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Some Rearrangement Inequalities on Space of Homogeneous Type

Tiejun Chen

Yiyang Medical College Hunan Pro of China, Yiyang, China
Email: cwwlove@sina.com

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Abstract

Let ω be a A_∞ Muckenhoupt weight. In this paper we get the estimate of rearrangement f_ω^* in homogeneous space that is $f_\omega^*(t) \le 2\left(M_\lambda^\# f\right)_\omega^*(2t) + f_\omega^*(2t)$ $(0 < t < \infty)$. The similar estimate is obtained only on space of R^n.

Keywords

Rearrangement, Homogeneous Space, A_∞ Weight

1. Introduction

We first recall some basic notions about the homogeneous space and the weights we are going to use.

Definition 1 [1]**.** (Homogeneous space *X*). Let *X* be a set. A function *d*: $X \times X \to [0,\infty)$ is called a quasi-distance on *X* if the following conditions are satisfied:

1) for every *x* and *y* in *X*, $d(x,y) \ge 0$, and $d(x,y) = 0$ if and only if *x* = *y*,

2) for every *x* and *y* in *X*, $d(x,y) = d(y,x)$,

3) there exists a constant *K* such that $d(x,y) \le K\left(d(x,z) + d(z,y)\right)$ for every *x*, *y* and *z* in *X*.

Let μ be a positive measure on the σ-algebra of subsets of *X* generated by the *d*-balls $B(x,r) = \{y : d(x,y) < r\}$, with $x \in X$ and *r* > 0. Then a structure (*X*, *d*, μ), with *d* and μ as above, is called a space of homogeneous type.

We say that (*X*, *d*, μ) is a space of homogeneous type regular in measure if μ is regular, that is for every measurable set *E*, given $\varepsilon > 0$, there exists an open set *G* such that $E \subset G$ and $\mu(G - E) < \varepsilon$. In what follows we always assume that the space (*X*, *d*, μ) is regular in measure.

A non-negative locally integrable on homogeneous space *X* function $\omega(x)$ is called a weight. With any weight function we call the measure $\omega(E)=\int_E \omega(x)\mathrm{d}x$. Given a measurable function *f* on homogeneous space *X*, define its non-increasing rearrangement f_ω^* with respect to a weight ω similar to (see [1], p. 32).

$$f_\omega^*(t)=\sup_{\omega(E)=t}\inf\left|f(x)\right| \quad \left(0<t<\omega\left(R^n\right)\right). \tag{1}$$

Definition 2 (A_∞ weight) [2]. A weight ω is in Muckenhoupt's class A_∞ respect to μ if there are positive constants *C* and ε such that the inequality:

$$\frac{\omega(E)}{\omega(B)}\le C\left(\frac{\mu(E)}{\mu(B)}\right)^{\varepsilon}$$

holds for every ball *B* and every measurable set $E\subset B$. The infimum of such *C* will be denoted by $[\omega]A_\infty$.

2. Basic Lemmas

Denote doubling condition *D*, a weight $\omega\in D$ if and only if for any ball holds $\omega(2B)\le C_2\omega(B)$. Clearly if $\omega\in A_\infty$ then $\omega\in D$.

Lemma 1 [3]. Let (*X*, *d*, μ) be a space of homogeneous type. Let $\mathrm{B}=\{B_\alpha:\alpha\in\Gamma\}$ be a family of balls in *X* such that $E=\bigcup_{\alpha\in\Gamma}B_\alpha$ is measurable and $\mu(E)<\infty$. Then there exists a disjoint sequence $\{B(x_i,r_i)\}\subset\mathrm{B}$, possibly finite, such that $E\subset\bigcup_{i=1}B(x_i,Cr_i)$ for some constant *C*. Moreover, every $B\in\mathrm{B}$ is contained in some $B(x_i,Cr_i)$.

Lemma 2. (C-Z decomposition) [4] [5]. Let (*X*, *d*, μ) be a space of homogeneous type such that the open balls are open sets. Let f be a nonnegative integrable function defined on *X*, then for every $\lambda\ge m_X(f)$ ($m_X(f)=0$ if $\omega(X)=\infty$), there exist a sequence of disjoint balls $B_i=B(x_i,r_i)$ such that if $B_i=B(x_i,Cr_i)$, *C* is the constant in Lemma [1] then

1) $m_{\widetilde{B(f)}}\le\lambda<m_{B_i}(f)$,

2) $m_{\widetilde{B(f)}}\le\lambda$ for every ball *B* centered at $x\in X\setminus\bigcup_i\widetilde{B_i}$, holds $m_B(f)\le\lambda$.

Lemma 3. $\omega\in D$ and $0<\lambda<1$, If *X* is a ball and $E\subset X$ is an arbitrary measurable set of positive measure with $\omega(E)$ $\lambda\omega(X)$, there exist mutually disjoint balls $\{B_i\}\subset X$ such that

B_i cover *E* and

$$\omega(E\cap B_i)/\omega(B_i)>S\ge\omega\left(E\cap\widetilde{B_i}\right)/\omega\left(\widetilde{B_i}\right).$$

Proof: If

$$\lambda\ge mx(f)=\frac{1}{\omega(X)}\int_X f(x)\omega(x)\mathrm{d}\mu(x).$$

Letting $f(x)=\chi E(x)$, then

$$m_{\widetilde{B_i}}(\chi E)\le\lambda<mB_i(\chi E)$$

then

$$\omega(E\cap B_i)/\omega(B_i)>S\ge\omega\left(E\cap\widetilde{B_i}\right)/\omega\left(\widetilde{B_i}\right).$$

For every ball *B* centered at $x\in X\setminus U_i\widetilde{B_i}$

$$m_B(\chi E)\le\lambda$$

i.e.

$$\frac{1}{\omega(B)}\int_B\chi E(x)\omega(x)\mathrm{d}u(x)\le\lambda,$$

$$\omega(E\cap B)/\omega(B)\le\lambda.$$

If $E \subset \bigcup_i \widetilde{B}_i$ there exist $x_0 \in E$ and $x_0 \in X - \widetilde{B}_i$, now exists r_0 such that $B(x_0, r_0) \subset E$, then

$$\frac{\omega\left(B(x_0, r_0) \cap E\right)}{\omega\left(B(x_0, r_0)\right)} = 1 \le \lambda,$$

this is a contradiction.

Then $E \subset U_i \widetilde{B}_i$ and

$$\lambda\omega(B_i) < \omega(E \cap B_i), \quad \omega\left(E \cap \widetilde{B}_i\right) \le \lambda\omega\left(\widetilde{B}_i\right).$$

3. Inequalities Conclusion

Theorem 1. $\omega \in A_\infty, f \ge 0, f \in L_0^1(X)$, then $f_\omega^*(t) \le 2\left(M_{\lambda\omega}^{\#} f\right)_\omega^*(2t) + f_\omega^*(2t) \quad \left(0 < t < \omega(X)/5C_1\right)$.

Proof: The proof is similar to Lerner [5]-[7],

$$|C| \le \inf_{x \in B}\left(|f - C| + |f|\right)_\omega^*\left(\omega(B)\right) \le \left((f - C)_{\chi B}\right)_\omega^*\left(\lambda\omega(B)\right) + \left(f\chi_B\right)_\omega^*\left((1-\lambda)\omega(B)\right).$$

$$\left(f\chi_B\right)_\omega^*\left(\lambda\omega(B)\right) \le \left((f - C)\chi_B\right)_\omega^*\left(\lambda\omega(B)\right) + |C| \le 2\left((f - C)\chi_B\right)_\omega^*\left(\lambda\omega(B)\right) + \left(f\chi_B\right)_\omega^*\left((1-\lambda)\omega(B)\right).$$

From [6], We get two collections of balls $\left\{B_i^s : i \in N \text{ with } s = \lambda, \lambda/2, \cdots\right\}$, then

$$\omega\left(E \cap B_i^s\right)/\omega\left(B_i^s\right) > S \ge \omega\left(E \cap \widetilde{B_i^s}\right)/\omega\left(\widetilde{B_i^s}\right).$$

Fix X, with $0 < t \le \frac{1}{5C_2}\omega(X)$, $\lambda < \frac{1}{5C_2}$, for all E, $\omega(E) = t$ there is $\omega(E) \le \frac{1}{5C_2}\omega(X)$, then exist disjoint balls $\left\{B_i^s \subset X\right\}$, hold

$$\omega\left(E \cap B_i^s\right) > s\omega\left(B_i^s\right), \quad \omega\left(E \cap \widetilde{B_i^s}\right) > s\omega\left(\widetilde{B_i^s}\right).$$

Which contains

$$\omega\left(E \cap B_i^{\lambda/2}\right) > \frac{1}{10C_2}\omega\left(B_i^{\lambda/2}\right), \quad \omega\left(E \cap \widetilde{B_i^\lambda}\right) > \frac{1}{5C_2}\omega\left(\widetilde{B_i^\lambda}\right).$$

Then

$$\sum_i \omega\left(\widetilde{B_i^\lambda}\right) \ge \frac{1}{\lambda}\sum_i \omega\left(E \cap \widetilde{B_i^\lambda}\right) \ge 5C_2 \sum_i \omega\left(E \cap \widetilde{B_i^\lambda}\right) = 5C_2\omega E = 5C_2 t.$$

Select from B_i^λ the balls B_i', $i \in F$ which are not contained in Ω, $\Omega = \left\{x \in X : M_{\lambda\omega}^{\#} f(x) > \left(M_{\lambda\omega}^{\#} f\right)(2t)\right\}$. That is for all $i \in F, B_i' \cap \Omega^c \ne \phi$. There exist $x_0 \in B_i', x_0 \in \Omega^c$, then

$$\inf_{x \in B_i'} M_{\lambda\omega}^{\#} f(x) \le M_{\lambda\omega}^{\#} f(x_0) \le \left(M_{\lambda\omega}^{\#} f\right)_\omega^*(2t).$$

Note that $\omega(\Omega) \le 2t$,

$$\sum_i \omega(B_i') \ge \sum_i \omega(B_i) - \omega(\Omega) \ge 5t - 3t = 2t.$$

Since

$$U_i \widetilde{B}_i - \Omega = U_i\left(\widetilde{B}_i - \Omega\right) = \left(U_{i \in F}\left(\widetilde{B}_i - \Omega\right)\right) \cup \left(U_{i \in F'}\left(\widetilde{B}_i - \Omega\right)\right) = U_{i \in F}\left(\widetilde{B}_i - \Omega\right) \subset U_{i \in F}\widetilde{B}_i.$$

Then

$$\sum_i C_2\omega(B_i) \ge \sum_i \omega\left(\widetilde{B}_i\right) \ge 5C_2 t,$$

i.e.

$$\sum_i \omega(B_i) \ge 5t.$$

$$\inf_i \left(f \cdot \chi B_i' \right)_\omega^* \left(\left(1 - \frac{1}{5C_2} \right) \omega\left(B_i' \right) \right) \le f_\omega^* \left(1 - \frac{1}{5C_2} 3t \right) \le f_\omega^* \left(2t \right).$$

We have

$$\inf_{x \in E} \left| f(x) \right| \le \inf_i \inf_{x \in E \cap B_i'} \left| f(x) \right| \le \inf_i \left(f \chi B_i' \right)_\omega^* \left(\omega\left(B_i' \cap E \right) \right)$$
$$\le \inf_i \left(f \chi B_i' \right)_\omega^* \left(\omega\left(B_i' \right) / 5C_2 \right) \le 2\left(M_{\lambda\omega}^{\#} f \right)_\omega^* (2t) + f_\omega^* (2t).$$

Taking supremum over all $E \subset X$ with $\omega(E) = t$, we get the argument .

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