

Numerical Solution of Second-Order Linear Fredholm Integro-Differential Equations by Trigonometric Scaling Functions

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Abstract

The main aim of this paper is to apply the Hermite trigonometric scaling function on $[0, 2\pi]$ which is constructed for Hermite interpolation for the linear Fredholm integro-differential equation of second order. This equation is usually difficult to solve analytically. Our approach consists of reducing the problem to a set of algebraic linear equations by expanding the approximate solution. Some numerical example is included to demonstrate the validity and applicability of the presented technique, the method produces very accurate results, and a comparison is made with existing results. An estimation of error bound for this method is presented.

Keywords

Numerical Technique, Fredholm Integro-Differential Equations, Hermite Trigonometric Wavelets, Operational Matrix, Error Estimates

1. Introduction

In this paper we solve the Fredholm Linear Integro-Differential Equations as

$$\mu_0(x) + \mu_1(x)u'(x) + \mu_2(x)u''(x) = g(x) + \int_0^1 k(x,t)u(t)dt, \quad 0 \leq x \leq 1, \quad u(0) = u_0, \quad u(1) = u_1 \quad (1)$$

where $\mu_i(x)$, $f(x)$, and $K_i(x,t)$ are given functions that have suitable derivatives, and u_0 and u_1 are given real constants. In most situations, it is difficult to obtain exact solution of the above integration. Hence various approximation method have been proposed and studied. The purpose of the present paper is to develop a trigonometric Hermite wavelet approximation for the computing of the problem [1].

Systems of integro-differential equations have a major role in the fields of science, physical phenomena, and engineering, such as nano-hydrodynamics, glass-forming process, dropwise condensation, wind ripple in the desert, and modeling the competition between tumor cells and the immune system. The concept of a system of integro-differential equations has motivated a huge amount of research work in recent years. A lot of attention has been devoted to the study of differential-difference equations, e.g. equations containing shifts of the unknown function and its derivatives, and also integro-differential-difference equations. For instance, see [2] [3]. There are several numerical methods for solving system of linear integro-differential equations, for example, the rationalized Haar functions method [4], Galerkin methods with hybrid functions [5], the spline approximation method [6], the Chebyshev polynomial method [7], the spectral method [8], the CAS wavelet method [9], Runge-Kutta methods [10], the Adomian decomposition methods [11], and the interested reader can see [12] [13] for more published research works in the subject.

Our approach consists of reducing the problem to a set of linear equations by trigonometric scaling functions which is constructed for Hermite interpolation. A difficulty of using wavelet for the representation of integral operators is that quadrature leads to potentially high cost with sparse matrix. This fact particularly encourages us in efforts to devote to some appropriate wavelet bases to simplify the computation expense of the representation matrix, which is important to improve the wavelet method. Recently, the trigonometric interpolant wavelet has arisen in the approximation of operators [14]-[16]. Quack [17] has constructed a multiresolution analysis (MRA). Chen [18] [19] presented the feasibility of trigonometric wavelet numerical methods for Stokes problem and Hadamard integral equation.

The organization of the rest of this paper is as follows: Section (0) describe the trigonometric scaling function on $[0, 2\pi]$, and construct the operational matrix of derivative for these function. Section (0) summarizes the application of trigonometric scaling functions to the solution of Problem (1). Thus, a set of linear equations is formed and a solution of the considered problem is introduced. In Section (0), we report our computational results and demonstrate the accuracy of the proposed numerical schemes by presenting numerical examples. Note that we have computed the numerical results by MATLAB programming.

2. Interpolatory Hermite Trigonometric Wavelets

In this section, we will give a brief introduction of Quack's work on the construction of Hermite interpolatory trigonometric wavelets and their basic properties. More details can be found in (see [17]).

For all $j \in \mathbb{N} = \mathbb{N} \cup \{0\}$, two scaling functions $\varphi_{j,0}^0(x)$ and $\varphi_{j,0}^1(x)$ are defined as

$$\varphi_{j,0}^0(x) = \frac{1}{2^{2j+1}} \sum_{l=0}^{2^{j+1}-1} D_l(x) \tag{2}$$

$$\varphi_{j,0}^1(x) = \frac{1}{2^{2j+1}} \left(\tilde{D}_{2^{j+1}}(x) + \frac{1}{2} \sin(2^{j+1}x) \right) \tag{3}$$

where the Dirichlet kernel $D_l(x)$ and its conjugate kernel $\tilde{D}_l(x)$ are defined as

$$D_l(x) = \frac{1}{2} + \sum_{k=1}^l \cos kx = \begin{cases} \frac{\sin\left(l + \frac{1}{2}\right)x}{2 \sin \frac{x}{2}} & x \notin 2\pi\mathbb{Z} \\ l + \frac{1}{2} & x \in 2\pi\mathbb{Z} \end{cases}$$

$$\tilde{D}_l(x) = \sum_{k=1}^l \sin kx = \begin{cases} \frac{\cos \frac{x}{2} - \cos\left(l + \frac{1}{2}\right)}{2 \sin \frac{x}{2}} & x \notin 2\pi\mathbb{Z} \\ 0 & x \in 2\pi\mathbb{Z} \end{cases}$$

Obviously, $D_l(x), \tilde{D}_l(x) \in T_l$, where T_l is the linear space of trigonometric polynomials with degree not exceeding l . The equally spaced nodes on the interval $[0, 2\pi)$ with a dyadic step are denoted by $t_{j,l} = \frac{l\pi}{2^j}$, for

any $j \in \mathbb{N}_0$ and $l = 0, 1, \dots, 2^{j+1} - 1$, where \mathbb{N}_0 is the set of all non-negative integers. Let $\varphi_{j,l}^s(x) = \varphi_{j,0}^s(x - t_{j,l})$, for $s = 0, 1$, and $l = 0, 1, \dots, 2^{j+1} - 1$.

Lemma 1 (See [17].) For $j \in \mathbb{N}_0$, we have

$$\varphi_{j,0}^0(x) = \begin{cases} \frac{1}{2^{2j+2}} \frac{\sin^2(2^j x)}{\sin^2\left(\frac{x}{2}\right)} & x \notin 2\pi\mathbb{Z} \\ 1 & x \in 2\pi\mathbb{Z} \end{cases}$$

$$\varphi_{j,0}^1(x) = \begin{cases} \frac{1}{2^{2j+2}} (1 - \cos(2^{j+1}x)) \cot\left(\frac{x}{2}\right) & x \notin 2\pi\mathbb{Z} \\ 0 & x \in 2\pi\mathbb{Z} \end{cases}$$

and their derivations are given by

$$\left(\varphi_{j,0}^0(x)\right)' = \begin{cases} \frac{1}{2^{j+2}} \frac{\sin(2^{j+1}x)}{\sin^2\left(\frac{x}{2}\right)} - \frac{1}{2^{2j+2}} \frac{\sin^2(2^j x) \cot\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} & x \notin 2\pi\mathbb{Z} \\ 0 & x \in 2\pi\mathbb{Z} \end{cases}$$

$$\left(\varphi_{j,0}^1(x)\right)' = \begin{cases} \frac{1}{2^{2j+3}} \frac{\cos(2^{j+1}x) - 1}{\sin^2\left(\frac{x}{2}\right)} + \frac{1}{2^{j+1}} \sin(2^{j+1}x) \cot\left(\frac{x}{2}\right) & x \notin 2\pi\mathbb{Z} \\ 1 & x \in 2\pi\mathbb{Z} \end{cases}$$

Theorem 1 (Interpolatory properties of the scaling functions). (See [17]) For $j \in \mathbb{N}_0$, the following interpolatory properties hold for each $l, k = 0, 1, \dots, 2^{j+1} - 1$

$$\begin{aligned} \varphi_{j,l}^0(x_{j,k}) &= \delta_{k,l}, & \left(\varphi_{j,l}^0(x_{j,k})\right)' &= 0, \\ \varphi_{j,l}^1(x_{j,k}) &= 0, & \left(\varphi_{j,l}^1(x_{j,k})\right)' &= \delta_{k,l}. \end{aligned} \tag{4}$$

From above we can take wavelet functions $\varphi_{j,l}^0(x), \varphi_{j,l}^1(x), l = 0, 1, \dots, 2^{j+1} - 1$ as scaling functions. Now, we can define the scaling function spaces V_j . Then we have

Definition 2 (Scaling functions space). For all $j \in \mathbb{N}_0$ define the wave space V_j as follows

$$V_j = \text{span}\{\varphi_{j,n}^0(t), \varphi_{j,n}^1(t), n = 0, 1, \dots, 2^{j+1} - 1\}$$

As a first step of studying the spaces V_j , the following result identifies the trigonometric polynomials which from alternative bases of these spaces.

Now a Hermite-type project operator can be introduced by means of the scaling functions. For all $j \in \mathbb{N}_0$ the Hermite projection operators L_j mapping any real-valued differentiable 2π -periodic function f into the space V_j is defined as

$$f(x)L_j f(x) \simeq \sum_{k=0}^{2^{j+1}-1} [a_k \varphi_{j,k}^0(x) + b_k \varphi_{j,k}^1(x)] = C^T \Phi \tag{5}$$

where $a_k = f(t_{j,k})$, $b_k = f'(t_{j,k})$, C , and Φ are vectors with dimension $2^{j+2} \times 1$. The following properties of the operators L_j are therefore obvious:

$$L_j \in T_{2^{j+1}}$$

$$L_j f(x_{j,k}) = f(x_{j,k}) \text{ and } (L_j f)'(x_{j,k}) = f'(x_{j,k}), k \in \mathbb{Z}$$

$$L_j f = f \text{ for all } f \in V_j.$$

Theorem 3 Let $f(x) \in L^2_{2\pi}$, and its trigonometric wavelet approximation is $L_j f$, then we have

$$\|f(x) - L_j f(x)\|_{L^2_{2\pi}} \leq C 2^{-2(j+1)}$$

where C is a positive constant value.

Proof. See [17].

Lemma 2 (The operational matrix of scaling function derivative). (See [20]) The differentiation of vector Φ in 5 can be expressed as [20]

$$\Phi' = D_\phi \Phi$$

where D_ϕ is $2^{j+2} \times 2^{j+2}$ operational matrix of derivative for trigonometric scaling function. Suppose

$$(\phi_{j,m}^s(x))' = \sum_{k=0}^{2^{j+1}-1} [a_{k,m}^s \phi_{j,k}^0(x) + b_{k,m}^s \phi_{j,k}^1(x)] \tag{6}$$

where $m = 0, 1, \dots, 2^{j+1} - 1$ and $s = 0, 1$. So the matrix D_ϕ can be represented as a block matrix as

$$D = \begin{bmatrix} A^0 & B^0 \\ A^1 & B^1 \end{bmatrix}$$

where A^s and B^s are $2^{j+1} \times 2^{j+1}$ matrices. The entries of matrices A^s and B^s may be find by using

$$A^s = (a_{k,m}^s) = (\phi_{j,k}^s)'(x_{j,m}), \quad B^s = (b_{k,m}^s) = (\phi_{j,k}^s)''(x_{j,m})$$

where A^0 is a $2^{j+1} \times 2^{j+1}$ zero matrix, A^1 is a $2^{j+1} \times 2^{j+1}$ identity matrix. Using $x_{j,m} = \frac{m\pi}{2^j}$ we get

$$\phi_{j,k}^s(x_{j,m}) = \phi_{j,0}^s\left(\frac{(m-k)\pi}{2^j}\right) = \phi_{j,0}^s(x_{j,m-k}) \tag{7}$$

Using 7 and $B^s = (b_{k,m}^s) = (\phi_{j,k}^s(x_{j,m}))'' = (\phi_{j,0}^s(x_{j,m-k}))''$ $s = 0, 1$ we get

$$B^0 = (b_{k,m}^0) = (\phi_{j,0}^0)''(x_{j,m-k}) = \begin{cases} \frac{1}{2} \frac{\cos((m-k)2\pi)}{\sin^2\left(\frac{m-k}{2^{j+1}}\pi\right)} & k \neq m \\ \frac{-1}{2^{2j+1}} \sum_{p=0}^{2^{j+1}-1} \sum_{k=1}^p \frac{p(p+1)(2p+1)}{6} & k = m \end{cases} \tag{8}$$

and

$$B^1 = (b_{k,m}^1) = (\phi_{j,0}^1)''(x_{j,m-k}) \begin{cases} -\cot\left(\frac{m-k}{2^{j+1}}\pi\right) & k \neq m \\ 0 & k = m \end{cases} \tag{9}$$

for $k, m = 0, 1, \dots, 2^{j+1} - 1$.

3. Procedure Solution Using the Trigonometric Scaling Function

In this section, we first give the computational schemes for Equation (1) with the Newton-Cotes formulas. For either one of these rules, we can make a more accurate approximation by breaking up the interval $[0,1]$ into some number N of subintervals. This is called a composite rule, extended rule, or iterated rule. For example, the composite trapezoidal rule for the discretization form of (1) can be stated as

$$\sum_{i=0}^2 \mu_i(x) u^{(i)}(x) = g(x) + \frac{1}{N} \left(\frac{k(x,0)u(0)}{2} + \sum_{k=1}^{N-1} \left(k \left(x, \frac{k}{N} \right) u \left(\frac{k}{N} \right) \right) + \frac{k(x,1)u(1)}{2} \right), \quad (10)$$

$$0 \leq x \leq 1, \quad u(0) = u_0, \quad u(1) = u_1$$

where the subintervals have the form $[kh, (k+1)h]$, with $h = \frac{1}{N}$ and $k = 0, 1, \dots, N-1$. By introducing a basis $\{\phi_{j,k}\}$ for the subspace V_j , the coefficients vector $u(x)$ of the discrete solution is defined by

$$GL_j u(x) = \sum_{k=0}^{2^{j+1}-1} [a_k \phi_{j,k}^0(x) + b_k \phi_{j,k}^1(x)] = C^T \Phi \quad (11)$$

where C is $2^{j+2} \times 1$ unknown vector defined similar to (5). By substituting $L_j u(x)$ and using Lemma (2) in (1) we have a linear system. Now for determining unknown coefficients vector C or a_k and b_k , we choose collocation method with choosing collocation points as

$$t_\tau = a + \frac{\tau(b-a)}{2^{j+2}} = \frac{\tau\pi}{2^{j+1}}; \quad \tau = 0, 1, \dots, 2^{j+2} - 1 \quad (12)$$

Thus we have

$$\begin{aligned} & \sum_{i=1}^2 \mu_i(t_\tau) L_j u^{(i)}(t_\tau) - \frac{1}{N} \sum_{k=1}^{N-1} \left(k \left(t_\tau, \frac{k}{N} \right) L_j u \left(\frac{k}{N} \right) \right) \\ & = \frac{1}{2N} (k(t_\tau, 0) L_j u_0 + k(t_\tau, 1) L_j u_1) + g(t_\tau) - \mu_0(t_\tau), \end{aligned} \quad (13)$$

By using Lemma 2 and after summarizing Equation (13) can be rewritten as the matrix form $ZX = F$ where $k = 0, 1, \dots, 2^{j+1} - 1$, $i, j = 1, \dots, 2^{j+1}$, and $\tau = 0, 1, \dots, 2^{j+2} - 1$. Now, let us calculate the entries $z_{m,s}^{m,s}$, $m, s = 1, 2$ and $f(t_\tau)$ in the system matrix.

$$Z_{2^{j+2} \times 2^{j+2}} = \begin{bmatrix} Z_{1,1} & Z_{1,2} \\ Z_{2,1} & Z_{2,2} \end{bmatrix}$$

$$Z_{1,1} = \{\mu_1\} B^0 - \left\{ \frac{1}{N} \sum_{k=1}^{N-1} \left(k \left(t_\tau, \frac{k}{N} \right) L_j u \left(\frac{k}{N} \right) \right) \right\} I = (z_{i,j}^{1,1})_{2^{j+1} \times 2^{j+1}}$$

$$Z_{2,1} = \{\mu_1\} B^1 + \left\{ \mu_0 - \frac{1}{N} \sum_{k=1}^{N-1} \left(k \left(t_\tau, \frac{k}{N} \right) L_j u \left(\frac{k}{N} \right) \right) \right\} I = (z_{i,j}^{2,1})_{2^{j+1} \times 2^{j+1}}$$

$$Z_{1,2} = B^0 \{(\mu_0 + \mu_1 B^1)\} - \left\{ \frac{1}{N} \sum_{k=1}^{N-1} \left(k \left(t_\tau, \frac{k}{N} \right) L_j u \left(\frac{k}{N} \right) \right) \right\} I = (z_{i,j}^{1,2})_{2^{j+1} \times 2^{j+1}}$$

$$Z_{2,2} = \{\mu_0\} B^1 + \mu_1 (B^0 + B^{12}) - \left\{ \frac{1}{N} \sum_{k=1}^{N-1} \left(k \left(t_\tau, \frac{k}{N} \right) L_j u \left(\frac{k}{N} \right) \right) \right\} I = (z_{i,j}^{2,2})_{2^{j+1} \times 2^{j+1}}$$

$$X_{2^{j+2} \times 1} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad X_1 = (a_k)_{2^{j+1} \times 1} \quad X_2 = (b_k)_{2^{j+1} \times 1}$$

$$F = (f(t_\tau))_{2^{j+2} \times 1} \quad f(t_\tau) = \frac{1}{2N} (k(t_\tau, 0) L_j u_0 + k(t_\tau, 1) L_j u_1) + g(t_\tau) - \mu_0(t_\tau)$$

where the matrix A^0 , A^1 , B^0 , and B^1 defined in Lemma (2), and I is a $2^{j+1} \times 2^{j+1}$ identity matrix. So the unknown function $u_j(x)$ can be found. Note that we find these function by MATLAB.

4. Numerical Example

To support our theoretical discussion, we applied the method presented in this paper to several examples. The

main objective here is to solve these two examples using the trigonometric scaling function and compare our results with exact solution.

Example 4 Consider the second-order the Fredholm Linear Integro-Differential Equation

$$y''(x) + 4xy' = -\frac{8x^4}{(x^2 + 1)^3} - 2 \int_0^1 \frac{t^2 + 1}{(x^2 + 1)^2} y(t) dt$$

with the mixed conditions $y(0) = 1$ and $y(1) = \frac{1}{2}$. The exact solution of this problem is $(x^2 + 1)^{(-1)}$. We

apply the suggested method with $J = 1$ and $J = 2$. The behavior of the approximate solution using the proposed method with $J = 1, J = 2$ and the exact solution are presented in **Figure 1**. In **Table 1**, we give the errors E_j of matrix A for different values of J . From this figure, it is clear that the proposed method can be considered as an efficient method to solve the linear integral equations. From **Table 1** we see the errors decrease rapidly as J increase.

In **Table 2** we compare the new method with $J = 1, J = 2$ and $J = 3$ together with the exact solution. For the purpose of comparison we defined the maximum error for $u_j(x)$ as

$$E_j = \|u_j(x) - u(x)\|_{\infty} = \max \{|u_j(x) - u(x)|, a \leq x \leq b\}$$

Example 5 Consider the following second-order the Fredholm Linear Integro-Differential Equation

$$x^2 u''(x) + 50xu' = \frac{1 - e^{(x+1)}}{x+1} (x^2 + 50x - 35)e^x + \int_0^1 e^{-xt} u(t) dt$$

with the initial conditions $u(0) = 1, u(1) = e$ and exact solution e^x . This problem is solved by the same methods applied in example (4). Results are shown in **Figure 2**. From this figure, it is clear that the proposed method can be considered as an efficient method to solve the linear integral equations. For the purpose of comparison in **Table 3** we give the errors E_j of matrix A for different values of J . From **Table 4** we see the errors decrease rapidly as J increase. In **Table 4** we compare the new method with $J = 1, J = 2$ and $J = 3$ together with the exact solution.

Table 1. The maximum error matrix A from Example 4.

J	$J = 1$	$J = 2$	$J = 3$	$J = 4$	$J = 5$
E_j	2.5×10^{-3}	3.5×10^{-5}	5.2×10^{-6}	2.6×10^{-7}	5.8×10^{-8}

Table 2. Error analysis and numerical results of Example 4.

t_i	Exact solution	u_j for $J = 1$	u_j for $J = 2$	u_j for $J = 3$
0	1.00000000e-00	1.00000000e-00	1.00000000e-00	1.00000000e-00
0.125	9.84643786e-01	9.81250761e-01	9.84057813e-01	9.84643237e-01
0.2500	9.41218794e-01	9.4075171e-01	9.41102894e-01	9.41218075e-01
0.3750	8.76735484e-01	8.76170942e-01	8.76731207e-01	8.76730782e-01
0.5000	8.00001246e-01	8.00180793e-01	8.00080476e-01	8.00001107e-01
0.6250	7.19101507e-01	7.18017064e-01	7.19852079e-01	7.19101672e-01
0.7500	6.40018961e-01	6.31094827e-01	6.4180973e-01	6.40018843e-01
0.8750	5.66434082e-01	5.50789164e-01	5.66027815e-01	5.66434729e-01

Table 3. The maximum error matrix A from Example 5.

J	$J = 1$	$J = 2$	$J = 3$	$J = 4$	$J = 5$
E_j	2.9×10^{-5}	5.2×10^{-6}	5.9×10^{-7}	2.2×10^{-7}	1.2×10^{-8}

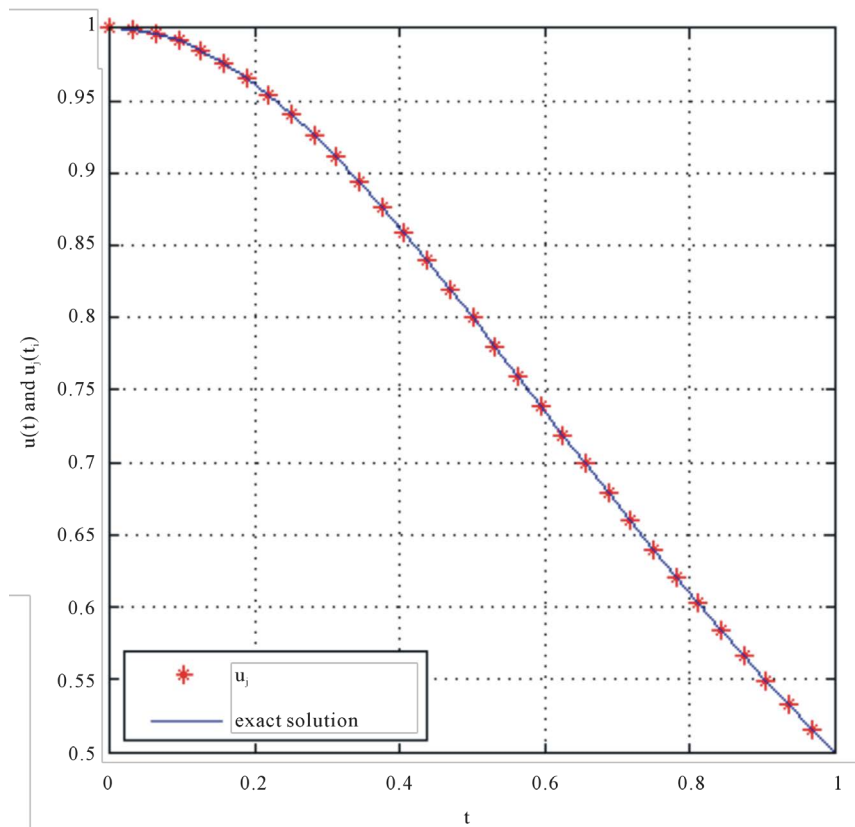
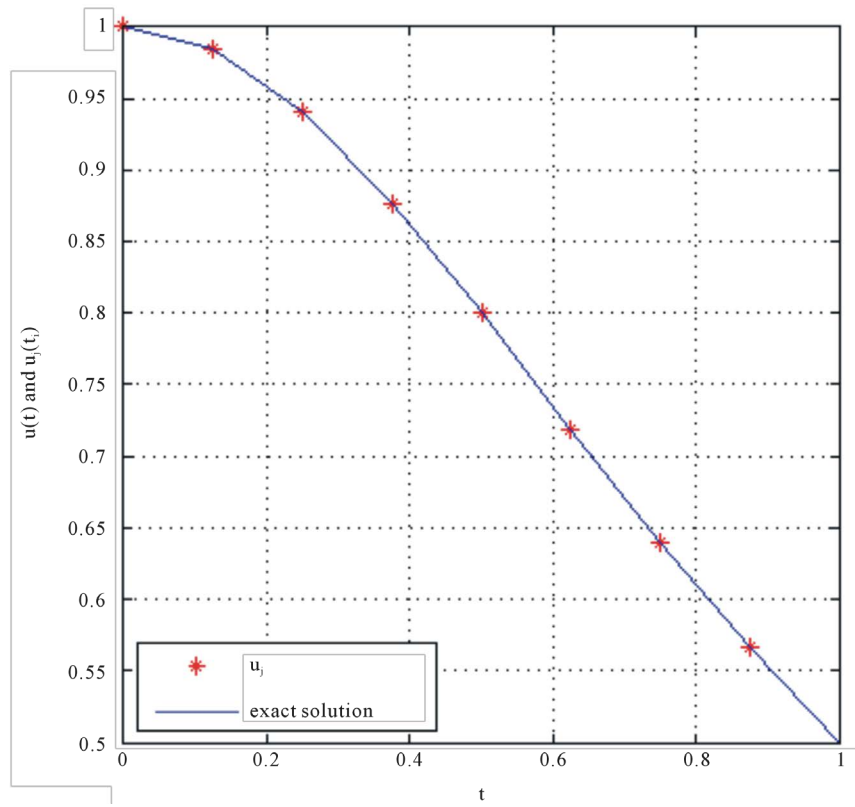


Figure 1. Result EX.4 for $J = 1$ and $N = 7$; Result EX.4 for $J = 2$ and $N = 7$.

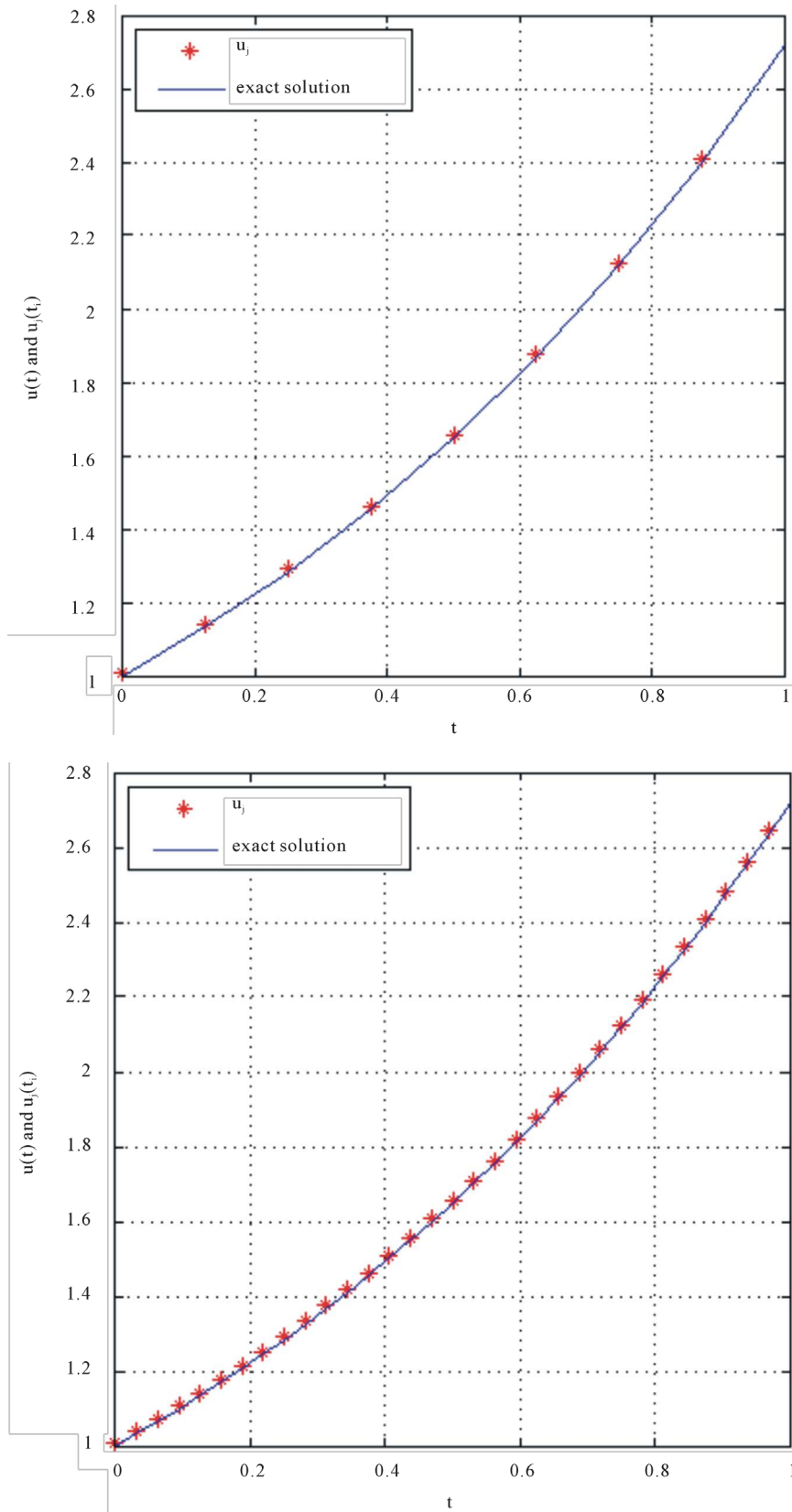


Figure 2. Result EX.5 for $J = 1$ and $N = 7$; Result EX.5 for $J = 2$ and $N = 7$.

Table 4. Error analysis and numerical results of Example 5.

t_i	Exact solution	u_j for $J = 1$	u_j for $J = 2$	u_j for $J = 3$
0	1.00000000	1.00104921	1.00002430	1.00000013
0.125	1.13310019	1.13048310	1.13307891	1.13310007
0.2500	1.28400186	1.28178029	1.28314720	1.28400134
0.3750	1.45500374	1.45304829	1.45456781	1.45500352
0.5000	1.64870391	1.64910273	1.64820756	1.64870372
0.6250	1.86820582	1.86705912	1.86835405	1.86820565
0.7500	2.11704076	2.11078316	2.11527804	2.11704052
0.8750	2.39893108	2.39719802	2.39884240	2.39893214

5. Conclusion

Our results indicate that the method with the trigonometric scaling bases can be regarded as a structurally simple algorithm that is conventionally applicable to the numerical solution of IDEs. In addition, although we have restricted our attention to linear Fredholm IDEs, we expect the method to be easily extended to more general IDEs. The presented method which is based on the trigonometric scaling function is proposed to find the approximate solution. A comparison of the exact solution reveals that the presented method is very effective and convenient. Nevertheless, as **Figure 1** and **Figure 2** illustrate, the error of the trigonometric scaling bases shows that the accuracy improves with increasing J , hence for better results, using number J is recommended. Also from the obtained approximate solution, we can conclude that the proposed method gives the solution in an excellent agreement with the exact solution. All computations are done using MATLAB programming.

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