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Images of Linear Block Codes over $F_q + uF_q + vF_q + uvF_q$

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ABSTRACT

In this paper, we considered linear block codes over $R_q = F_q + uF_q + vF_q + uvF_q, u^2 = v^2 = 0, uv = vu$ where $q = p^m$, $m \in \mathbb{N}$. First we looked at the structure of the ring. It was shown that R_q is neither a finite chain ring nor a principal ideal ring but is a local ring. We then established a generator matrix for the linear block codes and equipped it with a homogeneous weight function. Field codes were then constructed as images of these codes by using a basis of R_q over F_q. Bounds on the minimum Hamming distance of the image codes were then derived. A code meeting such bounds is given as an example.

Keywords: q-ary Images; Distance Bounds

1. Introduction

Let p be a prime number, $m \in \mathbb{N}$, $q = p^m$ and F_q denote the Galois field with q elements. During the late 1990s, C. Bachoc used linear block codes over $F_p + uF_p$, $u^2 = 0$ for constructing modular lattices. Its success motivated the study of linear block codes over the finite chain ring $F_p + uF_p$. And many of the results from these studies have been extended over finite chain rings of the form

$$F_q + uF_q + u^2F_q + \ldots + u^{r-1}F_q, u^r = 0, r \in \mathbb{N}.$$

Such rings can be seen as natural extensions of $F_q + uF_q$. Another ring extension of $F_q + uF_q$ is

$$R_q = F_q + uF_q + vF_q + uvF_q$$

where $u^2 = v^2 = 0$, $uv = vu$. Unlike $F_q + uF_q$, R_q is neither a finite chain ring nor a principal ideal ring. B. Yildiz and S. Karadeniz introduced linear block codes over the ring $F_2 + uF_2 + vF_2 + uvF_2$ in [6]. Self-dual codes, cyclic codes and constacyclic codes over this ring were also studied by these authors in [3,7,8]. In 2011, X. Xu and X. Liu studied the structure of cyclic codes over R_q in [5].

In this work, we will analyze linear block codes over R_q. The structure of the ring will be discussed in Section 2. The generator matrix of linear block codes over R_q and weight functions defined on R_q will be tackled in Section 3. The q-ary images of these linear block codes and bounds on its minimum Hamming distance will be presented in Sections 4 and 5, respectively. Lastly, a code meeting these bounds is given in Section 6.

2. Preliminaries and Definitions

Structure of the Ring $F_q + uF_q + vF_q + uvF_q$

Let R_q denote the ring $F_q + uF_q + vF_q + uvF_q$ whose elements can be uniquely written as $a + bu + cv + duv$ where $a, b, c, d \in F_q$. An element of R_q is a unit if and only if $a \neq 0$. The ring has $q+5$ ideals namely (0), $(uv), (v), (u,v), R_q, (u+jv)$ where $j \in F_q$. R_q is not a principal ideal ring since the maximal ideal (u,v) is generated by u and v. The cardinality of the ideals are $|(uv)| = q$, $|(v)| = |(u+jv)| = q^2$, $|(u,v)| = q^3$, and $|R_q| = q^4$. Its lattice of ideals is shown in **Figure 1**. As can be seen in the lattice of ideals, R_q is not a finite chain ring. But it is a local, Noetherian and Artinian ring. All zero divisors are the elements of $(u,v) \setminus (0)$ and its units are the elements of $R_q \setminus (u,v)$.

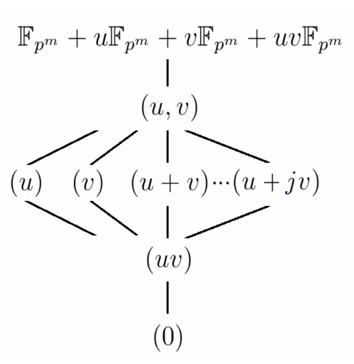

Figure 1. Lattice of ideals of $F_q + uF_q + vF_q + uvF_q$.

Clearly, the ring is isomorphic to $F_q[x,y]/(x^2, y^2, xy-yx)$. It is also isomorphic to the ring of all 4×4 matrices of the form

$$\begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

Moreover, R_q is Frobenius with generating character $\chi : R_q \to T, a+bu+cv+duv \mapsto e^{\frac{2\pi i}{p}tr(d)}$ where tr denotes the trace map on F_q and T is the multiplicative group of unit complex numbers.

Further, R_q is a vector space over F_q with dimension 4. A basis of R_q over F_q is given by the set $\{1,u,v,uv\}$ which we will refer to as the polynomial basis of R_q. Another basis considered in this work is

$$\{1+u+v+uv, 1+v+uv, 1+u+uv, 1+u+v\}.$$

3. Linear Block Codes over $F_q+uF_q+vF_q+uvF_q$

Any linear block code over a finite commutative ring R has a generator matrix which can be put in the following form

$$G=\begin{pmatrix} a_1 I_{k_1} & A_{1,2} & A_{1,3} & \cdots & A_{1,l+1} \\ & a_2 I_{k_2} & a_2 A_{2,3} & \cdots & a_2 A_{2,l+1} \\ & & \ddots & \cdots & \vdots \\ & & & a_l I_{k_l} & a_l A_{l,l+1} \end{pmatrix} \quad (1)$$

where $A_{i,j}$ are binary matrices for $i>1$ and are matrices over R_q for $i=1$. A code of this form has $\prod_{i=1}^{l}|a_i R|^{k_i}$ elements, where the $a_i's$ define the nonzero equivalence classes $[a_1],[a_2],\cdots,[a_l]$ under the equivalence relation on R defined by

$$a \sim b \Leftrightarrow \text{if } a=bu \text{ for a unit } u \text{ in } R$$

$a_i R = \{x \mid x = a_i r \text{ for some } r\in R\}$; and the blanks in G are to be filled with zeros.

A linear block code B of length n over R_q is an R_q-submodule of R_q^n. B has a generator matrix which can be put in the form shown in **Figure 2** where $A_{i,j}$ are $k_i\times k_j$ matrices over R_q, $D_{i,j}$ are $k_i \times k_j$ matrices over F_2 and the blank parts of $G[B]$ are to be filled with zeros. Moreover, B has $q^{4k_1}\cdot q^{2t}\cdot q^{k_{q+3}}$ codewords where $t=\sum_{i=2}^{q+2} k_i$. A linear block code over R_q is free if and only if $k_i = 0$ for all $i=2,\cdots,q+3$.

Now, we equip B with two weight functions namely the usual Hamming metric and a homogeneous weight function.

Lemma 2.1. *(T. Honold, [2])* Let R be a Frobenius ring with generating character χ, then every homogeneous weight w_{hom} on R can be expressed in terms of χ as follows

$$w_{\text{hom}} = \Gamma\left[1-\frac{1}{|R^\times|}\sum_{y\in R^\times}\chi(xy)\right] \quad (1)$$

where $R^\times$ is the group of units of R.

Theorem 2.1. A homogeneous weight w_{hom} on R_q is given by

$$w_{\text{hom}}(x)=\begin{cases}\Gamma & \text{if } x\in R_q\setminus(uv)\\ \frac{q}{q-1}\Gamma & \text{if } x\in(uv)\setminus(0)\\ 0 & \text{otherwise}\end{cases} \quad (2)$$

Proof: Let $x=a+bu+cv+duv\in R_q$. Now, using the previous lemma, every homogeneous weight on R_q can be expressed as

$$w_{\text{hom}} = \Gamma\left[1-\frac{1}{(q-1)q^3}\sum_{y\in R^\times}\chi(xy)\right]$$

Case 1. Let $x\in R_q^\times$. There are $(q-1)q^2$ units having the same d, for any $d\in F_q$. But there are p^{m-1} elements of F_q that has trace j, for any $j\in F_p$. Hence,

$$G[B]=\begin{pmatrix} I_{k_1} & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & \cdots & \cdots & \cdots & A_{1,q+4} \\ & vI_{k_2} & vD_{2,3} & vD_{2,4} & vD_{2,5} & \cdots & \cdots & \cdots & vD_{2,q+4} \\ & & uI_{k_3} & uD_{3,4} & uD_{3,5} & \cdots & \cdots & \cdots & uD_{3,q+4} \\ & & & (u+v)I_{k_4} & (u+v)D_{4,5} & \cdots & \cdots & \cdots & (u+v)D_{4,q+4} \\ & & & & \ddots & \cdots & \cdots & \cdots & \vdots \\ & & & & & (u+jv)I_{k_r} & (u+jv)D_{r,r+1} & \cdots & (u+jv)D_{r,q+4} \\ & & & & & & \ddots & \cdots & \vdots \\ & & & & & & & uvI_{k_{q+3}} & uvD_{q+3,q+4} \end{pmatrix}$$

Figure 2. Generator Matrix of Linear Block Codes over $F_q+uF_q+vF_q+uvF_q$.

$$\sum_{y\in R_q^\times}\chi(xy)=(q-1)q^2\left(p^{m-1}\right)\sum_{j\in F_p}e^{\frac{2\pi i}{p}j}.$$

But $\sum_{j\in F_p}e^{\frac{2\pi i}{p}j}=0$. So, $w_{\text{hom}}=\Gamma$.

Case 2. Let $x\in(uv)\setminus(0)$. For every $a\in F_q^\times$, there are q^3 units of the form $y=a+bu+cv+duv$. Now, p^{m-1} of these have the same trace value *j*, for any $j\in F_p$ while there are $p^{m-1}-1$ of them with trace zero. Hence,

$$\sum_{y\in R_q^\times}\chi(xy)=q^3\left(p^{m-1}\right)\sum_{j\in F_q^\times}e^{\frac{2\pi i}{p}j}+q^3\left(p^{m-1}-1\right).$$

But $\sum_{j\in F_q^\times}e^{\frac{2\pi i}{p}j}=-1$. So, $w_{\text{hom}}=\frac{q}{q-1}\Gamma$.

Case 3. Let $x\in(u,v)\setminus(uv)$. There are $q-1$ elements of $(u,v)\setminus(uv)$ that have the same coefficient for *uv*. For each element $x\in(u,v)\setminus(uv)$ appears *q* copies in the multiset $\left\{xy\middle|y\in R_q^\times,x\in(u,v)\setminus(uv)\right\}$. Moreover, there are p^{m-1} elements of F_q that has trace *j*, for any $j\in F_p$. Hence

$$\sum_{y\in R_q^\times}\chi(xy)=(q-1)q\left(p^{m-1}\right)\sum_{j\in F_q}e^{\frac{2\pi i}{p}j}=\Gamma.$$

We extend this to R_q^n naturally: if $x=(x_1,x_2,\cdots,x_n)$ then $w_{\text{hom}}(x)=\sum_{i=1}^{n}w_{\text{hom}}(x_i)$. The homogeneous (resp. Hamming) distance between any distinct vectors $x,y\in R_q^n$, denoted by $d_{\text{hom}}(x,y)$ (resp. $d_H(x,y)$), is defined as $w_{\text{hom}}(x-y)$ (resp. $w_H(x-y)$). We will denote the minimum homogeneous distance (resp. Hamming) distance by a linear block code over R_q by d_{hom} (resp. d_H).

4. The *q*-ary Images of Linear Block Codes over $F_q+uF_q+vF_q+uvF_q$

Let b_1,b_2,b_3,b_4 be distinct elements of an ordered basis of R_q. Then any element of R_q can be written in the form $\sum_{i=1}^{4}a_ib_i, a_i\in F_q$. Define the mapping

$$\phi:R_q\to F_q$$
$$\sum_{i=1}^{4}a_ib_i\mapsto(a_1,a_2,a_3,a_4)$$

We then extend ϕ to R_q^n coordinate-wise: if $x=(x_1,x_2,\cdots,x_n)$ and $x_i=\sum_{j=1}^{4}a_{i,j}b_j$ then

$$\phi(x)=\left(a_{1,1},\cdots,a_{1,4},a_{2,1},\cdots,a_{2,4},\cdots,a_{n,1},\cdots,a_{n,4}\right).$$

It is easy to show that ϕ is an F_q-module isomorphism.

Theorem 4.1. If B is a linear block code over R_q of length *n*, then $\phi(B)=\left\{\phi(x)\middle|x\in B\right\}$ is a linear block code over F_q with length $4n$.

Proof: First we show that for every $x\in B,\phi(x)\in F_q^{4n}$. Let $x=(x_1,x_2,\cdots x_n)\in B$, Since $\phi(x_i)\in F_q^4$ for any $i=1,2,\cdots,n$, then $\phi(x)\in F_q^{4n}$. Next we show that $\phi(B)$ is a subspace of F_q^{4n}. Let $s\in F_q$ and let $y,y_1\in\phi(B)$. Then there exist $x,x_1\in B$ such that $y=\phi(x)$ and $y_1=\phi(x_1)$. But $sy+y_1=\phi(sx+x_1)$ since ϕ is a module homomorphism. Since $sx+x_1\in B$, $sy+y_1\in\phi(B)$. Thus, $\phi(B)$ is a subspace of F_q^{4n}.

Theorem 4.2. Let $G[B]$ be the generator matrix of *B* given in **Figure 2**. Then $G[\phi(B)]$ has a generator matrix that is permutation-equivalent to the matrix given in **Figure 3**.

$$\begin{pmatrix}
\phi(I_{k_1}) & \phi(A_{1,2}) & \phi(A_{1,3}) & \cdots & \cdots & \cdots & \phi(A_{1,q+4})\\
\phi(vI_{k_1}) & \phi(vA_{1,2}) & \phi(vA_{1,3}) & \cdots & \cdots & \cdots & \phi(vA_{1,q+4})\\
\phi(uI_{k_1}) & \phi(uA_{1,2}) & \phi(uA_{1,3}) & \cdots & \cdots & \cdots & \phi(uA_{1,q+4})\\
\phi(uvI_{k_1}) & \phi(uvA_{1,2}) & \phi(uvA_{1,3}) & \cdots & \cdots & \cdots & \phi(uvA_{1,q+4})\\
 & \phi(vI_{k_2}) & \phi(vD_{2,3}) & \cdots & \cdots & \cdots & \phi(vD_{2,q+4})\\
 & \phi(uvI_{k_2}) & \phi(uvD_{2,3}) & \cdots & \cdots & \cdots & \phi(uvD_{2,q+4})\\
 & & \ddots & \cdots & \cdots & \cdots & \vdots\\
 & & & \phi((u+jv)I_{k_l}) & \phi((u+jv)D_{l,l+1}) & \cdots & \phi((u+jv)D_{l,l+1})\\
 & & & \phi(uvI_{k_l}) & \phi(uvD_{l,l+1}) & \cdots & \phi(uvD_{l,l+1})\\
 & & & & \ddots & \cdots & \vdots\\
 & & & & & \phi(uvI_{k_{q+3}}) & \phi(uvD_{q+3,q+4})
\end{pmatrix}$$

Figure 3. Generator Matrix of $\phi(B)$.

Proof: Let B have a generator matrix given in **Figure 2**. Then for every $c \in B$, c can be expressed as yG where $y \in R_q^k$, $k = \sum_{i=1}^{4} k_i$, that is, $c = \sum_{i=1}^{k} s_i z_i$ where $s_i \in R_q$ and the z_i's are the k rows of $G[B]$. Using any basis of R_q, c can further be written

$$\sum_{i=1}^{k}\sum_{j=1}^{4} a_{i,j,l} z_i + \sum_{i=1}^{k}\sum_{j=1}^{4} b_{i,j} u z_i + \sum_{i=1}^{k}\sum_{j=1}^{4} c_{i,j} v z_i + \sum_{i=1}^{k}\sum_{j=1}^{4} d_{i,j} uv z_i$$

Now,

$$\phi(c) = \sum_{i=1}^{k}\sum_{j=1}^{4} a_{i,j}\phi(z_i) + \sum_{i=1}^{k}\sum_{j=1}^{4} b_{i,j}\phi(uz_i) + \sum_{i=1}^{k}\sum_{j=1}^{4} c_{i,j}\phi(vz_i) + \sum_{i=1}^{k}\sum_{j=1}^{4} d_{i,j}\phi(uvz_i).$$

Hence, $S = \{\phi(z_i), \phi(uz_i), \phi(vz_i), \phi(uvz_i) \mid i = 1, 2, \cdots, k\}$ spans $\phi(B)$. But

- $vz_i = 0$ whenever $i = k_1 + 1, \cdots, k_1 + k_2$ or $i = k - k_{q+3} + 1, \ldots, k$;
- $uz_i = 0$ whenever $i = k_1 + k_2 + 1, \cdots, k_1 + k_2 + k_3$ or $i = k - k_{q+3} + 1, \ldots, k$;
- $uvz_i = 0$ whenever $i > k_1$; and
- $uz_i = jvz_i$ for some $j \in F_q^{\times}$ whenever $i = \sum_{i=1}^{l-1} k_i + 1, \ldots, \sum_{i=1}^{l} k_i$ for some l.

Define the set β as the resulting set once the undesirable cases listed above are deducted from the set S. Notice that the elements of β are the rows of the matrix given in **Figure 3** we will denote by M. Now, define B_l as the matrix that consists of the rows $4k_1 + 2\sum_{i=2}^{l-1} k_i + 1, \ldots, 4k_1 + 2\sum_{i=2}^{l} k_i$ of M so that M can be written in the form $\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_{q+3} \end{pmatrix}$.

We wish to show that the rows of M are linearly independent. Without loss of generality, let $k_i = 1$ for all i. Consider a row of B_i. Clearly, it cannot be expressed as a linear combination of rows from any of the B_j's, $j > i$. We know that $\phi(1), \phi(u), \phi(v), \phi(uv)$ are linearly independent and so any nonzero linear combination of these vectors is not the zero vector. Thus, any row of B_i cannot be written as a linear combination of rows of any of the B_j's, $j \le i$. Hence, the rows of M are linearly independent.

The succeeding theorems are direct consequences of Theorem 4.2.

Corollary 4.3. If B is free with rank k, then $\phi(B)$ is free with rank $4k$.

Corollary 4.4. Let B be a free rate-k/n linear block code over R_2 with generator matrix $(I \quad A)$, then the generator matrix of the q-ary image of B with respect to the basis $\{1+u+v+uv, 1+v+uv, 1+u+uv, 1+u+v\}$ is permutation-equivalent to

$$\begin{pmatrix} 0 & I_k & I_k & I_k & E+F+H & D+E & D+F & D+H \\ I_k & 0 & I_k & 0 & D+E & 0 & D & E \\ I_k & I_k & 0 & 0 & D+F & D & 0 & F \\ I_k & 0 & 0 & I_k & D & 0 & 0 & D \end{pmatrix}$$

where $A = D + Eu + Fv + Huv$.

5. Distance Bounds of the Images of Linear Block Codes over $F_q + uF_q + vF_q + uvF_q$

The minimum distance of a code gives a simple indication of the *goodness* of a code. A field code can correct at most $\left\lfloor \frac{\delta - 1}{2} \right\rfloor$ errors where δ is its minimum Hamming distance. Hence, we are interested with upper bounds of the minimum Hamming distance of the images of the linear block codes over R_q. For the succeeding discussions, we let B be a rate-k/n linear block code over R_q. Also, we denote by δ the minimum Hamming distance of $\phi(B)$.

Theorem 5.1. *(Singleton-type Bound)* Let B be free. Then

$$\delta \le 4(n-k) + 1. \qquad (3)$$

The above theorem is a direct consequence of **Corollary 4.3** and the Singleton Bound for codes over fields while the next theorem is a direct consequence of the Plotkin Bound for codes over fields.

Theorem 5.2. *(Plotkin-type Bound)* Let B be free. Then

$$\delta \le \left\lfloor \frac{q^{4k-1}}{q^{4k} - 1}(q-1)(4n) \right\rfloor. \qquad (4)$$

The next bound is in terms of the average homogeneous weight Γ on F_q and the minimum Hamming distance of B.

Theorem 5.3. *(Rains-type bound)* For a code B,

$$d_H \le \delta \le 4d_H. \qquad (4)$$

Proof: Note that δ is bounded above by $4n$. If for every $x \in B, w_H(B) = d_H$ then $\delta \le 4d_H$. Now, δ is bounded below by d_H since 1 is the minimum nonzero value of the Hamming weight on F_q. Thus, inequality (4) holds.

Now, we use the concept of subcodes of B generated by x as defined by V. Sison and P. Sole in [4]. The subcode of B generated by $x \in B$, denoted by B_x, is the set

$\{ax|a\in R\}$. A generalization of the Rabizzoni bound was derived in [4]. Here we prove a parallel bound for linear block codes over R_q. The proof presented here is based on the proof in [4].

Lemma 5.4. Let $x\in B, x\neq 0$. B_x is free if and only if $|B_x|=q^4$.

Proof: $(\Rightarrow)$ Let B_x be free then the equation $ax=0$ has only the trivial solution. In particular, $(a-b)x=0\Rightarrow a=b$, that is, $a\neq b$ implies $ax\neq bx$. Thus, $|B_x|=q^4$. $(\Leftarrow)$ Let $|B_x|=q^4$. Then for any nonzero *a* and *b*, $a\neq b\Rightarrow ax\neq bx$. That is, $(a-b)x=0\Rightarrow a=b$. But *x* generates B_x by definition. So, B_x is free.

The next statement is a direct consequence of the cardinality of the ideals of R_q

Corollary 5.5. Let $x\in B$. Then

- $x\in(uv)^n\setminus(0)^n$ if and only if $|B_x|=p^m$;
- $x\in(u+jv)^n\setminus(uv)^n$ or $x\in(v)^n\setminus(uv)^n$ if and only if $|B_x|=p^{2m}$;
- $x\in(u,v)^n\setminus S$ if and only if $|B_x|=p^{3m}$ where $S=\bigcup_{j\in F_q}(u+jv)^n\cup(v)^n$.

Theorem 5.5. (*Rabizzoni-type Bound*) Let *x* be a minimum Hamming weight codeword. Then

$$\delta\le\delta_x\le\left\lfloor\frac{|B_x|}{|B_x|-1}\frac{q-1}{q}4d_H\right\rfloor. \quad (5)$$

Moreover, if $|B_x|$ is free, then

$$\delta\le\delta_x\le\left\lfloor\frac{q^3}{q^4-1}(q-1)4d_H\right\rfloor. \quad (6)$$

Proof: Let *x* be a minimum Hamming weight codeword in *B* then consider subcode B_x. Let δ_x denote the minimum Hamming distance of $\phi(B_x)$. The minimum Hamming distance of B_x is still d_H since B_x is a subcode of *B*. Also $\phi(B_x)$ is a subcode of $\phi(B)$ with $\delta\le\delta_x$. The effective length of $\phi(B_x)$ is $4d_H$ coming from the d_H nonzero positions in *x*. Direct application of the Rabizzoni bound results to inequality (5) holds. By Lemma 5.4, inequality (6) follows.

6. Example

Consider the free rate-1/4 self-orthogonal code *B* over R_2 generated by $G=(1\quad 1+v\quad 1+u+v\quad 1+u+uv)$. If $G=(I\quad A)$ then $I_k=1, D=(1\quad 1\quad 1), E=(0\quad 1\quad 1)$, $F=(1\quad 1\quad 0)$ and $H=(0\quad 0\quad 1)$. A codeword in *B* either has homogeneous weight 0,4 or 8. The minimum Hamming distance of *B* is 4. The binary image of *B* was obtained with respect to the basis

$$\{1+u+v+uv, 1+v+uv, 1+u+uv, 1+u+v\}.$$

Table 1. Comparison of bounds for δ.

Singleton-type	$\delta=8\le 13$
Plotkin-type	$8\le 8=\lfloor 8.5\bar{3}\rfloor$
Rains-type	$4\le 8\le 16$
Rabizzoni-type	
$\lvert B_x\rvert=16$	$8\le 8$
$\lvert B_x\rvert=4$	$8\le 10.\bar{6}$
$\lvert B_x\rvert=2$	$8\le 16$

Using Corollary 4.4, $G[\phi(B)]$ is permutation-equivalent to

$$\begin{pmatrix}0&1&1&1&1&0&0&1&0&0&0&0&1&1&1&0\\1&0&1&0&1&0&0&0&0&0&1&1&1&0&1&1\\1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&0\\1&0&0&1&1&1&1&0&0&0&0&0&0&1&1&1\end{pmatrix}.$$

The image code has a minimum Hamming distance of 8 and is self-orthogonal. In **Table 1**, we can see that *B* meets the upper bound of the Plotkin-type and Rabizzoni-type bound.

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