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Periodic Solution for Neutral Type Neural Networks

Wenxiang Zhang, Yan Yan, Zhanji Gui, Kaihua Wang[*]

School of Mathematics and Statistics, Hainan Normal University, Haikou, Hainan
Email: [*]kaihuawang@qq.com

ABSTRACT

The principle aim of this paper is to explore the existence of periodic solution of neural networks model with neutral delay. Sufficient and realistic conditions are obtained by means of an abstract continuous theorem of k-set contractive operator and some analysis technique.

1. Introduction

Man-made neural networks have been widely used in the fields of pattern recognition, image processing, association, optimal computation, and others. However, owing to the unavoidable finite switching speed of amplifiers, time delays in the electronic implementations of analog neural networks are inevitable, which may cause undesirable dynamic network behaviors such as oscillation and instability. Thus, it is very important to investigate the dynamics of delay neural networks.

The existence of periodic oscillatory solutions of neural networks model has been studied by many researchers [1-6]. Some authors [3-5] used the well-known Hopf bifurcation theory to discuss the bifurcating periodic solutions. However, the usual Hopf bifurcation theory cannot be applied to non-autonomous system. In [6], Li and Lu applied the theory of coincidence degree to non-autonomous neural networks system and obtained some new criteria for the existence of periodic solutions.

We consider the following model for neutral type neural networks with periodic coefficient:

$$\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)g_j(\dot{x}_j(t - \tau_{ij}(t))) + I_i(t), \quad (1)$$

where n is the number of neurons in the network, $x_i(t)$ ($i = 1, 2, \ldots, n$) corresponds to the state of the ith unit at time t, $a_i(t)$ denotes the neuron firing rate, $b_{ij}(t)$ represents the neutral delayed connection weight.

$$g(x) = (g_1(x_1), g_2(x_2), \ldots, g_n(x_n))^T : \ R^n \to R^n$$

is the activation function. $I(t) = (I_1(t), I_2(t), \ldots, I_n(t))^T$ is the ω-periodic external input to the ith neuron.

Throughout this paper, we assume that:

(H1) Functions $g_j(u)$ ($j = 1, 2, \ldots, n$) are globally Lipschitz continuous with the Lipschitz constant $L_j > 0$, that is, $\left| g_j(u_1) - g_j(u_2) \right| \le L_j \left| u_1 - u_2 \right|$, for all $u_1, u_2 \in R$.

(H2) Functions $\tau_{ij}(t)$ ($i, j = 1, 2, \ldots, n$) are nonnegative, bounded and continuously differentiable defined on R^+ and $\dot{\tau}_{ij} < 1$, where $\dot{\tau}_{ij}(t)$ express the derivative of $\tau_{ij}(t)$.

(H3) There exist constant $\alpha_j \ge 0$, $\beta_j > 0$, such that $\forall x \in R$, $\left| g_j(x) \right| \le \alpha_j \left| x \right| + \beta_j$, $j = 1, 2, \ldots, n$.

Remark 1.1 *It is easy to verify that if condition (H1) is satisfied then condition (H3) holds.*

The organization of this paper is as follows. Preliminaries will be given in the next section. In section 3, we will study the existence of periodic solutions of system (1.1) by the abstract continuous theorem of k-set contractive operator.

2. Preliminaries

In order to study Equation (1.1), we should make some preparations. Let E be a Banach space. For a bounded subset $A \subset E$, let

$$\alpha_E(A) = \inf\{\delta > 0 \mid \text{there is a finite number of}$$
$$\text{subsets } A_i \subset A, \text{such that } A = \cup_{i=1} A_i,$$
$$\text{and } \text{diam}(A_i) \le \delta\}$$

denotes the (Kuratoskii) measure of noncompactness, where $\text{diam}(A_i)$ denotes the diameter of set A_i. Let X, Y be two Banach spaces and Ω be a bounded open subset of X. A continuous and bounded map $N : \overline{\Omega} \to Y$ is called k-set contractive if for any bounded set $A \subset \Omega$ we have $\alpha_Y(N(A)) \le k\alpha_X(A)$, where k is a constant. In addition, for a Fredholm operator $L : X \to Y$ with index zero, according to [7], we may define

$$l(L) = \sup\{r \ge 0 \mid r\alpha_X(A) \le \alpha_Y(L(A)), \text{ for all}$$
$$\text{bounded subset } A \subset X\}.$$

Lemma 2.1 *Let* $L : X \to Y$ *be a Fredholm operator with index zero, and* $a \in Y$ *be a fixed point. Suppose that* $N : \Omega \to Y$ *is a k-set contractive with* $k < l(L)$ *, where* $\Omega \subset X$ *is bounded, open, and symmetric about* $0 \in \Omega$ *. Furthermore, we also assume that*

1) $Lx \ne \lambda Nx + \lambda r$, *for* $x \in \partial\Omega$, $\lambda \in (0,1)$;

2) $[QN(x) + Qr, x] \cdot [QN(-x) + Qr, x] < 0$, *for* $x \in \ker L \cap \partial\Omega$, *where* $[\cdot, \cdot]$ *is a bilinear form on* $Y \times X$ *and* Q *is the project of* Y *onto* $\operatorname{Coker}(L)$.

Then there is a $x \in \bar\Omega$, *such that* $Lx - Nx = r$.

In order to use Lemma (2.1) for studying Equation (1.1), we set

$$Y = C_\omega = \{x(t) = (x_1(t),\dots,x_n(t)) \in C(R, R^n) :$$
$$x_i(t+\omega) \equiv x_i(t), i = 1,\dots,n\},$$

with the norm defined by $|x|_0 = \sum_{i=1}^n \max_{t\in[0,\omega]} |x_i(t)|$, and

$$X = C_\omega^1 = \{x(t) = (x_1(t),\dots,x_n(t)) \in C^1(R, R^n) :$$
$$x_i(t+\omega) \equiv x_i(t), i = 1,\dots,n\},$$

with the norm $\|x\| = \max\{|x|_0, |\dot x|_0\}$. Then C_ω, C_ω^1 are all Banach spaces. Let $L : C_\omega^1 \to C_\omega$ defined by

$$Lx = \frac{dx}{dt} = (\dot x_1,\dots,\dot x_n)^T; \quad N : C_\omega^1 \to C_\omega,$$

$$N\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -a_1(t)x_1(t) + \sum_{j=1}^n b_{1j}(t)g_j(\dot x_j(t - \tau_{1j}(t))) \\ \vdots \\ -a_n(t)x_n(t) + \sum_{j=1}^n b_{nj}(t)g_j(\dot x_j(t - \tau_{nj}(t))) \end{bmatrix} \quad (2.1)$$

It is easy to see from [8] that L is a Fredholm operator with index zero. Clearly, Equation (1.1) has a ω -periodic solution if and only if $Lx = Nx + r$ for some $x \in C_\omega^1$, where $r := I(t) = (I_1(t),\dots,I_n(t))^T$.

Lemma 2.2 [9] *The differential operator* L *is a Fredholm operator with index zero, and satisfies* $l(L) \ge 1$.

Lemma 2.3 *If* $k = \max\{\sum_{j=1}^n b_{ij}^+ L_j, i = 1,\dots,n\} < 1$, *here* $b_{ij}^+ = \max_{t\in[0,\omega]} |b_{ij}(t)|$, $N : \Omega \to C_\omega$ *is a k-contractive map.*

Proof. Let $A \subset \bar\Omega$ be a bounded subset and let $\eta = \alpha_{C_\omega^1}(A)$. Then, for any $\varepsilon > 0$, there is a finite family of subsets A_i satisfying $A = \cup_{i=1}^n A_i$ with $\operatorname{diam}(A_i) \le \eta + \varepsilon$. Now let

$$V_i(t, x_i, y_1,\dots,y_n) = a_i(t)x_i - \sum_{j=1}^n g_j(y_j).$$

Since $V_i(t, x_i, y_1,\dots,y_n)$ are uniformly continuous on any compact subset of $R \times R^{n+1}$, A and A_i are precompact in C_ω^0 , it follows that there is a finite family of sub-

sets A_{ij} of A_i such that $A_i = \cup_{j=1} A_{ij}$ with

$$|V_i(t, x_i(t), \dot u_1(t - \tau_{i1}(t)),\dots,\dot u_n(t - \tau_{in}(t)))$$
$$-V_i(t, u_i(t), \dot u_1(t - \tau_{i1}(t)),\dots,\dot u_n(t - \tau_{in}(t)))| \le \varepsilon,$$

for any $x, u \in A_{ij}$. Therefore we have

$$|Nx - Nu|_0$$
$$= \sup_{t\in[0,\omega]} |V_i(t, x_i(t), \dot x_1(t - \tau_{i1}(t)),\dots,\dot x_n(t - \tau_{in}(t)))$$
$$-V_i(t, u_i(t), \dot u_1(t - \tau_{i1}(t)),\dots,\dot u_n(t - \tau_{in}(t)))|$$
$$\le \sup_{t\in[0,\omega]} |V_i(t, x_i(t), \dot x_1(t - \tau_{i1}(t)),\dots,\dot x_n(t - \tau_{in}(t)))$$
$$-V_i(t, x_i(t), \dot u_1(t - \tau_{i1}(t)),\dots,\dot u_n(t - \tau_{in}(t)))|$$
$$+ \sup_{t\in[0,\omega]} |V_i(t, x_i(t), \dot u_1(t - \tau_{i1}(t)),\dots,\dot u_n(t - \tau_{in}(t)))$$
$$-V_i(t, u_i(t), \dot u_1(t - \tau_{i1}(t)),\dots,\dot u_n(t - \tau_{in}(t)))|$$
$$\le \sum_{j=1}^n b_{ij}^+ |g_j(\dot x_j(t - \tau_{ij}(t))) - g_j(\dot u_j(t - \tau_{ij}(t)))| + \varepsilon$$
$$\le \sum_{j=1}^n b_{ij}^+ L_j |\dot x_j(t - \tau_{ij}(t)) - \dot u_j(t - \tau_{ij}(t))| + \varepsilon$$
$$\le k\eta + (k+1)\varepsilon.$$

As ε is arbitrary small, it is easy to see that $\alpha_{C_\omega^0}(N(A)) \le k\alpha_{C_\omega^1}(A)$.

Lemma 2.4 *Let* $\Phi \in C_\omega^0$, $\tau \in C_\omega^1$, *and* $\dot\tau(t) < 1$, *then* $\Phi(v(t)) \in C_\omega^0$, *where* $v(t)$ *is the inverse function of* $t - \tau(t)$.

Throughout this paper, we assume that $\tau_{ij} \in C_\omega^1$, $\dot\tau_{ij}(t) < 1$ $(i, j = 1,\dots,n)$. So that $\tau_{ij}(t)$ has a unique inverse, and we set $v_{ij}(t)$ to represent the inverse of function $t - \tau_{ij}(t)$. Meanwhile, we denote

$$\bar h := (1/\omega)\int_0^\omega h(s)ds \quad \text{and} \quad \|h\|_2 := \left(\int_0^\omega |h(s)| ds\right)^{1/2},$$

3. Main Results

Set

$$p_{ij}^2 = \max\left\{\frac{1}{1 - \dot\tau_{ij}(v_{ij}(t))} : t \in R\right\},$$
$$A = \operatorname{diag}(a_1^l, a_2^l,\dots,a_n^l), \quad B = (\tilde b_{ij})_{n\times n}, \quad \tilde b_{ij} = b_{ij}^m \alpha_j p_{ij},$$
$$C = \operatorname{diag}(a_1^m, a_2^m,\dots,a_n^m), \quad H = (\tilde h_{ij})_{n\times n},$$
$$\tilde h_i = \sum_{j=1}^n b_{ij}^m \beta_j \sqrt\omega + \|I_i\|_2.$$

Theorem 3.1 *Assume that (H1), (H2) hold, furthermore, assume that (H4)* $A - BC$ *is non-singular M-matrix, then Equation* (1.1) *has at least one positive* ω -*periodic solution.*

Proof. We consider the operator Equation

$$Lx = \lambda Nx + \lambda r, \ \lambda \in (0,1) \quad (3.1)$$

Corresponding to Equation (3.1), we have

$$\dot{x}_i(t) = \lambda \left[-a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)g_j(\dot{x}_j(t - \tau_{ij}(t))) + I_i(t) \right] \tag{3.2}$$

Suppose that $(x_1(t),\ldots,x_n(t))^T$ is a solution of system (3.2) for a parameter $\lambda \in (0,1)$. Multiplying the ith Equation of system (3.2) by x_i and integrating over $[0,\omega]$ gives

$$
\begin{aligned}
a_i^l \|x_i\|_2^2 &\le \int_0^\omega x_i(t) \left[\sum_{j=1}^n b_{ij}(t)g_j(\dot{x}_j(t - \tau_{ij}(t))) + I_i(t) \right] dt \\
&\le \int_0^\omega x_i(t) \left[\sum_{j=1}^n b_{ij}^m \alpha_j |\dot{x}_j(t - \tau_{ij}(t))| + \sum_{j=1}^n b_{ij}^m \beta_j + I_i(t) \right] dt \\
&\le \sum_{j=1}^n b_{ij}^m \alpha_j \left(\int_0^\omega |x_i(t)|^2 dt \right)^{1/2} \left(\int_0^\omega |\dot{x}_j(t - \tau_{ij}(t))|^2 dt \right)^{1/2} \\
&\quad + \sum_{j=1}^n b_{ij}^m \beta_j \sqrt{\omega} \left(\int_0^\omega |x_i(t)|^2 dt \right)^{1/2} \\
&\quad + \left(\int_0^\omega |x_i(t)|^2 dt \right)^{1/2} \left(\int_0^\omega |I_i(t)|^2 dt \right)^{1/2}.
\end{aligned}
$$

And according to Lemma 2.4, we have

$$
\begin{aligned}
&\sqrt{\int_0^\omega |\dot{x}_j(t - \tau_{ij}(t))|^2 dt} \\
&= \sqrt{\int_0^\omega \frac{|\dot{x}_j(s)|^2}{1 - \dot{\tau}_{ij}(v_{ij}(s))} ds} \\
&\le p_{ij} \|\dot{x}_j\|_2.
\end{aligned}
$$

Thus

$$a_i^l \|x_i\|_2 \le \sum_{j=1}^n b_{ij}^m \alpha_j p_{ij} \|\dot{x}_j\|_2 + \sum_{j=1}^n b_{ij}^m \beta_j \sqrt{\omega} + \|I_i\|_2 \tag{3.3}$$

Then we may rewrite Equation (3.3) as

$$AX \le BY + H, \tag{3.4}$$

where $X = (\|x_1\|_2,\ldots,\|x_n\|_2)^T$, $Y = (\|\dot{x}_1\|_2,\ldots,\|\dot{x}_n\|_2)^T$.

Multiplying the ith Equation of system (3.2) by $\dot{x}_i$ and integrating over $[0,\omega]$ gives

$$
\begin{aligned}
\|\dot{x}_i\|_2^2 &\le \int_0^\omega \dot{x}_i(t) \left[\begin{array}{l} -a_i(t)x_i(t) \\ + \sum_{j=1}^n b_{ij}(t)g_j(\dot{x}_j(t - \tau_{ij}(t))) + I_i(t) \end{array} \right] dt\partial, \\
&\le \int_0^\omega \dot{x}_i(t) \left[\begin{array}{l} a_i^m |x_i(t)| + \sum_{j=1}^n b_{ij}^m \alpha_j |\dot{x}_j(t - \tau_{ij}(t))| \\ + \sum_{j=1}^n b_{ij}^m \beta_j + I_i(t) \end{array} \right] dt, \\
&\le a_i^m \left(\int_0^\omega |\dot{x}_i(t)|^2 dt \right)^{1/2} \left(\int_0^\omega |x_i(t)|^2 dt \right)^{1/2} \\
&\quad + \sum_{j=1}^n b_{ij}^m \alpha_j \left(\int_0^\omega |\dot{x}_i(t)|^2 dt \right)^{1/2} \left(\int_0^\omega |\dot{x}_j(t - \tau_{ij}(t))|^2 dt \right)^{1/2} \\
&\quad + \sum_{j=1}^n b_{ij}^m \beta_j \sqrt{\omega} \left(\int_0^\omega |\dot{x}_i(t)|^2 dt \right)^{1/2} \\
&\quad + \left(\int_0^\omega |\dot{x}_i(t)|^2 dt \right)^{1/2} \left(\int_0^\omega |I_i(t)|^2 dt \right)^{1/2}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
\|\dot{x}_i\|_2 &\le a_i^m \|x_i\|_2 + \sum_{j=1}^n b_{ij}^m \alpha_j p_{ij} \|\dot{x}_j\|_2 \\
&\quad + \sum_{j=1}^n b_{ij}^m \beta_j \sqrt{\omega} + \|I_i\|_2
\end{aligned} \tag{3.5}
$$

Then we may rewrite Equation (3.5) as

$$Y \le CX + H. \tag{3.6}$$

Substituting (3.6) into (3.4), we obtain

$$(A - BC)X \le BH + H. \tag{3.7}$$

It follows from Lemma 2.2 of paper [10] that $X \le H^+ \triangleq (h_1^+,\ldots,h_n^+)^T$. That is

$$\|x_i\|_2 \le h_i^+, \quad i = 1,\ldots,n, \tag{3.8}$$

where $H^+ \triangleq (h_1^+,\ldots,h_n^+)^T = (A - BC)^{-1}(BH + H)$

Substituting (3.8) into (3.6), we obtain

$$\|x_i\|_2 \le h_i^+, \quad i = 1,\ldots,n. \tag{3.9}$$

It is easy to see that there exist two positive constants $N_i(i = 1,2)$ such that

$$|x|_0 < N_1, \quad |\dot{x}|_0 < N_2. \tag{3.10}$$

Let $\Omega = \{x \in C_\omega^1 : |x|_0 < N_1, |\dot{x}|_0 < N_2\}$, and define a bounded bilinear form $[\cdot,\cdot]$ on $C_\omega \times C_\omega^1$ by

$$[y,x] = \int_0^\omega y(t)x(t)dt.$$

Also we define $Q : y \to \mathrm{Co\,ker}(L)$ by $y \to \int_0^\omega y(t)dt$. Obviously,

$$\{x \mid x \in \ker L \cap \partial\Omega\} = \{x \mid x \equiv N_1, \text{ or } x \equiv -N_1\}.$$

Without loss of generality, we may assume that $x \equiv N_1$. Thus

$$
\begin{aligned}
&[QN(x) + Qr, x] \cdot [QN(-x) + Qr, x] = \omega^2 \times N_1^2 \\
&\times \left[\begin{array}{c} [-\bar{a}_1 N_1 + \sum_{j=1}^n \bar{b}_{1j}g_j(0) + \bar{I}_1][\bar{a}_1 N_1 + \sum_{j=1}^n \bar{b}_{1j}g_j(0) + \bar{I}_1] \\ \vdots \\ [-\bar{a}_n N_1 + \sum_{j=1}^n \bar{b}_{nj}g_j(0) + \bar{I}_n][\bar{a}_n N_1 + \sum_{j=1}^n \bar{b}_{nj}g_j(0) + \bar{I}_n] \end{array} \right]
\end{aligned} \tag{3.11}
$$

If $N_1 > \max\limits_{1 \le i \le n} \left\{ \dfrac{\sum_{j=1}^n \bar{b}_{ij}\beta_j + \bar{I}_i}{\bar{a}_i} \right\} > 0$, then

$$\bar{a}_i N_1 > \sum_{j=1}^n \bar{b}_{ij}\beta_j + \bar{I}_i > \sum_{j=1}^n \bar{b}_{ij}g_j(0) + \bar{I}_i \tag{3.12}$$

So from (3.12) we get

$$
[QN(x_i) + Q(r_i), x_i] \cdot [QN(-x_i) + Q(r_i), x_i] < 0,
$$
$$i = 1,\cdots,n$$

Therefore, by using Lemma 2.1, we obtain that Equation (1.1) has at least one positive ω-periodic solution. The proof is complete.

4. Acknowledgements

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