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A Review and Update of Analytical and Numerical Solutions of the Terzaghi One-Dimensional Consolidation Equation

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Abstract

Practical resolution of consolidation problems that we often face requires an extensive and solid knowledge of the different parameters highlighted by the Terzaghi one-dimensional consolidation theory. This theory, with its assumptions, leads to a partial differential equation of second order in space and first order in time of pore water pressure. Analytical and numerical resolutions of this equation allow determining the water pressure variation before and after the application of a charge. Numerical modeling has enabled the simulation of the whole results obtained by the two methods of resolution (pressure, degree of consolidation, time factor, among others) to have a physical analysis and a lawful observation that lead to a suitable understanding of the phenomenon of Terzaghi one-dimensional consolidation.

Keywords

Pore Pressure, Numerical Solution, Analytical Solution, Modeling, Consolidation, Time Factor

1. Introduction

The unidimensional consolidation of soils has been described by Terzaghi using partial differential equations [1]. The resolution of these equations can be performed analytically and/or numerically [2] [3]. In this work, we resolved Terzaghi partial differential equations using Fourier method and finite difference respectively for analytical and numerical solutions. At a further step we used numerical modeling to represent graphically the two solutions in order to validate the obtained results.

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We consider an example of a clay layer drained on the faces and submitted instantly to the initial conditions to constant stress (*load*) [4] (**Figure 1**). It is assumed that the clay layer thickness is $2H$ with $H = 8$ m subject to its surface to a total stress $\Delta\sigma_v = 30$ kPa and with a consolidation coefficient estimated to 2 m^2/year.

Solving the Terzaghi one-dimensional consolidation equation, from the hydro mechanical modeling of the solid [2] [3], and adapted to the parameters of the problem allows for the following simulations:

$$\frac{\partial \Delta u(z,t)}{\partial t} = c_v \frac{\partial^2 \Delta u(z,t)}{\partial z^2} \quad (1)$$

where Δu is the pore water pressure, c_v is the coefficient of consolidation, z is the depth and t corresponds to the time.

2. Analytical Method

To solve this kind of first-order partial differential equation with respect to time and second-order with respect to space, we must combine two boundary conditions and initial condition for the interstitial pressure.

- Boundary conditions (for all time t)

-At the bottom of the layer, $z = 0$, we have:

$$\Delta u(0,t) = 0$$

-On the surface of the layer, $z = 2H$, then:

$$\Delta u(2H,t) = 0$$

- Initial condition (for $t = 0$)

$$\Delta u(z,0) = \Delta\sigma_v \text{ except for } z = 0 \text{ and } z = 2H$$

If we apply the Fourier sine transform F_s on the left and right members of this equation:

$$F_s\left(\frac{\partial \Delta u}{\partial t}\right) = F_s\left(c_v \frac{\partial^2 \Delta u}{\partial z^2}\right) \quad (2)$$

where

$$F_s\left(c_v \frac{\partial^2 \Delta u}{\partial z^2}\right) = c_v F_s\left[\frac{\partial^2 \Delta u}{\partial z^2}\right] = -c_v \frac{n\pi}{2H} F_c\left[\frac{\partial \Delta u}{\partial z}\right] \quad (3)$$

$$F_s\left(c_v \frac{\partial^2 \Delta u}{\partial z^2}\right) = -c_v \frac{n\pi}{2H}\left[\Delta u(2H,t)\cos n\pi - \Delta u(0,t) + \frac{n\pi}{2H} F_s\left[\Delta u(z,t)\right]\right] \quad (4)$$

The relation (2) is as follows:

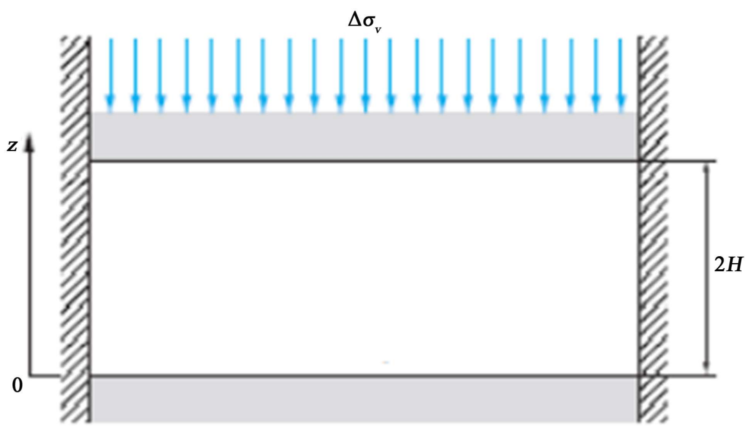


Figure 1. Example of a clay layer drained on the two faces (Modified from [4]).

$$\frac{\partial F_s\left[\Delta u(z,t)\right]}{\partial t}=-c_v\left(\frac{n\pi}{2H}\right)^2 F_s\left[\Delta u(z,t)\right] \tag{5}$$

$$\ln F_s\left[\Delta u(z,t)\right]=-c_v\left(\frac{n\pi}{2H}\right)^2 t+\ln F_s[0]\Rightarrow \ln\left(\frac{F_s\left[\Delta u(z,t)\right]}{F_s[0]}\right)=-c_v\left(\frac{n\pi}{2H}\right)^2 t \tag{6}$$

$$\frac{F_s\left[\Delta u(z,t)\right]}{F_s[0]}=\mathrm{e}^{-c_v\left(\frac{n\pi}{2H}\right)^2 t}\Rightarrow F_s\left[\Delta u(z,t)\right]=F_s[0]\mathrm{e}^{-c_v\left(\frac{n\pi}{2H}\right)^2 t} \tag{7}$$

Since the method used is the sine Fourier transform based on the odd frequency (sinusoidal) of the interstitial pressure, the development into Fourier series will not affect the coefficients a_n.

$$u(z,t)=\sum_{n=0}^{\infty} b_n \sin\frac{n\pi z}{2H} \tag{8}$$

where

$$b_n=\frac{2}{2H}F_s\left[\Delta u(z,t)\right]$$

By substituting, the equation can be rewritten into this form:

$$\Delta u(z,t)=\sum_{n=0}^{\infty}\frac{2}{2H}F_s\left[\Delta u(z,t)\right]\sin\frac{n\pi z}{2H}=\sum_{n=0}^{\infty}\frac{1}{H}F_s[0]\mathrm{e}^{-c_v\left(\frac{n\pi}{2H}\right)^2 t}\sin\frac{n\pi z}{2H} \tag{9}$$

$$F_s[0]=F_s\left[\Delta u(z,0)\right]=\int_0^{2H}\Delta u(z,0)\sin\frac{n\pi z}{2H}\mathrm{d}z=\Delta\sigma_v\int_0^{2H}\sin\frac{n\pi z}{2H}\mathrm{d}z=-\Delta\sigma_v\frac{2H}{n\pi}\left[\cos\frac{n\pi z}{2H}\right]_0^{2H}$$

$$F_s[0]=F_s\left[\Delta u(z,0)\right]=\frac{4\Delta\sigma_v H}{(2m+1)\pi} \tag{10}$$

By simplifying and separating constants we obtain the analytical solution of the Terzaghi's equation from the Fourier method:

$$\Delta u(z,t)=\frac{4\Delta\sigma_v}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\exp\left[-(2m+1)^2\pi^2\frac{c_v t}{4H^2}\right]\sin\left[\frac{(2m+1)}{2}\pi\frac{z}{H}\right] \tag{11}$$

Assuming that the $\infty = 100$, which is acceptable in numerical analysis [5], we can evaluate the numerical value of Δu (*which is the exact solution*) [6] [7]. For this, we make a loop for each point (z_i, t_i) and calculate $\Delta u_{\text{exact}}(z_i,t_i)$. By acting on the value $t_{\max}$ of the consolidation duration, the following results are obtained (**Figure 2(a)** and **Figure 2(b)**):

For a longer time of consolidation (e.g. 20 *years*), we obtain the following results (**Figure 3(a)** and **Figure 3(b)**):

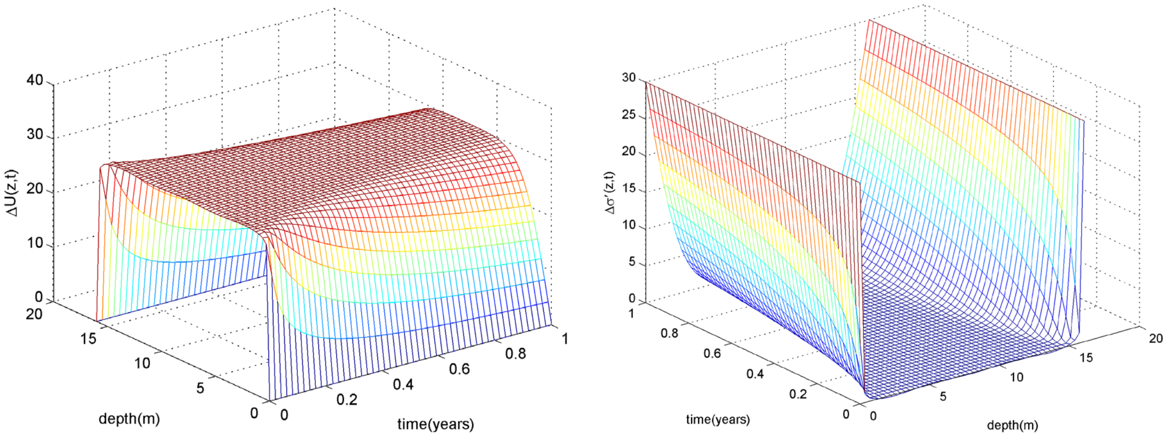
For an infinite time, the phenomenon of pressure dissipation becomes more and more clear and the effective stresses are more important (**Figure 4(a)** and **Figure 4(b)**); this means that the load is transmitted to solid grains.

The degree of consolidation and the time factor are derived from the exact solution of the consolidation equation.

$$U_v(T_v)=1-\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\exp\left[-(2m+1)^2\pi^2\frac{T_v}{4}\right] \tag{12}$$

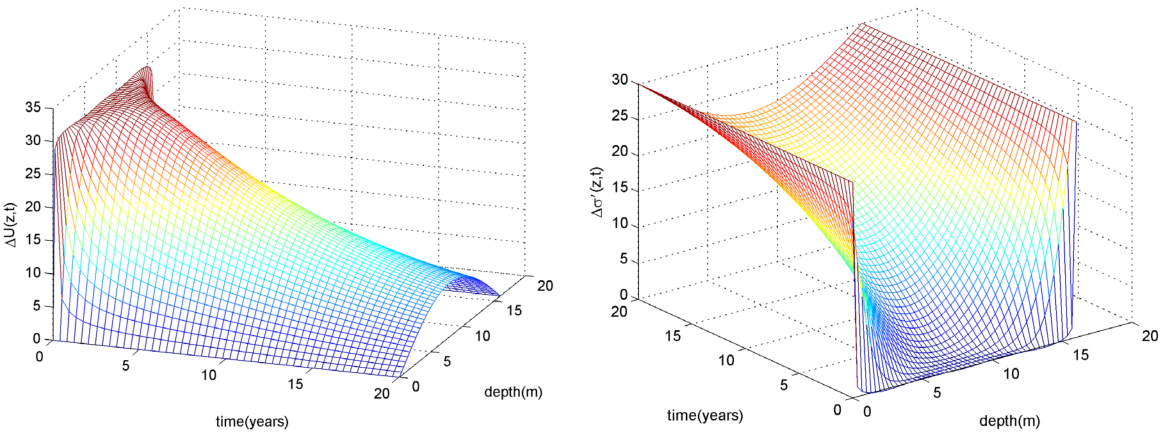
Hence the representation of the function $U_v(T_v)$ and the inverse function $T_v(U_v)$ gives (**Figure 5**):

The dimensionless term which is the ratio between the value of $\Delta u(z, t)$ at (t) and its initial value $\Delta u_0 = \Delta\sigma_v$ are simulated based on the reduced depth $Z = z/H$ and for different values of T_v.



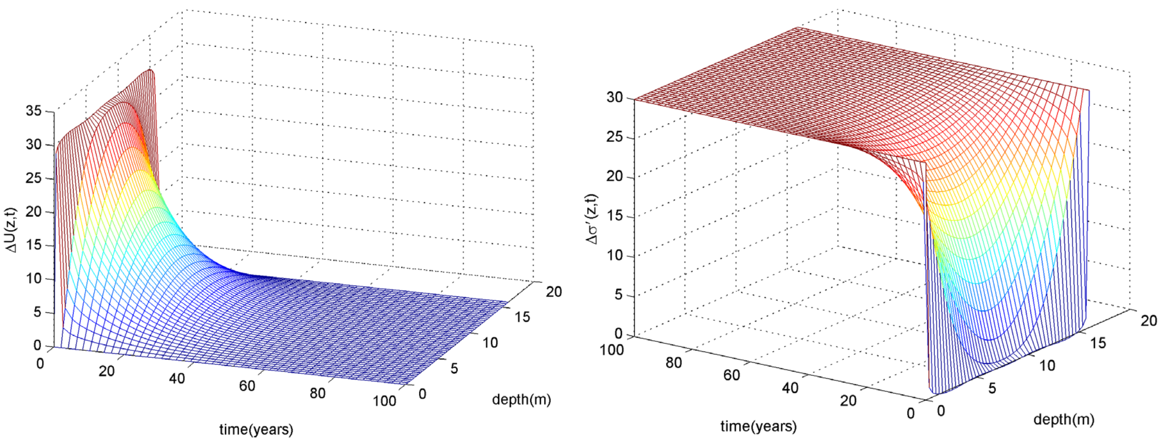
(a) (b)

Figure 2. Evolution of the pore water pressure (a) and the effective stress (b) as a function of depth and time (we consider here $t_{\max}$ = 1 year).



(a) (b)

Figure 3. Evolution of the pore water pressure (a) and the effective stress (b) as a function of depth and time (we consider here $t_{\max}$ = 20 years).



(a) (b)

Figure 4. Evolution of the pore water pressure (a) and the effective stress (b) as a function of depth and time (we consider here $t_{\max}$ = 100 years).

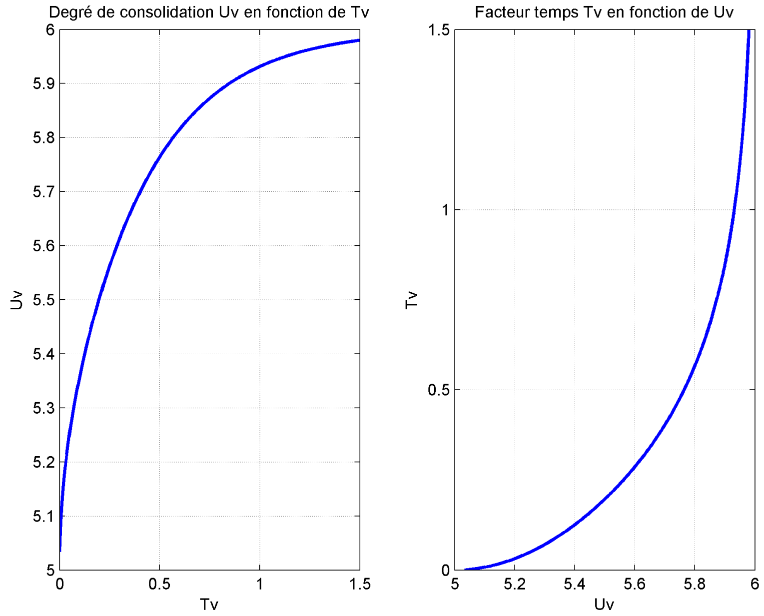


Figure 5. Respective changes of U_v and T_v, one according to others.

$$\frac{\Delta u(z,t)}{\Delta u_0}=\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\exp\left[-(2m+1)^2\pi^2\frac{T_v}{4}\right]\sin\left[\frac{(2m+1)}{2}\pi z\right] \tag{13}$$

Thus, we can see that the ratio $\frac{\Delta u(z,t)}{\Delta u_0}$ reaches a maximum for $T_v = 0$ and $Z = 0.2$ after an increasing and linear evolution for Z comprise between 0 and 0.1 (**Figure 6**). Then for other values of T_v with time steps ranging from 0.05 to 0.2, we obtain the other isochronous which respectively follow the first one, then with a time step of 0.1, the last isochronous; like the first isochronoous they show all ratio values that deviate more and more to 1 (*which is the maximum value*) when the time factor T_v tends to 1.

3. Numerical Method

The principle of this method consists in substituting the function $\Delta u(z, t)$ of the interstitial pressure at the point M at time t in a discrete function $\Delta\tilde{u}(z,t)$. It requires the selection of a mesh with Δz as a space step and Δt as time step. u_{ik} or u_i^k is the interstitial pressure of water at the node (i, k). Which means that at the node $z_i = i\Delta z$ and at $t = k\Delta t$.

$$f_i'=\frac{f_{i+1}-f_i}{h}+O(h)\Rightarrow\left(\frac{\partial\Delta u}{\partial t}\right)_{ik}=\frac{\Delta u_i^{k+1}-\Delta u_i^k}{\Delta t}+O(\Delta t)$$

$$f_i''=\frac{f_{i-1}-2f_i+f_{i+1}}{h^2}+O(h^2)=\left(\frac{\partial^2\Delta u}{\partial z^2}\right)_{ik}=\frac{\Delta u_{i-1}^k-2\Delta u_i^k+\Delta u_{i+1}^k}{\Delta z^2}+O(\Delta z^2)$$

This form leads to address the resolution by an explicit scheme which uses a discretisation at z_i node and at iteration n. By analogy to the consolidation equation the equality between the first order scheme in time and the second order centered scheme in space has been set

Hence it may be evaluated to obtain:

$$\frac{\Delta u_i^{k+1}-\Delta u_i^k}{\Delta t}=c_v\frac{\Delta u_{i-1}^k-2\Delta u_i^k+\Delta u_{i+1}^k}{\Delta z^2} \tag{14}$$

Given $\theta=c_v\frac{\Delta t}{\Delta z^2}$, this equation can be rewritten in the form that gives the interstitial pressure of the water at

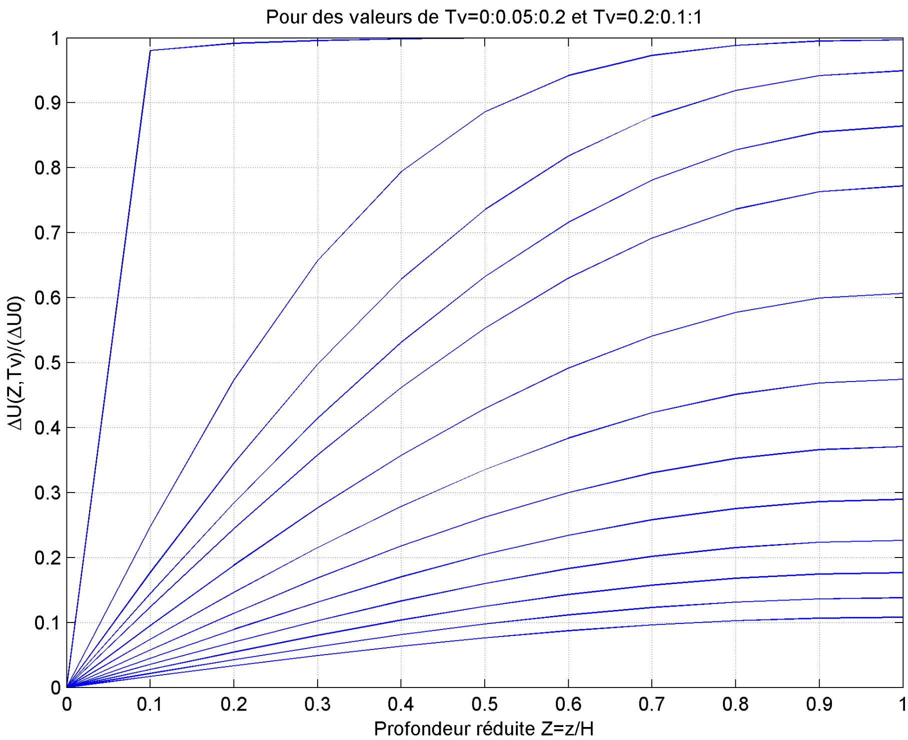


Figure 6. Isochrones of pore pressure for different values of T_v as a function of Z.

iteration $k+1$:

$$\Delta u_i^{k+1} = \theta \Delta u_{i-1}^k + \Delta u_i^k - 2\theta \Delta u_i^k + \theta \Delta u_{i+1}^k$$

$$\Delta u_i^{k+1} = \theta \Delta u_{i-1}^k + (1-2\theta)\Delta u_i^k + \theta \Delta u_{i+1}^k \quad \text{where } i = 1:N-1 \tag{15}$$

So the matrix of excess water pore pressure from the resolution is given by the following finite differential method:

$$\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \vdots \\ \Delta u_{N-2} \\ \Delta u_{N-1} \end{bmatrix}^{k+1} = \begin{bmatrix} 1-2\theta & \theta & 0 & \cdots & 0 \\ \theta & 1-2\theta & \theta & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \theta & 1-2\theta & \theta \\ 0 & 0 & 0 & \theta & 1-2\theta \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \vdots \\ \Delta u_{N-2} \\ \Delta u_{N-1} \end{bmatrix}^{k} + \theta \begin{bmatrix} \Delta u_g \\ 0 \\ \vdots \\ 0 \\ \Delta u_d \end{bmatrix} \tag{16}$$

Computed results in matrix form obtained by the numerical solution give also performances that converge towards an analytical solution.

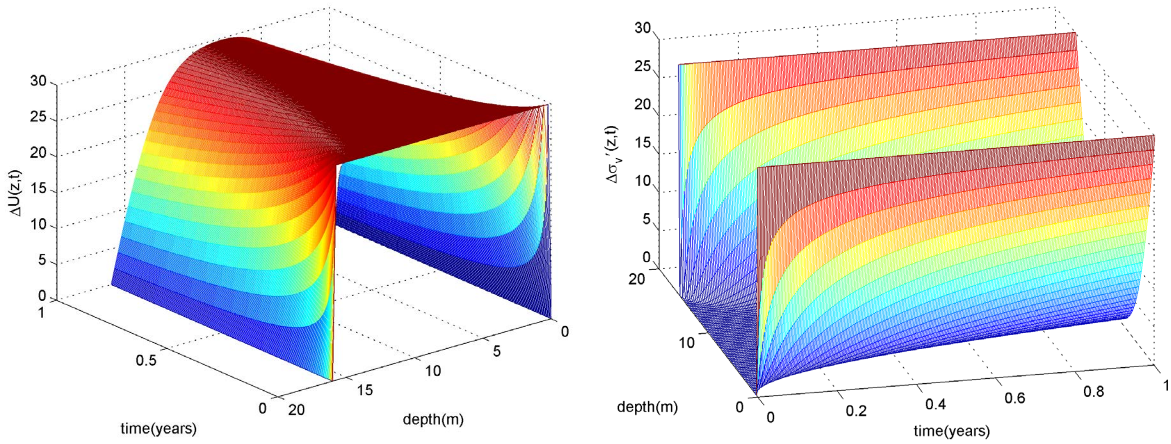
For greater stability, θ must be less than 1/2 [8]; however, we assume that θ = 1/4 and apply the other parameters of the problem for obtaining the following figures (**Figure 7(a)** and **Figure 7(b)**):

For a longer time of consolidation (e.g. 20 *years*), we obtain the following results (**Figure 8(a)** and **Figure 8(b)**). With these results, we can notice the phenomenon of dissipation that appears in a clear way:

For an infinite time, the phenomenon of pressure dissipation becomes more clear and the effective stress more important (**Figure 9(a)** and **Figure 9(b)**). In fact, the load is transmitted to solid grains.

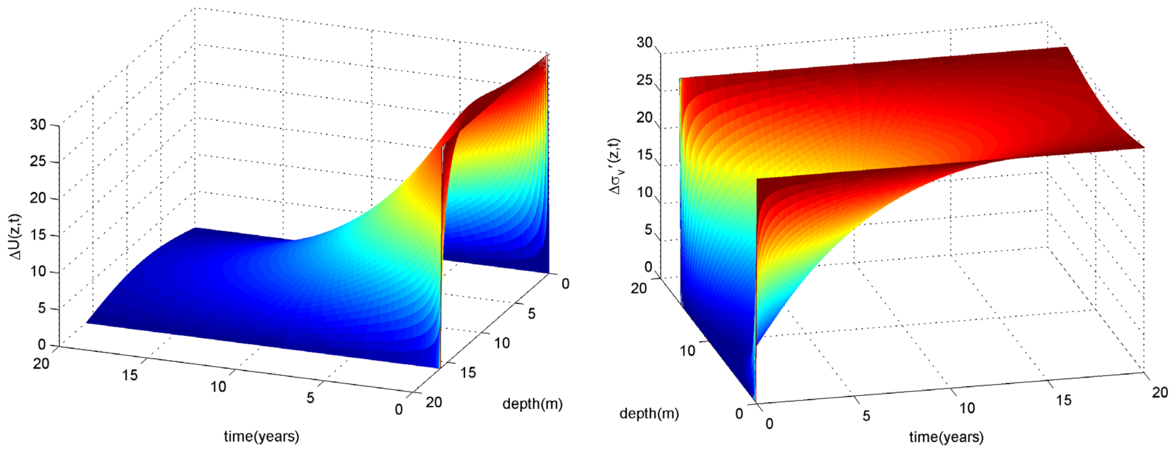
These results show without any ambiguity that the water interstitial pressure is canceled through the soil layer thickness in the same way as noticed in the analytical resolution:

The pore water pressure obviously vanishes over time with perfect coherence with Terzaghi's relation ($\Delta\sigma_v = \Delta\sigma' + \Delta u$). The load is more and more transferred towards the solid grains. Then the effective stress tends to the value of the initial stress which is applied at the soil surface.



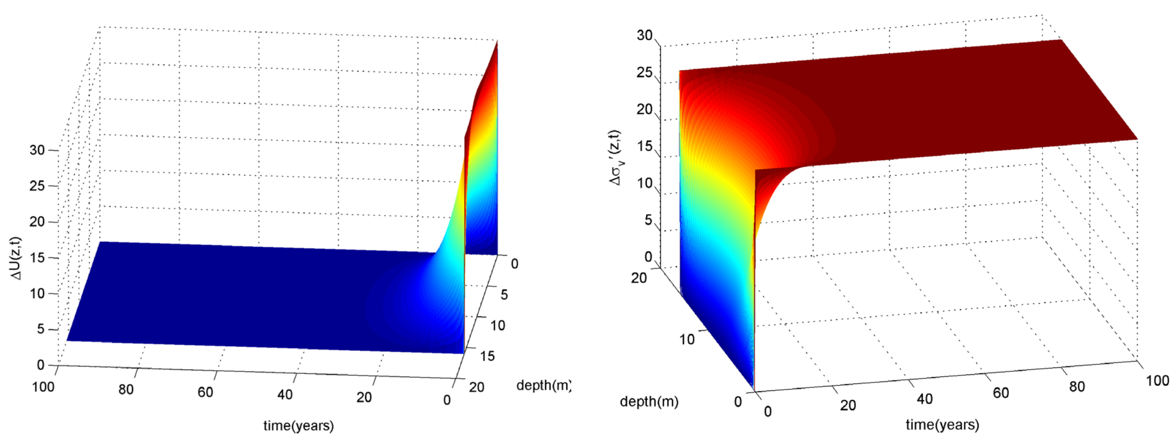
(a) (b)

Figure 7. Evolution of the pore water pressure (a) and the effective stress (b) as a function of depth and time (we consider here tmax = 1 year).



(a) (b)

Figure 8. Evolution of the pore water pressure (a) and the effective stress (b) as a function of depth and time (we consider here t_{max} = 20 years).



(a) (b)

Figure 9. Evolution of the pore water pressure (a) and the effective stress (b) as a function of depth and time (we consider here t_{max} = 100 years).

4. Conclusions

We found for the different parameters used in our simulations that we almost have the same changes in the numerical and analytical resolutions [6] [7]. However, we notice more accuracy on the analytical resolution due to the fact that the numerical resolution gives an approximate solution while the finite differential method imposes some specific conditions that lead to a stable resolution; for the example $\theta < \frac{1}{2}$ for the finite differential explicit method. Nevertheless $\theta = c_v \frac{\Delta t}{\Delta z^2}$ implies the choice of time and space step which combination will respect the explicit scheme rule [8].

The problem of Terzaghi one-dimensional consolidation can be easily solved by analytical and numerical methods; and the solutions resulting from this resolution may also be an interesting subject for numerical simulation [9] in order to highlight more clearly the physical interpretation necessary to better understand soil consolidation.

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Annexes

(1) Analytical solution

-Pore water pressure:

```
function simulation(N,Cv,H,Tmax)
lamda=4*30/pi; %hpa
[z,t]=meshgrid(linspace(0,2*H,50),linspace(0,Tmax,50));
V=0; % V represente la surpression
for m=0:N
V=V+lamda*(1/(2*m+1)*exp((2*m+1)^2*(pi^2)*(Cv*t/(4*H^2))).*sin((2*m+1)*pi*z/(2*H)));
end
mesh(z,t,V)
xlabel depth(m); ylabel time(years); zlabel \DeltaU(z,t)
for i=1:15
view(-100+24*(i-1),30)
m(:,i)=getframe;
end
```

-effective stress:

```
function simulation_efective_stress(N,Cv,H,Tmax)
lamda=4*30/pi; %hpa
[z,t]=meshgrid(linspace(0,2*H,50),linspace(0,Tmax,50));
V=0;
for m=0:N
V=V+lamda*(1/(2*m+1)*exp((2*m+1)^2*(pi^2)*(Cv*t/(4*H^2))).*sin((2*m+1)*pi*z/(2*H)));
end
q=30-V; % q=delta_sigma_prime_v : effective stress
mesh(z,t,q)
xlabel depth(m); ylabel time(years); zlabel \Delta\sigma\prime(z,t)
axis([0 20 0 20 0 30])
```

(2) Numerical solution

-Pore water pressure:

```
clc; clear;
k=4; %k=cv=
dx=0.25;
r=1/4; dt=dx*dx*r/k;
Tmax=100;
a=0;b=16;
cla=0;clb=0;
nx=(b-a)/dx;
nt=Tmax/dt;
x=0:dx:b; t=0:dt:Tmax;
%[x,t]=meshgrid(linspace(0,2*b,50),linspace(0,Tmax,50));
for i=1:nx-1
N(i)=0;
end
N(1)=r*cla;
N(nx-1)=r*clb;
for i=1:nx-2
M(i,i)=1-2*r;
M(i,i+1)=r;
M(i+1,i)=r;
end
M(nx-1,nx-1)=1-2*r;
```

```
for i=1:nx+1
Ci(i)=30;
end
for i=1:nx-1
h(i)=Ci(i+1);
end
j=1;
h=h';
while(j < nt + 2)
for i=1:nx-1
w(i,j)=h(i);
end
h=M*h+N';
j=j+1;
end
for i=nx:-1:2
for j=nt+1:-1:1
w(i,j)=w(i-1,j);
end
end
for j=1:nt+1
w(1,j)=0;
w(nx+1,j)=0;
end
mesh(t,x,w);
xlabel time(t); ylabel depth(m); zlabel \DeltaU(z,t)
for i=1:15
view(-150+24(i-1),30)
m(:,i)=getframe;
end
```

-effective stress:

```
clc; clear;
k=4; %k=cv=
dx=0.25;
r=1/4; dt=dx*dx*r/k;
Tmax=100;
a=0;b=16;
cla=0;clb=0;
nx=(b-a)/dx;
nt=Tmax/dt;
x=0:dx:b; t=0:dt:Tmax;
%[x,t]=meshgrid(linspace(0,2*b,50),linspace(0,Tmax,50));
for i=1:nx-1
N(i)=0;
end
N(1)=r*cla;
N(nx-1)=r*clb;
for i=1:nx-2
M(i,i)=1-2*r;
M(i,i+1)=r;
M(i+1,i)=r;
end
M(nx-1,nx-1)=1-2*r;
```

```
for i=1:nx+1
Ci(i)=30;
end
for i=1:nx-1
h(i)=Ci(i+1);
end
j=1;
h=h';
while(j < nt + 2)
for i=1:nx-1
w(i,j)=h(i);
end
h=M*h+N';
j=j+1;
end
for i=nx:-1:2
for j=nt+1:-1:1
w(i,j)=w(i-1,j);
end
end
for j=1:nt+1
w(1,j)=0;
w(nx+1,j)=0;
end
q=30-w; % q=deltasigmaprime ( i.e effective stress)
mesh(t,x,q)
xlabel time(t); ylabel depth(m); zlabel \Delta\sigma\prime_v(z,t)
for i=1:5
view(-80+24(i-1),30)
m(:,i)=getframe;
end
```


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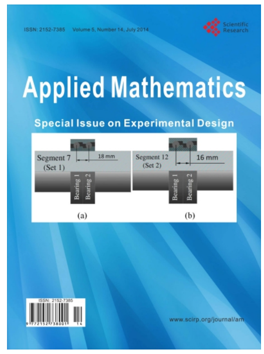
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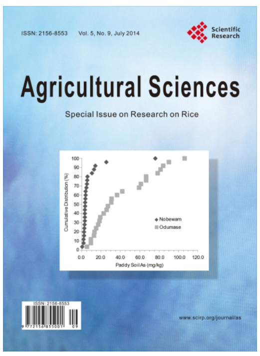
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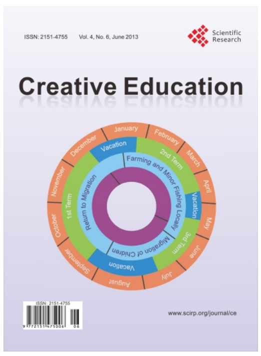
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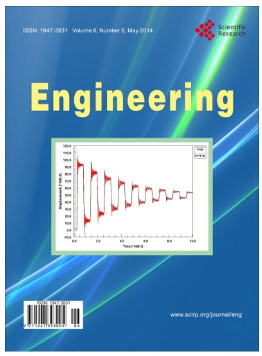
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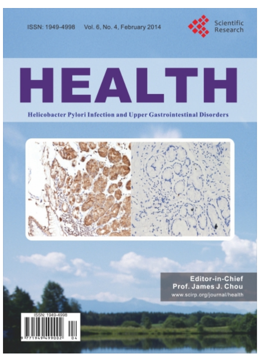
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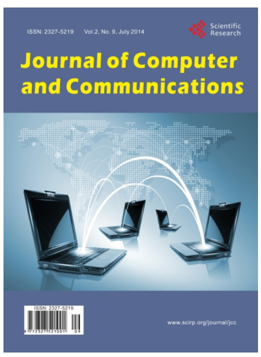
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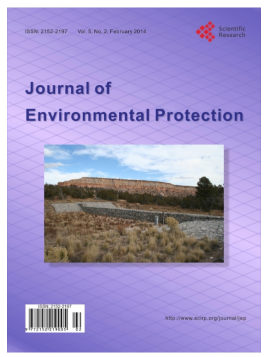
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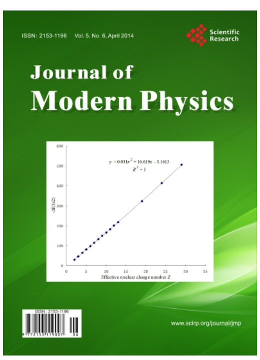
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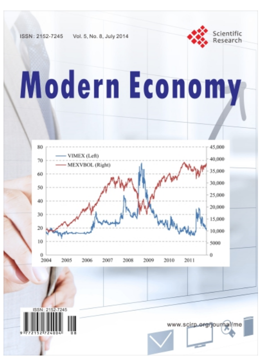
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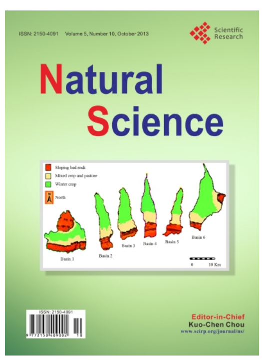
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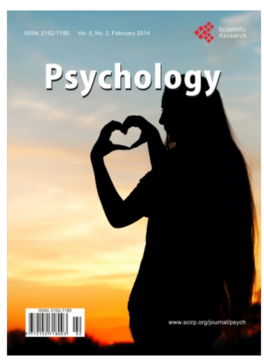
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