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Mean Square Solutions of Second-Order Random Differential Equations by Using the Differential Transformation Method

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Abstract
The differential transformation method (DTM) is applied to solve the second-order random differential equations. Several examples are represented to demonstrate the effectiveness of the proposed method. The results show that DTM is an efficient and accurate technique for finding exact and approximate solutions.

Keywords
Random Differential Equations, Stochastic Differential Equation, Differential Transformation Method

1. Introduction

The ordinary differential equations which contain random constant or random variables are well known topics which are called the random ordinary differential equations. The subject of second-order random differential equations is one of much current interests due to the great importance of many applications in engineering, biology and physical phenomena (see, e.g. Chil'es and Delfiner [1], Cort'es *et al*. [2], Soong [3] and references therein). Recently, several first-order random differential models have been solved by using the Mean Square Calculus [2] [4]-[11]. Variety scientific problems have been modeled by using the nonlinear second-order random differential equations. However, most of these equations cannot be solved analytically. Thus, accurate and efficient numerical techniques are needed. There are several semi-numerical techniques which have been considered to obtain exact and approximate solutions of linear and nonlinear differential equations, such as adomian decomposition method (ADM) [12], variational iteration method (VIM) [13] and homotopy perturbation method (HPM) [14]. We observe that semi-numerical methods are very prevalent in the current literature, cf. [12]-[14].

The object of this work is to describe how to implement the differential transformation method (DTM) for

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finding exact and approximate solutions of the second-order random differential equations. To this end, the second-order random differential equations and the concept of the differential transformation method are presented in Section 2. In Section 3, we consider the statistical functions of the mean square solution of the second-order random differential equation. Section 4 is devoted to numerical examples.

2. Differential Transform Method

The differential transform method (DTM) has been used by Zhou [15]. This method is effective to obtain exact and approximate solutions of linear and nonlinear differential equations. To describe the basic ideas of DTM, we consider the second order random differential equation,

$$L\left[x(t)\right]+N\left[x(t),A\right]=g(t), \tag{1}$$

$$x(0)=y_0,\ \left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=0}=y_1 \tag{2}$$

where $L\left[x(t)\right]=\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}$, $x(t)$ is an unknown function, $N\left[x(t),A\right]$ is a nonlinear operator, $g(t)$ is the source in homogeneos term, and A, y_0 and y_1 are random variables.

We now write the differential transform of function as

$$X(k)=\frac{1}{k!}\left.\frac{\mathrm{d}^kx(t)}{\mathrm{d}t^k}\right|_{t=0} \tag{3}$$

In fact, $x(t)$ is a differential inverse transform of the form

$$x(t)=\sum_{k=0}^{\infty}X(k)t^k \tag{4}$$

It is clear from (3) and (4) that the concept of differential transform is derived from Taylor series expansion. That is

$$x(t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left.\frac{\mathrm{d}^kx(t)}{\mathrm{d}t^k}\right|_{t=0}t^k \tag{5}$$

Original function	Transformed function
$x(t)\pm y(t)$	$X(k)\pm Y(k)$
$cx(t)$	$cX(k)$
$y(t)=\frac{\mathrm{d}x(t)}{\mathrm{d}t}$	$Y(k)=(k+1)X(k+1)$
$y(t)=\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}$	$Y(k)=(k+1)(k+2)X(k+2)$
$u(t)=x(t)y(t)$	$U(k)=\sum_{s=0}^{k}X(s)Y(k-s)$
$x(t)=t^s$	$X(k)=\begin{cases}1 & k=s\\ 0 & k\neq s\end{cases}$
$y(t)=t\frac{\mathrm{d}x(t)}{\mathrm{d}t}$	$Y(k)=kX(k)$
$y(t)=t\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}$	$Y(k)=k(k+1)X(k+1)$
$y(t)=t^2\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}$	$Y(k)=k(k-1)X(k)$

Differential transform for some functions.

Notes that, the derivatives in differential transform method does not evaluate symbolically.

In keeping with Equations (3) and (4), let $X(k)$, $Y(k)$ and $U(k)$, respectively, are the transformed functions of $x(t)$, $y(t)$ and $u(t)$. The fundamental mathematical operations of differential transformation are listed in the following table.

3. Statistical Functions of the Mean Square Solution

Before proceeding to find the computation of the main statistical functions of the mean square solution of Equations (1) and (2) we briefly clarify some concept, notation, and results belonging to the so-called L_P calculus. The reader is referred to the books by Soong [3], Loeve [16], and Wong and Hajek [17]. Throughout the paper, we deal with the triplet Probabilistic space (Ω, F, P). Thus, suppose $L_2 = L_2(\Omega, F, P)$ is the set of second order random variables. Then the random variable $X:\Omega \to R \in L_2$, if $E\left[X^2\right] < \infty$, where $E[\bullet]$ is an expectation operator. The norm on is denoted by $\|\bullet\|_2$. For example, for the random variable X we define $\|X\|_2 = \left(E\left[X^2\right]\right)^{\frac{1}{2}}$, in such way that $\left(L_2, \|X\|_2\right)$ is a Banach space. In addition, let T (real interval) represent the space of times, we say that $\{X(t), t \in T\}$ is a second order stochastic process, if the random variable $X(t) \in L_2$ for each $t \in T$.

A sequence of second order random variables $\{X_n, n \geq 0\}$ converges to $X(t) \in L_2$, if

$$\lim_{n\to\infty}\|X_n - X\| = \lim_{n\to\infty}\left(E\left[X_n - X^2\right]^{\frac{1}{2}}\right) = 0.$$

To proceed from (4), we truncate the expansion of at the term as follows

$$x_N(t) = \sum_{k=0}^{N} X(k) t^k \tag{6}$$

By using the independence between A, y_0 and y_1 we have

$$E\left[x_N(t)\right] = \sum_{k=0}^{N} E\left[X(k)\right] t^k \tag{7}$$

$$V\left[x_N(t)\right] = \sum_{j=0}^{N}\sum_{i=0}^{N} \operatorname{cov}\left(X(i), X(j)\right) t^{i+j} \tag{8}$$

where $\operatorname{cov}\left(X(i), X(j)\right) = E\left(X(i)X(j)\right) - E\left[X(i)\right]E\left[X(j)\right], \forall i, j = 0, 1, \cdots, N.$

The following Lemma guarantee the convergent of the sequence $E\left[X_N(t)\right]$ to $E\left[X(t)\right]$ and the sequence $V\left[X_N(t)\right]$ to $V\left[X(t)\right]$ if the sequence the $X_N(t)$ converges to $X(t)$.

Lemma [5]: Let $\{X_N\}$ and $\{Y_N\}$ be two sequences of 2-r.vs X and Y, respectively, *i.e.*, $\lim_{N\to\infty} X_N = X$ and $\lim_{N\to\infty} Y_N = Y$ then $\lim_{N\to\infty} E\left[X_N Y_N\right] = E[XY]$. If $X_N = Y_N$, then $\lim_{N\to\infty} E\left[X_N^2\right] = E\left[X^2\right]$, $\lim_{N\to\infty} E\left[X_N\right] = E[X]$ and $\lim_{N\to\infty} V\left[X_N\right] = V[X]$.

4. Numerical Examples

In this section, we adopt several examples to illustrate the using of differential transform method for approximating the mean and the variance.

Example 1: Consider random initial value problem $\dfrac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + A^2 x(t) = 0$, $x(0) = Y_0$ and $\left.\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=0} = Y_1$ where $A^2 \sim Be(\alpha = 2, \beta = 1)$ and independently of the initial conditions Y_0 and Y_1 which satisfy $E\left[Y_o\right] = 1$, $E\left[Y_o^2\right] = 2$, $E\left[Y_1\right] = 1$, $E\left[Y_1^2\right] = 3$ and $E\left[Y_0 Y_1\right] = 0$.

The approximate mean and variance are

$$E\left[x(t)\right] = 1 + t - \frac{1}{3}t^2 - \frac{1}{9}t^3 + \frac{1}{48}t^4 + \frac{1}{240}t^5 - \frac{1}{1800}t^6 - \frac{1}{12600}t^7 + \frac{1}{120960}t^8 + \cdots$$

$$V\left[x(t)\right]=1-2t+\frac{4}{3}t^2+\frac{8}{9}t^3-\frac{19}{72}t^4-\frac{67}{540}t^5+\frac{103}{4050}t^6+\frac{41}{4725}t^7-\frac{247}{172800}t^8-\cdots$$

Figure 1 explain the Graph of the expectation approximation solution by using DTM with n = 18, while **Figure 2** explain the Graph of variance approximation solution by using DTM with n = 18.

Example 2: Consider random initial value problem $\frac{d^2x(t)}{dt^2}+Atx(t)=0$, $X(0)=Y_0$ and $\left.\frac{dx(t)}{dt}\right|_{t=0}=Y_1$ where A is a Beta r.v. with parameters $\alpha=2$ and $\beta=3$, *i.e.* $A\sim Be(\alpha=2,\beta=3)$ and the initial conditions Y_o and Y_1 are independent r.v.'s such as $E\left[Y_o\right]=1$, $E\left[Y_o^2\right]=2$, $E\left[Y_1\right]=2$, $E\left[Y_1^2\right]=5$.

The approximate mean and variance are

$$E\left[x(t)\right]=1+2t-\frac{1}{15}t^3-\frac{1}{15}t^4+\frac{1}{900}t^6+\frac{1}{1260}t^7-\frac{1}{113400}t^9-\frac{1}{198450}t^{10}+\cdots$$

$$V\left[x(t)\right]=1+t^2-\frac{2}{15}t^3-\frac{1}{15}t^5+\frac{2}{225}t^6+\frac{1}{450}t^7+\frac{83}{25200}t^8-\frac{83}{283500}t^9-\frac{2}{18375}t^{10}-\cdots$$

Figure 3 explain the Graph of the expectation approximation solution by using DTM with n = 18, while **Figure 4** explain the Graph of variance approximation solution by using DTM with n = 18.

Example 3: Consider the problem $\frac{d^2x(t)}{dt^2}+2A\frac{dx(t)}{dt}+A^2x(t)=0$, $x(0)=Y_0$ and $\left.\frac{dx(t)}{dt}\right|_{t=0}=Y_1$ where A is a Beta r.v. with parameters $\alpha=2$ and $\beta=1$, *i.e.* $A\sim Be(\alpha=2,\beta=1)$ and independently of the initial conditions Y_o and Y_1 which are independent r.v.' satisfy $E\left[Y_o\right]=1$, $E\left[Y_o^2\right]=2$, $E\left[Y_1\right]=1$, $E\left[Y_1^2\right]=1$.

The approximate mean and variance are

$$E\left[x(t)\right]=1+t-\frac{11}{12}t^2+\frac{23}{60}t^3-\frac{13}{120}t^4+\frac{59}{2520}t^5-\frac{83}{20160}t^6+\frac{37}{60480}t^7-\frac{143}{1814400}t^8+\cdots$$

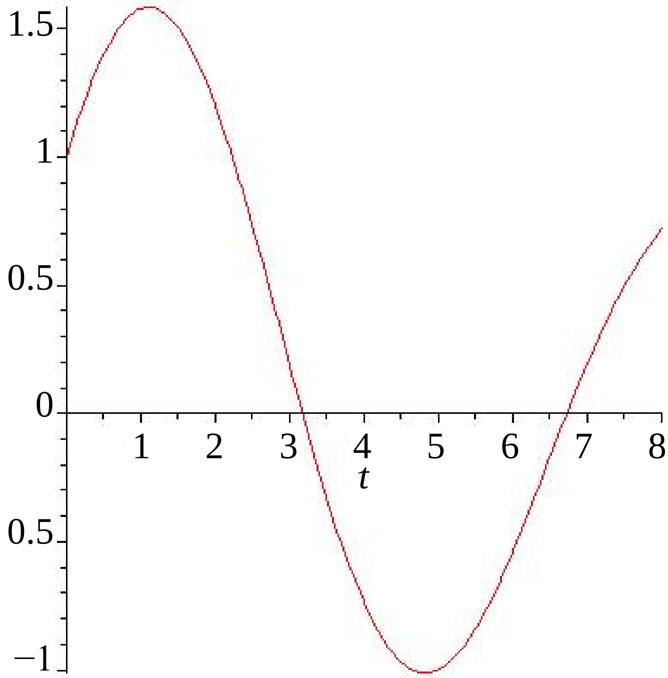

Figure 1. Graphs of the expectation approximation solution of the DTM with n = 18.

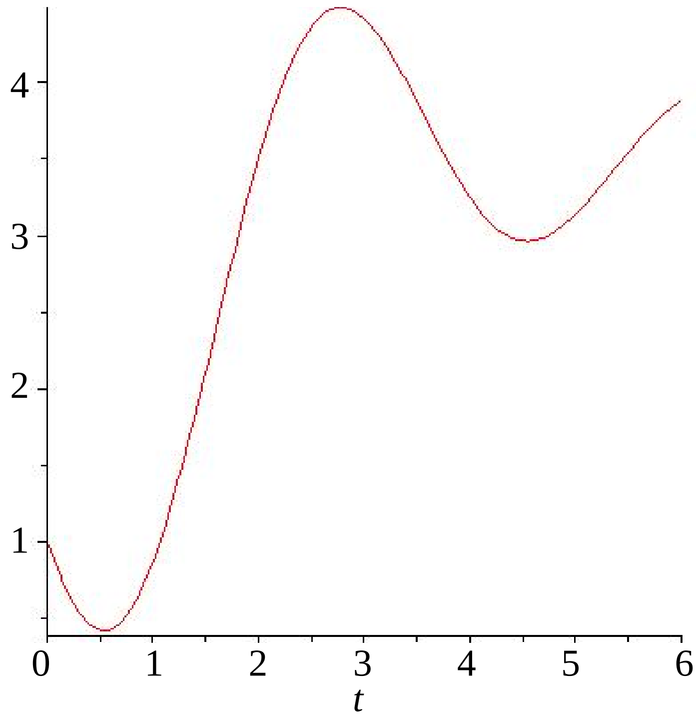

Figure 2. Graphs of variance approximation solution of the DTM with $n = 18$.

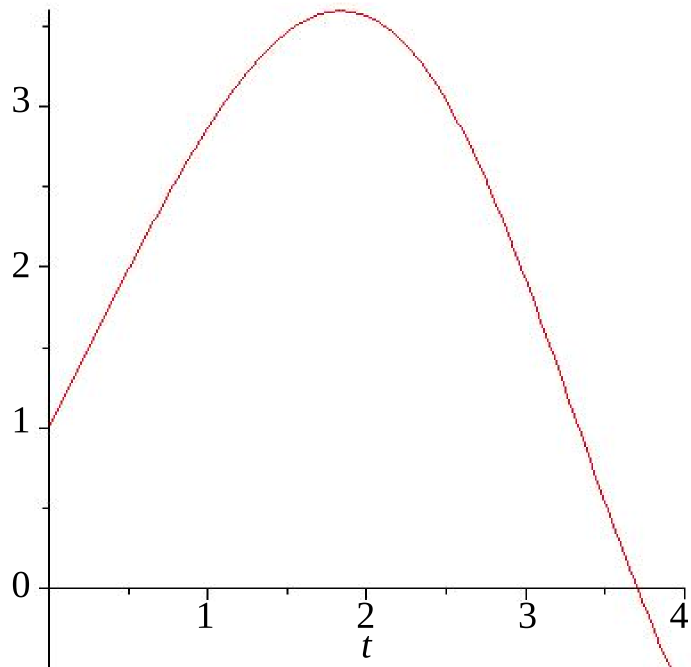

Figure 3. Graphs of the expectation approximation solution of the DTM with $n = 18$.

$$V\left[x(t)\right]=1-\frac{1}{2}t^2+\frac{4}{15}t^3+\frac{103}{720}t^4-\frac{649}{2520}t^5+\frac{8959}{50400}t^6-\frac{3133}{37800}t^7+\frac{517}{17280}t^8-\cdots$$

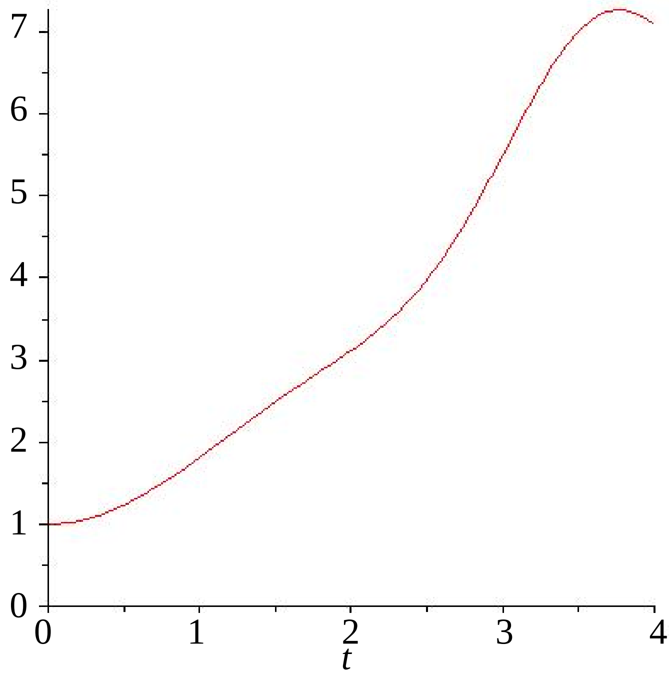

Figure 4. Graphs of variance approximation solution of the DTM with n = 18.

Figure 5 explain the Graph of the expectation approximation solution by using DTM with n = 18, while **Figure 6** explain the Graph of variance approximation solution by using DTM with n = 18.

Example 4: Consider the problem $\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + Atx(t) = 0$, $X(0) = Y_0$ and $\left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=0} = Y_1$ where A is a uniform r.v. with parameters $\alpha = 0$ and $\beta = 1$, *i.e.* $A \sim U(\alpha = 0, \beta = 1)$ and independently of the initial conditions Y_o and Y_1 which are independent r.v.'s satisfy $E[Y_o] = 1$, $E[Y_o^2] = 2$, $E[Y_1] = 1$, $E[Y_1^2] = 1$.

The approximate mean and variance are

$$E[x(t)] = 1 + t - \frac{11}{12}t^2 + \frac{23}{60}t^3 - \frac{13}{120}t^4 + \frac{59}{2520}t^5 - \frac{83}{20160}t^6 + \frac{37}{60480}t^7 - \frac{143}{1814400}t^8 + \cdots$$

$$V[x(t)] = 1 + \frac{1}{432}t^6 - \frac{1}{1440}t^8 + \frac{131}{1008000}t^{10} - \frac{41}{2177280}t^{12} + \frac{71471}{31294771200}t^{14} \cdots$$

Figure 7 explain the Graph of the expectation approximation solution by using DTM with n = 18, while **Figure 8** explain the Graph of variance approximation solution by using DTM with n = 18.

Example 5: Consider the problem $\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + Ax(t) = 0$, $X(0) = Y_0$ and $\left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=0} = Y_1$ where A is a uniform r.v. with parameters $\alpha = 0$ and $\beta = 2$, *i.e.* $A \sim U(\alpha = 0, \beta = 2)$ and independently of the initial conditions Y_o and Y_1 which are independent r.v.'s satisfy $E[Y_o] = 1$, $E[Y_o^2] = 4$, $E[Y_1] = 1$, $E[Y_1^2] = 2$.

The approximate mean and variance are

$$E[x(t)] = 1 + t - \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{18}t^4 + \frac{1}{90}t^5 - \frac{1}{360}t^6 - \frac{1}{2520}t^7 + \frac{1}{12600}t^8 + \frac{1}{113400}t^9 - \cdots$$

$$V[x(t)] = 3 - 2t^2 + \frac{13}{12}t^4 + \frac{1}{18}t^5 - \frac{61}{270}t^6 - \frac{2}{135}t^7 + \frac{599}{22680}t^8 + \frac{101}{56700}t^9 - \cdots$$

Figure 9 explain the Graph of the expectation approximation solution by using DTM with n = 18, while **Figure 10** explain the Graph of variance approximation solution by using DTM with n = 18.

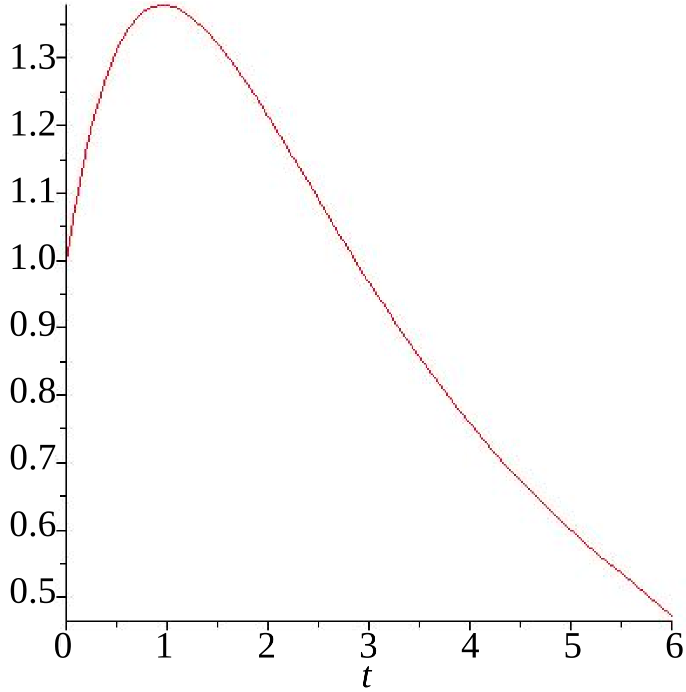

Figure 5. Graphs of the expectation approximation solution of the HAM with $n = 18$.

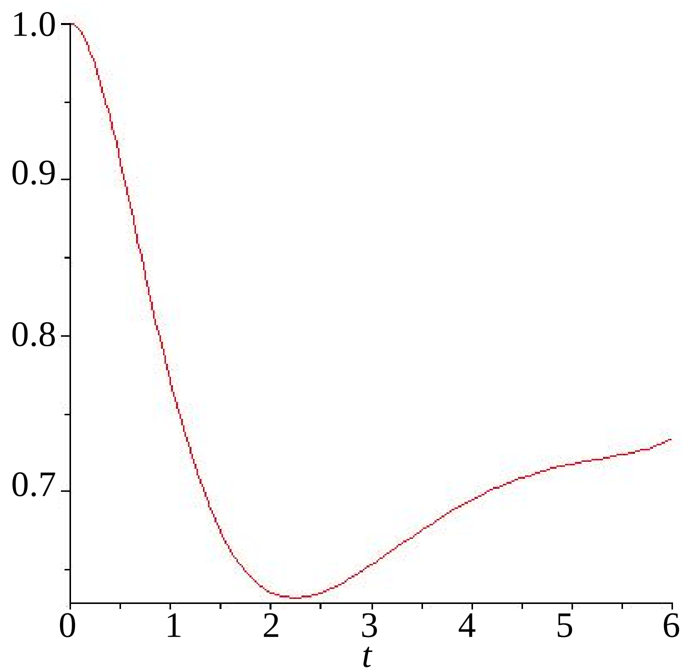

Figure 6. Graphs of variance approximation solution of the HAM with $n = 18$.

Example 6: Consider the problem $\frac{d^2x(t)}{dt^2} + Ax(t) = -x(t) + \sin(t)$, $x(0) = Y_0$ and $\left.\frac{dX(t)}{dt}\right|_{t=0} = Y_1$ where A is a uniform r.v. with parameters $\alpha = 1$ and $\beta = 2$, *i.e.* $A \sim U(\alpha = 1, \beta = 2)$ and independently of the

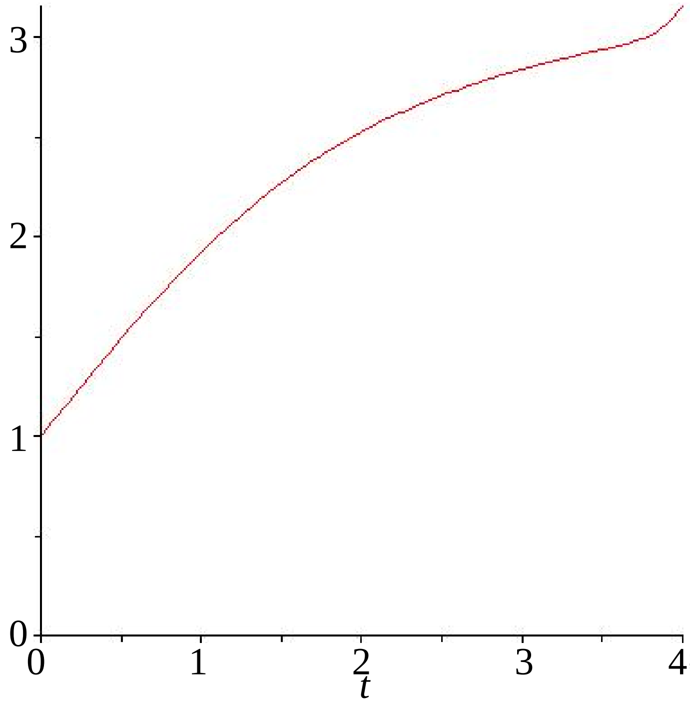

Figure 7. Graphs of the expectation approximation solution of the DTM with $n = 18$.

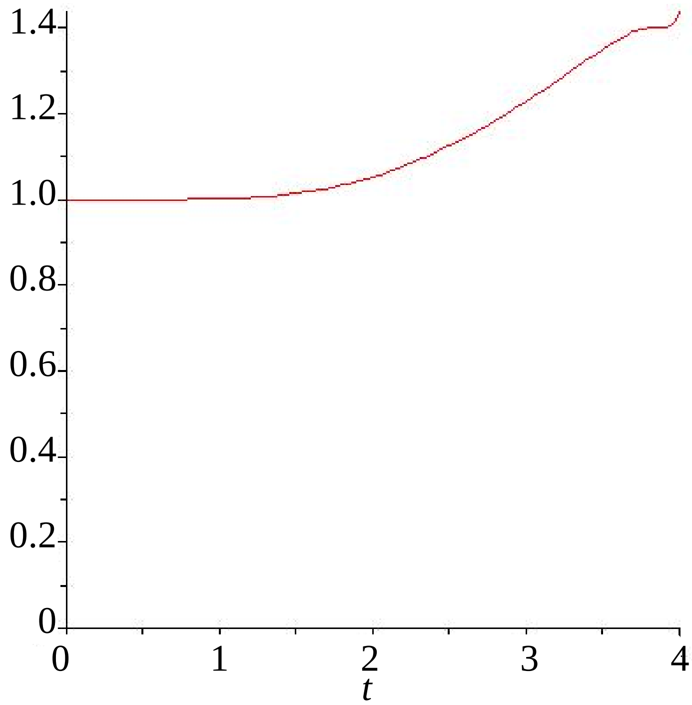

Figure 8. Graphs of variance approximation solution of the DTM with $n = 18$.

initial conditions Y_o and Y_1 which satisfy $E[Y_o]=1$, $E[Y_o^2]=2$, $E[Y_1]=1$, $E[Y_1^2]=6$ and $E[Y_0Y_1]=0$.

The approximate mean and variance are

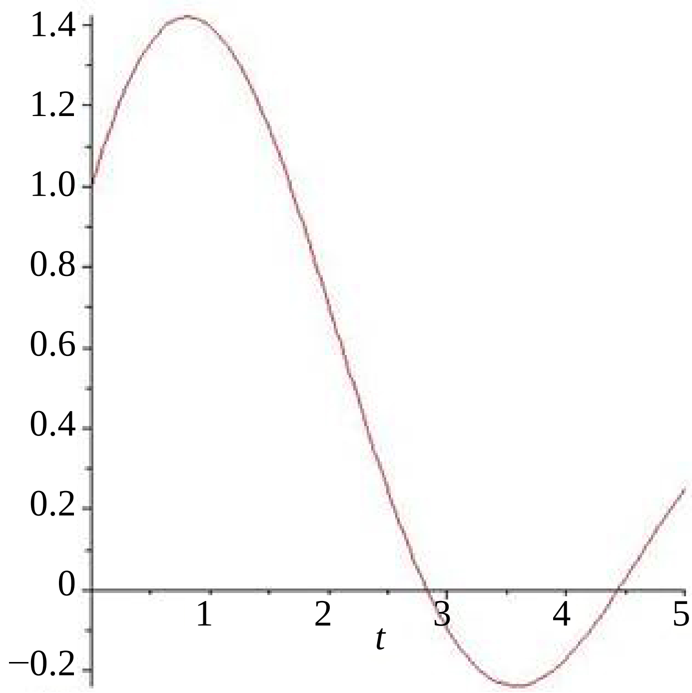

Figure 9. Graphs of the expectation approximation solution of the DTM with $n = 18$.

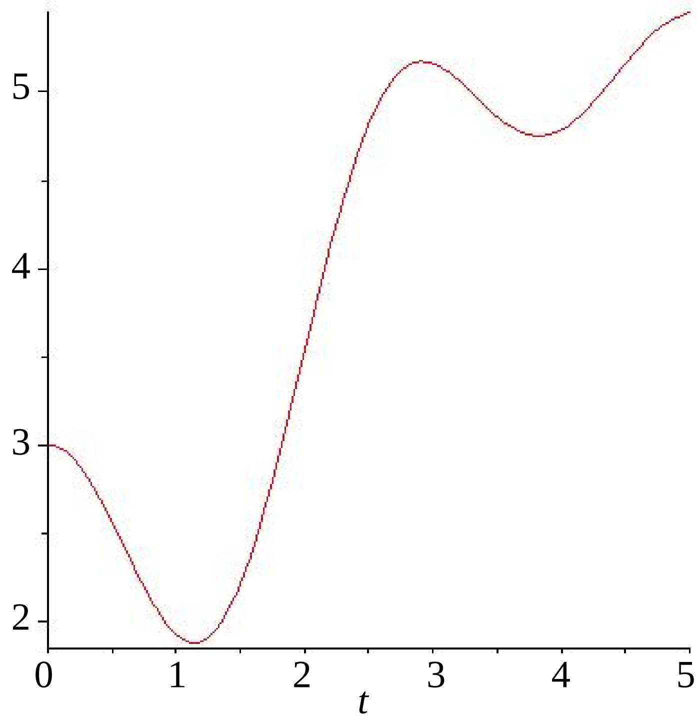

Figure 10. Graphs of variance approximation solution of the DTM with $n = 18$.

$$E\left[x(t)\right]=1+t-\frac{5}{4}t^2-\frac{1}{4}t^3+\frac{19}{72}t^4+\frac{17}{720}t^5-\frac{13}{576}t^6-\frac{11}{8640}t^7+\frac{211}{201600}t^8+\cdots$$

$$V\left[x(t)\right]=1-2t+\frac{5}{2}t^2+\frac{10}{3}t^3-\frac{293}{144}t^4-\frac{67}{40}t^5+\frac{185}{288}t^6+\frac{12221}{30240}t^7-\frac{198419}{1814400}t^8-\cdots$$

Figure 11 explain the Graph of the expectation approximation solution by using DTM with n = 18, while **Figure 12** explain the Graph of variance approximation solution by using DTM with n = 18.

5. Conclusion

In this paper, we successfully applied the differential transform method to solve the second-order random

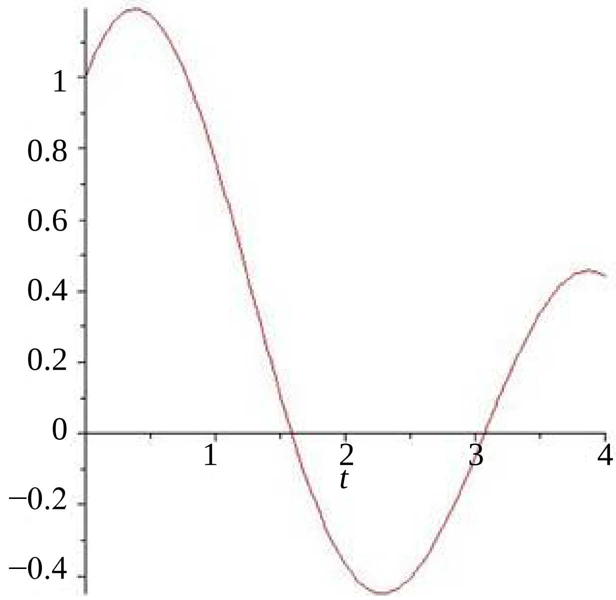

Figure 11. Graphs of the expectation approximation solution of the DTM with n = 18.

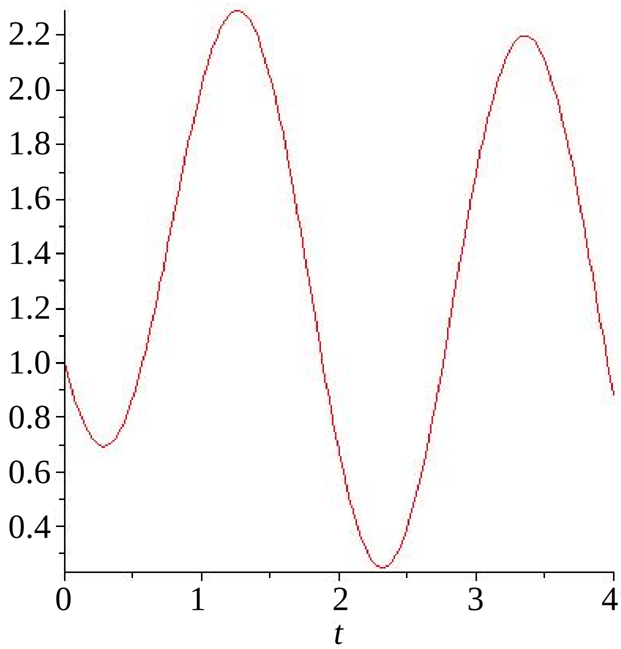

Figure 12. Graphs of variance approximation solution of the DTM with n = 18.

differential Equations (1)-(2) with coefficients which depend on a random variable A which has been assumed to be independent of the random initial conditions y_0 and y_1. This includes the computation of approximations of the mean and variance functions to the random solution. These approximations not only agree but also improve those provided by the Adomian Decomposition Method [12], Variational Iteration Method [13] and Homotopy Perturbation Method [14] as we have illustrated through different examples. Otherwise, the differential transform method is very effective and powerful tools for the second-order random differential equation because it is a direct way without using linearization, perturbation or restrictive assumptions.

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