

Nonnegative Solutions for a Riemann-Liouville Fractional Boundary Value Problem

Rodica Luca¹, Alexandru Tudorache²

¹Department of Mathematics, Gh. Asachi Technical University, Iasi, Romania

²Faculty of Computer Engineering and Automatic Control, Gh. Asachi Technical University, Iasi, Romania

Email: rluca@math.tuia.si.ro, alexandru.tudorache93@gmail.co

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Abstract

We investigate the existence of nonnegative solutions for a Riemann-Liouville fractional differential equation with integral terms, subject to boundary conditions which contain fractional derivatives and Riemann-Stieltjes integrals. In the proof of the main results, we use the Banach contraction mapping principle and the Krasnosel'skii fixed point theorem for the sum of two operators.

Keywords

Riemann-Liouville Fractional Differential Equations, Nonlocal Boundary Conditions, Nonnegative Solutions, Existence

1. Introduction

We consider the nonlinear fractional differential equation

$$(E) \quad D_{0+}^{\alpha} x(t) + f(t, x(t), Ax(t), Bx(t)) = 0, t \in (0, 1),$$

with the nonlocal boundary conditions

$$(BC) \quad x(0) = x'(0) = \dots = x^{(n-2)}(0) = 0, D_{0+}^{\beta_0} x(1) = \sum_{i=1}^m \int_0^1 D_{0+}^{\beta_i} x(t) dH_i(t),$$

where $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, $n, m \in \mathbb{N}$, $n \geq 3$, $\beta_i \in \mathbb{R}$ for all $i = 0, \dots, m$, $0 \leq \beta_1 < \beta_2 < \dots < \beta_m \leq \beta_0 < \alpha - 1$, $\beta_0 \geq 1$, D_{0+}^k denotes the Riemann-Liouville derivative of order k (for $k = \alpha, \beta_0, \beta_1, \dots, \beta_m$), the integrals from the boundary condition (BC) are Riemann-Stieltjes integrals with $H_i : [0, 1] \rightarrow \mathbb{R}, i = 1, \dots, m$ non-decreasing functions, $Ax(t) = \int_0^t K(t, s)x(s)ds$, and

$$Bx(t) = \int_0^1 H(t, s)x(s)ds \quad \text{for all } t \in [0, 1].$$

We study the existence of nonnegative solutions for problem (E)-(BC) by using the Banach contraction mapping principle and the Krasnosel'skii fixed point

theorem for the sum of two operators. Equation (E) supplemented with the multi-point boundary conditions

$$(BC_1) \quad x(0) = x'(0) = \dots = x^{(n-2)}(0) = 0, D_{0+}^p x(1) = \sum_{i=1}^m a_i D_{0+}^q x(\xi_i),$$

where $a_i, \xi_i \in \mathbb{R}$ for all $i=1, \dots, m$, ($m \in \mathbb{N}$), $0 < \xi_1 < \dots < \xi_m \leq 1$, $p, q \in \mathbb{R}$, $p \in [1, n-2]$, $q \in [0, p]$ was investigated in the paper [1]. The last condition in (BC_1) can be written as $D_{0+}^p x(1) = \int_0^1 D_{0+}^q x(t) dH_0(t)$, where H_0 is the step function defined by

$$H_0(t) = \left\{ 0, \text{ for } t \in [0, \xi_1); a_1, \text{ for } t \in [\xi_1, \xi_2); a_1 + a_2, \text{ for } t \in [\xi_2, \xi_3); \dots; \sum_{i=1}^m a_i, \text{ for } t \in [\xi_m, 1] \right\}$$

So (BC) is a generalization of (BC_1) , because in (BC) we have a sum of Riemann-Stieltjes integrals and various orders for the fractional derivatives. In the paper [2], the authors investigated the existence of nonnegative solutions for the Caputo fractional differential equation

$${}^c D^\alpha x(t) + f(t, x(t), Ax(t), Bx(t)) = 0, t \in (0, 1),$$

with the boundary conditions

$$(BC_2) \quad x(0) = b_0, x'(0) = b_1, \dots, x^{(n-3)}(0) = b_{n-3}, x^{(n-1)}(0) = b_{n-1}, x(1) = \mu \int_0^1 x(s) ds,$$

where $n-1 < \alpha \leq n$, $0 \leq \mu < n-1$, $n \geq 3$, $b_i \geq 0$ ($i=1, 2, \dots, n-3, n-1$), ${}^c D^\alpha$ is the Caputo fractional derivative, and the operators A and B are defined as the operators from our problem, given above. In the paper [3], the authors studied the existence and multiplicity of positive solutions for the Riemann-Liouville fractional differential equation $D_{0+}^\alpha x(t) + f(t, x(t)) = 0, t \in (0, 1)$, subject to the boundary conditions (BC) , where f is a sign-changing function that can be singular in the points $t = 0, 1$ and/or in the variable x . In addition, the methods used in the proofs of the main results in [3] are different than those used in the present paper, namely, in [3] the authors used various conditions which contain height functions of the nonlinearity defined on special bounded sets, and two theorems from the fixed point index theory. For some recent results on the existence, nonexistence and multiplicity of solutions for fractional differential equations and systems of fractional differential equations subject to various boundary conditions we refer the reader to the monographs [4] [5] and the papers [6]-[14]. We also mention the books [15]-[21], and the papers [22]-[28] for applications of the fractional differential equations in various disciplines.

2. Preliminary Results

We present in this section some auxiliary results from [3] that we will use in the proof of the main results. We consider the fractional differential equation

$$D_{0+}^\alpha x(t) + y(t) = 0, t \in (0, 1), \tag{1}$$

with the boundary conditions (BC) , where $y \in C(0, 1) \cap L^1(0, 1)$. We denote by

$$\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta_0)} - \sum_{i=1}^m \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta_i)} \int_0^1 s^{\alpha - \beta_i - 1} dH_i(s).$$

Lemma 1 If $\Delta \neq 0$, then the unique solution $x \in C[0, 1]$ of problem (1)-(BC) is given by

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{t^{\alpha-1}}{\Delta \Gamma(\alpha - \beta_0)} \int_0^1 (1-s)^{\alpha - \beta_0 - 1} y(s) ds - \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha - \beta_i - 1} y(\tau) d\tau \right) dH_i(s), t \in [0, 1]. \tag{2}$$

Lemma 2 If $\Delta \neq 0$, then the solution x of problem (1)-(BC) given by (2) can be written as

$$x(t) = \int_0^1 G(t, s) y(s) ds, t \in [0, 1], \tag{3}$$

where

$$G(t, s) = g_1(t, s) + \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^m \left(\int_0^1 g_{2i}(\tau, s) dH_i(\tau) \right), \tag{4}$$

and

$$g_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} (1-s)^{\alpha - \beta_0 - 1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1} (1-s)^{\alpha - \beta_0 - 1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{5}$$

$$g_{2i}(t, s) = \frac{1}{\Gamma(\alpha - \beta_i)} \begin{cases} t^{\alpha - \beta_i - 1} (1-s)^{\alpha - \beta_0 - 1} - (t-s)^{\alpha - \beta_i - 1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha - \beta_i - 1} (1-s)^{\alpha - \beta_0 - 1}, & 0 \leq t \leq s \leq 1, \end{cases}$$

for all $(t, s) \in [0, 1] \times [0, 1]$, $i = 1, \dots, m$.

By using some properties of the functions $g_1, g_{2i}, i = 1, \dots, m$ given by (5) from [29], we obtain the following lemma.

Lemma 3 We suppose that $\Delta > 0$. Then the function G given by (4) is a continuous function on $[0, 1] \times [0, 1]$ and satisfies the inequalities:

a) $G(t, s) \leq J(s)$ for all $t, s \in [0, 1]$, where

$$J(s) = h_1(s) + \frac{1}{\Delta} \sum_{i=1}^m \int_0^1 g_{2i}(\tau, s) dH_i(\tau), s \in [0, 1], \text{ and}$$

$$h_1(s) = \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha - \beta_0 - 1} (1 - (1-s)^{\beta_0}), s \in [0, 1];$$

b) $G(t, s) \geq t^{\alpha-1} J(s)$ for all $t, s \in [0, 1]$;

c) $G(t, s) \leq \sigma t^{\alpha-1}$, for all $t, s \in [0, 1]$, where

$$\sigma = \frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} \int_0^1 \tau^{\alpha - \beta_i - 1} dH_i(\tau).$$

Lemma 4 We suppose that $\Delta > 0$, $y \in C(0, 1) \cap L^1(0, 1)$ and $y(t) \geq 0$ for all $t \in (0, 1)$. Then the solution x of problem (1)-(BC) given by (3) satisfies the inequality $x(t) \geq t^{\alpha-1} \|x\|$ for all $t \in [0, 1]$, where $\|x\| = \sup_{t \in [0, 1]} |x(t)|$, and so $x(t) \geq 0$ for all $t \in [0, 1]$.

In the proof of our main theorems, we use the Banach contraction mapping principle and the Krasnosel'skii fixed point theorem for the sum of two opera-

tors presented below.

Theorem 1 (see [30]) *If (Y, d) is a nonempty complete metric space with the metric d , and $T : Y \rightarrow Y$ is a contraction mapping, then T has a unique fixed point $x^* \in Y$ ($Tx^* = x^*$).*

Theorem 2 ([31]) *Let M be a closed, convex, bounded and nonempty subset of a Banach space X . Let A_1 and A_2 be two operators such that*

- a) $A_1x + A_2y \in M$ for all $x, y \in M$;
- b) A_1 is a completely continuous operator (continuous, and compact, that is, it maps bounded sets into relatively compact sets);
- c) A_2 is a contraction mapping.

Then there exists $z \in M$ such that $z = A_1z + A_2z$.

3. Main Results

In this section we study the existence of nonnegative solutions for our problem (E)-(BC). We present now the assumptions that we will use in the sequel.

(I) $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, $n, m \in \mathbb{N}$, $n \geq 3$, $\beta_i \in \mathbb{R}$ for all $i = 0, \dots, m$, $0 \leq \beta_1 < \beta_2 < \dots < \beta_m \leq \beta_0 < \alpha - 1$, $\beta_0 \geq 1$, $H_i : [0, 1] \rightarrow \mathbb{R}$, $i = 1, \dots, m$ are nondecreasing functions, an

$$\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta_0)} - \sum_{i=1}^m \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta_i)} \int_0^1 s^{\alpha - \beta_i - 1} dH_i(s) > 0.$$

(II) $f : [0, 1] \times \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ is measurable with respect to t on $[0, 1]$, ($\mathbb{R}_+ = [0, \infty)$).

(III) There exist the functions $a, b, c \in L^1((0, 1), \mathbb{R}_+)$ such that

$$|f(t, x, y, z) - f(t, \bar{x}, \bar{y}, \bar{z})| \leq a(t)|x - \bar{x}| + b(t)|y - \bar{y}| + c(t)|z - \bar{z}|,$$

a.e. $t \in (0, 1)$ and for all $x, y, z, \bar{x}, \bar{y}, \bar{z} \in \mathbb{R}_+$.

(IV) There exists the function $g \in L^1((0, 1), \mathbb{R}_+)$ such that

$$|f(t, x, y, z)| \leq g(t), \text{ a.e. } t \in (0, 1), \forall x, y, z \in \mathbb{R}_+.$$

(V) $K \in C(E, \mathbb{R}_+)$, $E = \{(t, s) \in [0, 1] \times [0, 1], t \geq s\}$, and $H \in C([0, 1] \times [0, 1], \mathbb{R}_+)$.

We denote by $k_0 = \sup_{t \in [0, 1]} \int_0^t K(t, s) ds$ and $h_0 = \sup_{t \in [0, 1]} \int_0^1 H(t, s) ds$.

Theorem 3 *We suppose that assumptions (I)-(V) hold. If $\sigma\omega_1 < 1$, where $\omega_1 = \|a + k_0b + h_0c\|_{L^1}$, then problem (E)-(BC) has at least one nonnegative solution on $[0, 1]$.*

Proof. By (IV) we obtain that the function $s \rightarrow (t-s)^{\alpha-1} f(s, x(s), Ax(s), Bx(s))$ is Lebesgue integrable on $[0, t]$ for all $t \in [0, 1]$ and $x \in C([0, 1], \mathbb{R}_+)$, the function $s \rightarrow (1-s)^{\alpha-\beta_0-1} f(s, x(s), Ax(s), Bx(s))$ is Lebesgue integrable on $[0, 1]$ for all $x \in C([0, 1], \mathbb{R}_+)$, and the function $\tau \rightarrow (s-\tau)^{\alpha-\beta_i-1} f(\tau, x(\tau), Ax(\tau), Bx(\tau))$ is Lebesgue integrable on $[0, s]$ for all $s \in [0, 1]$, $x \in C([0, 1], \mathbb{R}_+)$ and $i = 1, \dots, m$.

We consider now the integral equation

$$\begin{aligned}
 x(t) = & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), Ax(s), Bx(s)) ds \\
 & + \frac{t^{\alpha-1}}{\Delta \Gamma(\alpha - \beta_0)} \int_0^1 (1-s)^{\alpha-\beta_0-1} f(s, x(s), Ax(s), Bx(s)) ds \\
 & - \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\beta_i-1} f(\tau, x(\tau), Ax(\tau), Bx(\tau)) d\tau \right) dH_i(s),
 \end{aligned} \tag{6}$$

or equivalently

$$x(t) = \int_0^1 G(t, s) f(s, x(s), Ax(s), Bx(s)) ds. \tag{7}$$

By Lemma 2 we easily deduce that if x is a solution of Equation (6) (or equivalently (7)), then x is a solution of problem (E)-(BC).

Let $r = \sigma \|g\|_{L^1}$. We define the operator \mathcal{T} on $\bar{B}_r = \{x \in C([0, 1], \mathbb{R}_+), \|x\| \leq r\}$ by

$$\begin{aligned}
 \mathcal{T}x(t) = & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), Ax(s), Bx(s)) ds \\
 & + \frac{t^{\alpha-1}}{\Delta \Gamma(\alpha - \beta_0)} \int_0^1 (1-s)^{\alpha-\beta_0-1} f(s, x(s), Ax(s), Bx(s)) ds \\
 & - \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\beta_i-1} f(\tau, x(\tau), Ax(\tau), Bx(\tau)) d\tau \right) dH_i(s),
 \end{aligned}$$

or equivalently

$$\mathcal{T}x(t) = \int_0^1 G(t, s) f(s, x(s), Ax(s), Bx(s)) ds.$$

If x is a fixed point of operator \mathcal{T} , then x is a solution of Equation (6) (or (7)), and hence x is a solution of problem (E)-(BC). Therefore we will study the existence (and uniqueness) of the fixed points of operator \mathcal{T} by using the Banach contraction mapping principle.

We firstly show that if $x \in \bar{B}_r$, then $\mathcal{T}x \in \bar{B}_r$. Indeed, we have

$$\begin{aligned}
 & |\mathcal{T}x(t + \varepsilon) - \mathcal{T}x(t)| \\
 &= \left| \int_0^1 [G(t + \varepsilon, s) - G(t, s)] f(s, x(s), Ax(s), Bx(s)) ds \right| \\
 &\leq \int_0^1 |G(t + \varepsilon, s) - G(t, s)| g(s) ds \\
 &\leq \max_{s \in [0, 1]} |G(t + \varepsilon, s) - G(t, s)| \cdot \|g\|_{L^1} \rightarrow 0, \text{ for } \varepsilon \rightarrow 0.
 \end{aligned}$$

Hence $\mathcal{T}x$ is a continuous function. By (I1), (I2) and Lemma 4, we obtain $\mathcal{T}x(t) \geq 0$ for all $t \in [0, 1]$, and then $\mathcal{T}x \in C([0, 1], \mathbb{R}_+)$.

In addition, for any $x \in \bar{B}_r$ and all $t \in [0, 1]$, we deduce

$$\begin{aligned}
 (\mathcal{T}x)(t) &= \int_0^1 G(t, s) f(s, x(s), Ax(s), Bx(s)) ds \\
 &\leq \int_0^1 \sigma t^{\alpha-1} f(s, x(s), Ax(s), Bx(s)) ds \\
 &\leq \sigma t^{\alpha-1} \int_0^1 g(s) ds \leq \sigma \|g\|_{L^1} = r,
 \end{aligned}$$

and then $\|\mathcal{T}x\| \leq r$ for all $x \in \bar{B}_r$, so $\mathcal{T} : \bar{B}_r \rightarrow \bar{B}_r$.

We show now that \mathcal{T} is a contraction mapping on \bar{B}_r . For $x_1, x_2 \in \bar{B}_r$, and

any $t \in [0, 1]$, by using (I3), we find

$$\begin{aligned} & |(\mathcal{I}x_1)(t) - (\mathcal{I}x_2)(t)| \\ & \leq \int_0^1 G(t, s) \left| f(s, x_1(s), Ax_1(s), Bx_1(s)) - f(s, x_2(s), Ax_2(s), Bx_2(s)) \right| ds \\ & \leq \int_0^1 G(t, s) \left[a(s) |x_1(s) - x_2(s)| + b(s) |Ax_1(s) - Ax_2(s)| \right. \\ & \quad \left. + c(s) |Bx_1(s) - Bx_2(s)| \right] ds \\ & \leq \|x_1 - x_2\| \sigma t^{\alpha-1} \int_0^1 [a(s) + k_0 b(s) + h_0 c(s)] ds \\ & \leq \sigma \|x_1 - x_2\| \cdot \|a + k_0 b + h_0 c\|_{L^1} = \sigma \omega_1 \|x_1 - x_2\|, \end{aligned}$$

because

$$\begin{aligned} Ax_1(s) - Ax_2(s) &= \int_0^s K(s, \tau) (x_1(\tau) - x_2(\tau)) d\tau \\ &\leq \sup_{\tau \in [0, 1]} |x_1(\tau) - x_2(\tau)| \int_0^s K(s, \tau) d\tau \\ &\leq k_0 \|x_1 - x_2\|, \forall s \in [0, 1], \\ Bx_1(s) - Bx_2(s) &= \int_0^1 H(s, \tau) (x_1(\tau) - x_2(\tau)) d\tau \\ &\leq \sup_{\tau \in [0, 1]} |x_1(\tau) - x_2(\tau)| \int_0^1 H(s, \tau) d\tau \\ &\leq h_0 \|x_1 - x_2\|, \forall s \in [0, 1], \end{aligned}$$

and $a, b, c \in L^1(0, 1)$.

Therefore we obtain the inequality

$$\|\mathcal{I}x_1 - \mathcal{I}x_2\| \leq \sigma \omega_1 \|x_1 - x_2\|.$$

Because $\sigma \omega_1 < 1$, we deduce that \mathcal{T} is a contraction mapping. By Theorem 1, we conclude that \mathcal{T} has a unique fixed point, which is a nonnegative solution of problem (E)-(BC).

In what follows, we denote by

$$\begin{aligned} \omega_2 &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (a(s) + k_0 b(s) + h_0 c(s)) ds, \\ \omega_3 &= \|a + k_0 b + h_0 c\|_{L^1} \left(\frac{1}{\Delta \Gamma(\alpha - \beta_0)} + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} (H_i(1) - H_i(0)) \right). \end{aligned} \tag{8}$$

Theorem 4 We suppose that assumptions (I1),

(I2)' $f : [0, 1] \times \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ is a continuous function and (I3), (I4), (I5) hold. If $\max\{\omega_2, \omega_3\} < 1$, then problem (E)-(BC) has at least one nonnegative solution on $[0, 1]$.

Proof. We define $R \geq R_0$, where

$$\begin{aligned} R_0 &= (1 - \omega_2)^{-1} \left[\frac{\tau_0}{\Gamma(\alpha + 1)} + \|g\|_{L^1} \left(\frac{1}{\Delta \Gamma(\alpha - \beta_0)} \right. \right. \\ & \quad \left. \left. + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} (H_i(1) - H_i(0)) \right) \right], \end{aligned}$$

$\tau_0 = \max\{f(t, 0, 0, 0), t \in [0, 1]\}$ and ω_2 is given by (8). We consider the set $\bar{B}_R = \{x \in C([0, 1], \mathbb{R}_+), \|x\| \leq R\}$. Then \bar{B}_R is a closed, convex and nonempty

set of $C([0,1], \mathbb{R}_+)$. We define the operators Q_1 and Q_2 on \bar{B}_R by

$$(Q_1x)(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), Ax(s), Bx(s)) ds, t \in [0,1],$$

$$(Q_2x)(t) = \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-\beta_0)} \int_0^1 (1-s)^{\alpha-\beta_0-1} f(s, x(s), Ax(s), Bx(s)) ds$$

$$- \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\beta_i-1} f(\tau, x(\tau), Ax(\tau), Bx(\tau)) d\tau \right) dH_i(s), t \in [0,1].$$

By (Ω) , $(\mathcal{D})'$ and Lemma 4, we have $(Q_1x)(t) + (Q_2x)(t) \geq 0$ for all $t \in [0,1]$. For any $x \in \bar{B}_R$, by using (\mathcal{B}) , we find

$$\begin{aligned} & |f(t, x(t), Ax(t), Bx(t))| \\ & \leq |f(t, x(t), Ax(t), Bx(t)) - f(t, 0, 0, 0)| + |f(t, 0, 0, 0)| \\ & \leq a(t)|x(t)| + b(t)|Ax(t)| + c(t)|Bx(t)| + \tau_0, \forall t \in [0,1]. \end{aligned}$$

Then for any $x \in \bar{B}_R$ and all $t \in [0,1]$, we obtain by using the above inequality

$$\begin{aligned} |Q_1x(t)| &= \left| -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), Ax(s), Bx(s)) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (a(s)|x(s)| + b(s)|Ax(s)| + c(s)|Bx(s)| + \tau_0) ds \\ &\leq \frac{\|x\|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (a(s) + k_0b(s) + h_0c(s)) ds + \frac{\tau_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \omega_2 R + \frac{t^\alpha \tau_0}{\Gamma(\alpha+1)} \leq \omega_2 R + \frac{\tau_0}{\Gamma(\alpha+1)}, \end{aligned}$$

because $(Ax)(t) \leq \sup_{t \in [0,1]} x(t) \int_0^t K(t,s) ds \leq k_0 \|x\|$ and

$$(Bx)(t) \leq \sup_{t \in [0,1]} x(t) \int_0^1 H(t,s) ds \leq h_0 \|x\| \text{ for all } t \in [0,1].$$

Therefore for any $z \in \bar{B}_R$ and all $t \in [0,1]$, we deduce by using (\mathcal{A}) that

$$\begin{aligned} & |Q_2z(t)| \\ & \leq \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-\beta_0)} \int_0^1 (1-s)^{\alpha-\beta_0-1} f(s, z(s), Az(s), Bz(s)) ds \\ & \quad + \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\beta_i-1} f(\tau, z(\tau), Az(\tau), Bz(\tau)) d\tau \right) dH_i(s) \\ & \leq \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha-\beta_0)} \int_0^1 (1-s)^{\alpha-\beta_0-1} g(s) ds \\ & \quad + \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\beta_i-1} g(\tau) d\tau \right) dH_i(s) \\ & \leq \frac{1}{\Delta\Gamma(\alpha-\beta_0)} \int_0^1 g(s) ds + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^1 \left(\int_0^1 g(\tau) d\tau \right) dH_i(s) \\ & = \|g\|_{L^1} \left(\frac{1}{\Delta\Gamma(\alpha-\beta_0)} + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)} (H_i(1) - H_i(0)) \right). \end{aligned}$$

Hence for $x, y \in \bar{B}_R$ and $t \in [0, 1]$, we find

$$\begin{aligned} & |Q_1x(t) + Q_2y(t)| \\ & \leq |Q_1x(t)| + |Q_2y(t)| \leq \omega_2R + \frac{\tau_0}{\Gamma(\alpha + 1)} \\ & \quad + \|g\|_{L^1} \left(\frac{1}{\Delta\Gamma(\alpha - \beta_0)} + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} (H_i(1) - H_i(0)) \right) \leq R. \end{aligned}$$

Therefore for $y_1, y_2 \in \bar{B}_R$ and $t \in [0, 1]$, by using (B) we obtain

$$\begin{aligned} & |Q_2y_1(t) - Q_2y_2(t)| \\ & \leq \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha - \beta_0)} \int_0^1 (1-s)^{\alpha-\beta_0-1} |f(s, y_1(s), Ay_1(s), By_1(s)) \\ & \quad - f(s, y_2(s), Ay_2(s), By_2(s))| ds \\ & \quad + \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\beta_i-1} |f(\tau, y_1(\tau), Ay_1(\tau), By_1(\tau)) \right. \\ & \quad \left. - f(\tau, y_2(\tau), Ay_2(\tau), By_2(\tau))| d\tau \right) dH_i(s) \\ & \leq \frac{1}{\Delta\Gamma(\alpha - \beta_0)} \int_0^1 (1-s)^{\alpha-\beta_0-1} (a(s)|y_1(s) - y_2(s)| + b(s)|Ay_1(s) - Ay_2(s)| \\ & \quad + c(s)|By_1(s) - By_2(s)|) ds \\ & \quad + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\beta_i-1} (a(\tau)|y_1(\tau) - y_2(\tau)| \right. \\ & \quad \left. + b(\tau)|Ay_1(\tau) - Ay_2(\tau)| + c(\tau)|By_1(\tau) - By_2(\tau)|) d\tau \right) dH_i(s) \\ & \leq \frac{\|y_1 - y_2\|}{\Delta\Gamma(\alpha - \beta_0)} \int_0^1 (1-s)^{\alpha-\beta_0-1} (a(s) + k_0b(s) + h_0c(s)) ds \\ & \quad + \frac{\|y_1 - y_2\|}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\beta_i-1} (a(\tau) + k_0b(\tau) + h_0c(\tau)) d\tau \right) dH_i(s). \end{aligned}$$

So, we deduce

$$\begin{aligned} & |Q_2y_1(t) - Q_2y_2(t)| \\ & \leq \frac{\|y_1 - y_2\|}{\Delta\Gamma(\alpha - \beta_0)} \int_0^1 (a(s) + k_0b(s) + h_0c(s)) ds \\ & \quad + \frac{\|y_1 - y_2\|}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} \left(\int_0^1 (a(\tau) + k_0b(\tau) + h_0c(\tau)) d\tau \right) (H_i(1) - H_i(0)) \\ & = \|y_1 - y_2\| \cdot \|a + k_0b + h_0c\|_{L^1} \left(\frac{1}{\Delta\Gamma(\alpha - \beta_0)} + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha - \beta_i)} (H_i(1) - H_i(0)) \right) \\ & = \omega_3 \|y_1 - y_2\|, \end{aligned}$$

where ω_3 is given by (8). Because $\omega_3 < 1$, we conclude that Q_2 is a contraction mapping.

By using assumptions (D)' and (E), we deduce that Q_1 is a continuous mapping. In addition, Q_1 is uniformly bounded on \bar{B}_R , because for any $x \in \bar{B}_R$, we find

$$\begin{aligned} |Q_1x(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), Ax(s), Bx(s)) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds \leq \frac{1}{\Gamma(\alpha)} \int_0^1 g(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \|g\|_{L^1}, \forall t \in [0,1], \end{aligned}$$

and then $\|Q_1x\| \leq \frac{1}{\Gamma(\alpha)} \|g\|_{L^1}$ for all $x \in \bar{B}_R$.

The operator Q_1 is also equicontinuous on \bar{B}_R . Indeed, let $x \in \bar{B}_R$, $t_1, t_2 \in [0,1]$, with $t_1 < t_2$. We have

$$\begin{aligned} &|Q_1x(t_2) - Q_1x(t_1)| \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} f(s, x(s), Ax(s), Bx(s)) ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, x(s), Ax(s), Bx(s)) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] f(s, x(s), Ax(s), Bx(s)) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s, x(s), Ax(s), Bx(s)) ds \\ &\leq \phi_0 \left\{ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \right\} \\ &= \frac{\phi_0}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha) \leq \frac{\phi_0(t_2 - t_1)}{\Gamma(\alpha)}, \end{aligned}$$

where $\phi_0 = \sup \{f(t, x, y, z), t \in [0,1], |x| \leq R, |y| \leq k_0R, |z| \leq h_0R\}$. Then we obtain that $|Q_1x(t_2) - Q_1x(t_1)| \rightarrow 0$ as $t_2 - t_1 \rightarrow 0$.

By using the Arzela-Ascoli theorem, we deduce that $Q_1(\bar{B}_R)$ is relatively compact. By Theorem 2, we conclude that operator $Q_1 + Q_2$ has at least one fixed point, and so problem (E)-(BC) has at least one nonnegative solution.

4. An Example

Let $\alpha = 10/3$ ($n = 4$), $\beta_0 = 11/5$, $m = 2$, $\beta_1 = 1/2$, $\beta_2 = 5/4$, $H_1(t) = t^2$ for all $t \in [0,1]$, $H_2(t) = \{0, \text{ for } t \in [0, 1/2); 1, \text{ for } t \in [1/2, 1]\}$.

We consider the fractional differential equation

$$(E_0) \quad D_{0+}^{10/3} x(t) + f(t, x(t), Ax(t), Bx(t)) = 0, t \in (0,1),$$

with the boundary conditions

$$(BC_0) \quad x(0) = x'(0) = x''(0) = 0, D_{0+}^{11/5} x(1) = 2 \int_0^1 t D_{0+}^{1/2} x(t) dt + D_{0+}^{5/4} x\left(\frac{1}{2}\right),$$

where $Ax(t) = \int_0^t K(t,s)x(s) ds$ and $Bx(t) = \int_0^1 H(t,s)x(s) ds$ for all $t \in [0,1]$, with $K(t,s) = e^{-s^2}$ for all $t, s \in [0,1]$ with $s \leq t$, and

$H(t, s) = e^{-2t}(s^3 + 1)$ for all $t, s \in [0, 1]$. Then we obtain $\Delta \approx 0.85599748 > 0$ and $\sigma \approx 1.24445843$. So assumptions (I1) and (I5) are satisfied.

We define the function

$$f(t, x, y, z) = \frac{t+1}{2} + \frac{e^{-t}x}{(1+k)(1+x)} + \frac{e^{-2t}y}{(1+k^2)(1+y)} + \frac{e^{-3t}z}{(1+k^3)(1+z)},$$

for all $t \in [0, 1]$ and $x, y, z \in \mathbb{R}_+$ with $k \geq 1$. We deduce that $k_0 = \frac{1}{3e}$ and

$h_0 = \frac{5}{4}$. Besides we obtain the inequalities

$$|f(t, x, y, z) - f(t, \bar{x}, \bar{y}, \bar{z})| \leq \frac{e^{-t}}{1+k}|x - \bar{x}| + \frac{e^{-2t}}{1+k^2}|y - \bar{y}| + \frac{e^{-3t}}{1+k^3}|z - \bar{z}|,$$

for all $t \in [0, 1]$, $x, y, z, \bar{x}, \bar{y}, \bar{z} \in \mathbb{R}_+$, and

$$|f(t, x, y, z)| \leq \frac{t+1}{2} + \frac{e^{-t}}{1+k} + \frac{e^{-2t}}{1+k^2} + \frac{e^{-3t}}{1+k^3}, \forall t \in [0, 1], x, y, z \in \mathbb{R}_+.$$

We define $a(t) = \frac{e^{-t}}{1+k}$, $b(t) = \frac{e^{-2t}}{1+k^2}$, $c(t) = \frac{e^{-3t}}{1+k^3}$, and

$$g(t) = \frac{t+1}{2} + \frac{e^{-t}}{1+k} + \frac{e^{-2t}}{1+k^2} + \frac{e^{-3t}}{1+k^3}, \text{ for all } t \in [0, 1]. \text{ We have } a, b, c \in L^1(0, 1)$$

and $g \in L^1(0, 1)$. So assumptions (I2), (I3), (I4) are also satisfied.

In addition, we find

$$\omega_1 = \frac{e-1}{e(1+k)} + \frac{e^2-1}{6e^3(1+k^2)} + \frac{5(e^3-1)}{12e^3(1+k^3)} \leq \frac{e-1}{2e} + \frac{e^2-1}{12e^3} + \frac{5(e^3-1)}{24e^3} \approx 0.540529,$$

and so $\omega_1 < 1/\sigma \approx 0.8035624$. Therefore, by Theorem 3, we conclude that problem (E₀)-(BC₀) has at least one nonnegative and nontrivial solution.

5. Conclusion

In this paper, we investigated the existence of nonnegative solutions for the Riemann-Liouville fractional differential equation with integral terms (E) supplemented with the boundary conditions (BC) which contain Riemann-Liouville fractional derivatives of different orders and Riemann-Stieltjes integrals, by using the Banach contraction mapping principle and the Krasnosel'skii fixed point theorem for the sum of two operators. For some future research directions, we have in mind the study of the existence, nonexistence and multiplicity of solutions or positive solutions for fractional differential equations subject to other boundary conditions.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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