G)=\min_{u\in V(G)}|N_e(u)|$. An edge $e=uv$ called equitable edge if $|\deg(u)-\deg(v)| \leq 1\}$, for more details about equitable domination number see [4]. Let S be a finite set and let $F=\{S_1, S_2, \cdots, S_n\}$ be a partition of S. Then the intersection graph $\Omega(F)$ of F is the graph whose vertices are the subsets in F and in which two vertices S_i and S_j are adjacent if and only if $S_i \cap S_j \neq \phi$, $i \neq j$. Kulli and Janakiram introduced new classes of intersection graphs in the field of domination theory see [5-8].

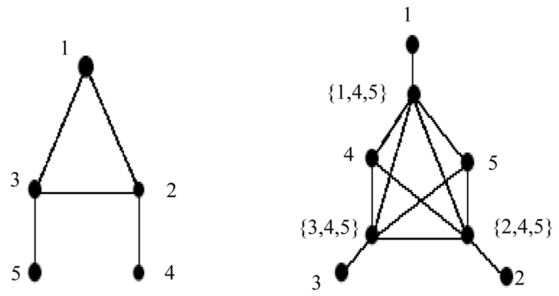
The purpose of this paper is to introduce a new class of intersection graphs in the field of domination theory as follows:

Let $G=(V,E)$ be a graph and S be the collection of minimal equitable dominating set of G. The middle equitable dominating graph of G is the graph denoted by $M_{ed}(G)$ with vertex set the disjoint union $V(G) \cup S$ and uv is an edge if and only if $u \cap v \neq \phi$ whenever $u, v \in S$ or $u \in v$ whenever $u \in V(G)$ and $v \in S$.

Example. Let *G* be a graph as in **Figure 1(a)**, then the equitable dominating sets are $\{1,4,5\}, \{2,4,5\}, \{3,4,5\}$ and the The middle equitable dominating graph shown in **Figure 1(b)**.

2. Main Results

Theorem 2.1. *A graph* G *with* $p \geq 3$ *vertices is without equitable edge* (*equitable edge-free graph*) *if and only*



(a) (b)

Figure 1. Example.

if $M_{ed}(G) \cong K_{1,p+1}$.

Proof. Suppose that $M_{ed}(G) \cong K_{1,p+1}$ and if its possible that G has equitable edge uv. Then G has at least two minimal equitable dominating set $(p \geq 3)$, a contradiction.

Conversely, assume that G is equitable edge-free graph with *p* vertices. Then clear all the vertices are equitable isolated vertices, that is there is only one minimal equitable dominating set contains all the vertices and according the definition of $M_{ed}(G)$, we get $M_{ed}(G) \cong K_{1,p+1}$.

Corollary 2.2. *If* $M_{ed}(G) \cong K_{1,2}$ *if and only if* $G \cong K_2$.

Corollary 2.3. *If* $M_{ed}(G) \cong K_{r,m}$, *where* $|r-m| \geq 2$, *then* $\gamma_e(M_{ed}(G)) = p+1$.

Corollary 2.4. *Let G be complete multi bipartite graph* $K_{n_1,n_2,\ldots n_t}$, *where* $|n_i - n_i \geq 2|$, *where* $i, j = 1, 2, \cdots t$, *then* $M_{ed}(G) \cong K_1, \sum_{i=1}^{t} n_i$.

Proposition 2.5. $M_{ed}(G) = pK_2$ *if and only if* $G = K_p$.

Proof. Suppose that $G = K_p$. Then clearly each vertex in G will form a minimal equitable dominating set. Hence $M_{ed}(G) = pK_2$.

Conversely, suppose $M_{ed}(G) = pK_2$ and $G \neq K_p$. Then there exists at least one minimal equitable dominating set S containing two vertices of G. Then S will form p_3 in $M_{ed}(G)$.

Proposition 2.6. *If* $G = K_p$, *then* $\gamma(M_{ed}(G)) = \gamma_e(M_{ed}(G)) = p$.

Proof. The proof coming directly from Proposition 2.5.

Theorem 2.7. *Let* G *be a graph with p vertices and* q *edges.* $M_{ed}(G)$ *is a graph with* $2P$ *vertices and p edges if and only if* $G \cong K_p$.

Proof. If $G \cong K_p$, then that is clear the $M_{ed}(G)$ is a graph with $2P$ vertices and *p* edges.

Conversely, let $M_{ed}(G)$ be $(2p, p)$ -graph. Since only the graph pK_2 is of 2*p* vertices and *p* edges, then by Proposition 2.5 $G \cong K_p$.

Theorem 2.8. *Let* G *be a graph with p vertices and* q *edges.* $M_{ed}(G)$ *is a graph with* $P+1$ *vertices and* p *edges if and only if* G *is equitable edge-free graph.*

Proof. Let G be equitable edge-free graph. Then by Theorem 2.1 $M_{ed}(G)$ is a graph with $P+1$ vertices and p edges.

Conversely, let $M_{ed}(G)$ be a graph with $P+1$ vertices and p edges that means $M_{ed}(G) \cong K_{1,p}$, and by Theorem 2.1 G is equitable edge-free graph.

Theorem 2.9. [1] *Let G be a graph, if D is maximal equitable independent set of G, then D also minimal equitable set.*

Theorem 2.10. *For any graph G with at least two vertices* $M_{ed}(G)$ *is connected if and only if* $\Delta_e(G) < p-1$.

Proof. Let $\Delta_e(G) < p-1$ and u, v be any two vertices if there is minimal equitable dominating set D containing *u* and *v*, the u, v are connected by the path uDv, and if there is no minimal equitable dominating containing both *u* and *v*. Then there exists a vertex w in $V(G)$ such that w is neither adjacent to *u* nor *v*. Let *D* and *D*, be two maximal equitable independent set containing *u*, *w* and *v*, *w* respectively since every maximal equitable independent set is minimal equitable dominating set by Theorem 2.9, *u* and *v* are connected by the path $uDD'v$. Thus $M_{ed}(G)$ is connected. Hence $M_{ed}(G)$ is connected.

Conversely, suppose $M_{ed}(G)$ is connected. let $\Delta_e(G) = p-1$ and let *u* be a vertex such that $d_e(u) = p-1$. Then $\{u\}$ is minimal equitable dominating set and G has at least two vertices , it is clear that G has no equitable isolated vertices, then $V-D$ containing minimal equitable dominating set say D'. Since $D \cap D' = \phi$ in $M_{ed}(G)$ there is no path joining u to any vertex of $V-D$, this implies that $M_{ed}(G)$ is disconnected, a contradiction. Hence $\Delta_e(G) < p-1$.

Corollary 2.11. *Let* $G = (V, E)$ *be a graph and* u, v *any two vertices in* $V(G)$. *Then* $d(u,v)_{M_{ed}(G)} \leq 3$, *where* $d(u,v)_{M_{ed}(G)}$ *is the distance between the vertex* u *and* v *in the graph* $M_{ed}(G)$.

Theorem 2.12. *For any graph* G, $K_t \subseteq M_{ed}(G)$ *if and only if* G *contains one equitable isolated vertex, where* t *is the number of minimal equitable dominating set in* G.

Proof. If G has equitable isolated vertex then the subgraph of $M_{ed}(G)$ which induced by the set of all minimal equitable dominating sets of G is complete graph K_t. Hence $K_p \subseteq M_{ed}(G)$.

Conversely, Suppose $K_p \subseteq M_{ed}(G)$, then it is clear that the vertices of K_t are the minimal equitable dominating sets of G. Therefor there exists at least one equitable isolated vertex in G.

Theorem 2.13. *For any graph* G, $M_{ed}(G)$ *is either connected or it has at least one component which is* K_2.

Proof. If $\Delta_{cn}(G) < p-1$, then by Theorem 2.10 M_{ed} is connected, and By Proposition 2.5, if G is complete $M_{ed} = pK_2$ which is connected. Hence we must prove the case $\delta_e(G) < \Delta_e(G) = p-1$.

Let $\{u_1 \cdots u_s\}$ be the set of vertices in G such that $d_e(u_i) = p-1$, where $i = 1, \cdots, t$, then it is clear $\{u_i\}$ is minimal equitable dominating set. Then each vertex u_i with the minimal equitable dominating set $\{u_i\}$ $i = 1, \cdots s$ form component isomorphic to K_2. Hence it has at least one component which is K_2.

Theorem 2.14. *For any graph* G *with* $\Delta_e(G) < p-1$, $\operatorname{diam}(M_{ed}(G)) \leq 5$.

Proof. Let G be any graph and $\Delta_e(G) < p-1$. Then by Theorem 2.10, $M_{ed}(G)$ is connected, suppose u, v be any two vertices in $M_{ed}(G)$, then we have the following cases:

Case 1: Let u, v be any two vertices in $V(G)$. Then by corollary 2.11, $d(u,v)_{M_{ed}(G)} \leq 3$.

Case 2: Suppose $u \in V(G)$ and $v = D$ be a minimal equitable dominating set in G, If $u \in D$ then $d(u,v) = 1$ and if $u \notin D$ then there exists a vertex $w \in D$ adjacent to u. Hence $d(u,v)_{M_{ed}(G)} = d(u,w)_{M_{ed}(G)} + d(w,v)_{M_{ed}(G)}$ and from corollary 2.11 we have $d(u,w)_{M_{ed}(G)} \leq 3$. Hence $d(u,v)_{M_{ed}(G)} \leq 4$.

Case 3: Suppose that both u and v are not in $V(G)$, and $u = D$, $v = D'$ are two minimal equitable dominating sets if D and D' are adjacent, then $d(u,v)_{M_{ed}(G)} = 1$, suppose that D and D' are not adjacent then every vertex $w \in D$ is adjacent to some vertex $x \in D'$ and vice versa. Hence $d(u,v) = d(u,w) + d(w,x) + d(x,v) \leq 5$ Hence $\operatorname{diam}(M_{ed}(G)) \leq 5$.

Theorem 2.15. *Let* G *be a graph. Then* $d_e(M_{ed}(G)) = 2$ *if and only if* $G = K_p$ *or* $G = \overline{K_2}$, *where* $d_e(M_{ed}(G))$ *is the equitable domatic number of the graph* $M_{ed}(G)$.

Proof. If $G = K_p$ or $G = \overline{K_2}$, then $M_{ed}(G) = pK_2$ or $G = K_{1,2}$ by Proposition 2.5. Hence $d_{cn}(M_{ed}(G)) = 2$. The converse is obvious.

Theorem 2.16. *For any graph* G *with at least one equitable isolated vertex,* $\beta_e(M_{ed}(G)) = \beta(M_{ed}(G)) = p$.

Proof. Let G be a graph of p vertices, and has at least one equitable isolated vertex w. Then from the definition of $M_{ed}(G)$ if u and v any vertices in G then u and v can not be adjacent in $M_{ed}(G)$, that is there is equitable independent set containing w and of size p, and this equitable independent will be the maximum equitable independent because w is adjacent for all the equitable independent sets. Therefore $\beta_e(G) = \beta(G) = p$.

Corollary 2.17. *For any graph* G *with at least one equitable isolated vertex,* $\alpha_e(M_{ed}(G)) = \alpha(M_{ed}(G)) = |S|$, *where* $|S|$ *is the number of minimal equitable dominating set in* G

Proof. We have for any graph G with p vertices $\alpha_{cn}(G) + \beta_{cn}(G) = \alpha(G) + \beta(G) = p$, and from [1]. Theorem 3.8.4 corollary is proved.

Theorem 2.18. *For any graph* G, $d_e(G) = \beta_e(M_{ed}(G))$, *if and only if* G *is complete graph.*

Proof. Let G be complete graph K_p. Then from Theorem 2.16 $\beta_e(M_{ed}(G)) = p$, and we have $d_e(G) = p$. Hence $d_e(G) = \beta_e(M_{ed}(G))$.

Conversely, suppose $d_e(G) = \beta_e(M_{ed}(G))$. From Theorem 2.16 $\beta_e(M_{ed}(G)) = p$ implies that $d_e(G) = p$. Hence $G = K_p$.

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