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Multicut L-Shaped Algorithm for Stochastic Convex Programming with Fuzzy Probability Distribution

Miaomiao Han
School of Mathematics and Physics
North China Electric Power University
Baoding, Hebei Province, China
Hanmiaomiao_1@163.com

Xinshun MA
School of Mathematics and Physics
North China Electric Power University
Baoding, Hebei Province, China
xsma@ncepubd.edu.cn

Abstract—Two-stage problem of stochastic convex programming with fuzzy probability distribution is studied in this paper. Multicut L-shaped algorithm is proposed to solve the problem based on the fuzzy cutting and the minimax rule. Theorem of the convergence for the algorithm is proved. Finally, a numerical example about two-stage convex recourse problem shows the essential character and the efficiency.

Keywords-stochastic convex programming; fuzzy probability distribution; two-stage problem; multicut L-shaped algorithm

1. Introduction

Stochastic programming is an important class of mathematical programming with random parameters, and has been widely applied to various fields such as economic management and optimization control[1]. Two-stage stochastic programming is a kind of mathematical programming where the decision variables and the decision process can be decomposed into two stages based on random parameters are observed before and after the specific value.

Stochastic linear programming as a basic issue has been studied widely, and many research results have been reported. In [2], two-stage problem of stochastic linear programming and the basic algorithm were first proposed and applied to the linear optimal control problem. Since then, a large variety of algorithms including Benders decomposition[2][3], stochastic decomposition[4], subgradient decomposition[5][6], nested decomposition[7], and disjunctive decomposition[8] for the two-stage stochastic linear programming had been developed. Among these methods, Benders decomposition also called the L-shaped method has become the main approach to deal with stochastic programming problems.

The theories and algorithms obtained before on stochastic linear programming all are based on a hypothesis that the probability distributions of random parameters have completely information. However, in many situations, due to lack of the date, the probability of a random event is not fully known, and need to get an approximate range with the help of experts' experience. Recently, model of the stochastic linear program with fuzzy probability distribution was proposed in [9], and the modified L-shaped algorithm was presented to solve the model.

Stochastic convex programming is an important class of stochastic nonlinear program and has more widely application than stochastic linear programming[10]. As a result, stochastic convex programming with fuzzy probability distribution will have more useful in many practical situations. In this paper, two-stage stochastic convex programming with fuzzy probability distribution and the solving method are studied, a numerical example shows the essential character and the efficiency.

2. Two-stage stochastic convex programming under fuzzy probability distribution

Let (Ω, Σ, P) be a probability space, sample space $\Omega = \{\omega_1, \cdots, \omega_N\}$ is a finite set, and $\Sigma = 2^{\Omega}$ is the σ-algebra composed by power set of Ω, $p_i = P\{\omega = \omega_i\}$. The two-stage stochastic convex programming problem is

$$\begin{aligned} &min\ f(x) + \sum_{i=1}^{N} p_i Q(x, \omega_i) \\ &s.t.\ \ Ax = b \\ &\qquad\ x \ge 0 \end{aligned} \tag{1}$$

where,

$$\begin{aligned} Q(x, \omega_i) = min\ & g(y, \omega_i) \\ s.t.\ \ & Wy = h(\omega_i) - T(\omega_i)x \\ & y \ge 0 \end{aligned} \tag{2}$$

$x \in \Re^{n_1}$ and $y \in \Re^{n_2}$ are decision vectors, $f(x)$ is convex function, A is $m_1 \times n_1$ matrix, $b \in \Re^{m_1}$ is known vector, W is $m_2 \times n_2$ recourse matrix, for each $\omega_i \in \Omega$, $g(y, \omega_i)$ is convex function on y, $h(\omega_i) \in \Re^{m_2}$ is vector, and $T(\omega_i)$ is

$m_2 \times n_1$ matrix. Where x and y are the first stage decision variable and the second stage decision variable respectively. The mathematical expected value $E[Q(x,\omega)] = \sum_{i=1}^{N} p_i Q(x,\omega_i)$.

When the random variable obeys fuzzy probability distribution, the scope of p_i is as follows[9]

$$\pi_\alpha = \{P \in \Re^N \mid d_i - (1-\alpha_i) l_i \le p_i \le d_i + (1-\alpha_i) l_i, \sum_{i=1}^{N} p_i = 1; p_i \ge 0, i = 1, \cdots, N\} \tag{3}$$

where $P = (p_1, p_2, \cdots, p_N)^T \in \Re^N$ consisted of probabilities, $l = (l_1, \cdots, l_N)^T$ denotes the vagueness level, and the level value $\alpha (0 \le \alpha_i \le 1)$ expresses the DM credibility degree of the partial information on probability distribution. The fuzzy probability distributions results in that mathematical expectation $E[Q(x,\omega)]$ is uncertain, here, $\max_{P\in\pi_\alpha} E_p[Q(x,\omega)] = \max_{P\in\pi_\alpha} \sum_{i=1}^{N} p_i Q(x,\omega_i)$ will be used to instead of $E[Q(x,\omega)]$, and then (1) can be expressed as follows

$$\begin{aligned} & \min\ f(x) + \max_{P\in\pi_\alpha} \sum_{i=1}^{N} p_i Q(x,\omega_i) \\ & s.t.\quad Ax = b \\ & \qquad x \ge 0 \end{aligned} \tag{4}$$

where $Q(x,\omega_i)$ is confirmed by(2).

Obviously, for a given x, there exists $\overline{P} = (\overline{p}_1, \overline{p}_2, \cdots, \overline{p}_N)^T \in \pi_\alpha$, such that $\sum_{i=1}^{N} \overline{p}_i Q(x,\omega_i) = \max_{P\in\pi_\alpha} \sum_{i=1}^{N} p_i Q(x,\omega_i)$.

I. MULTICUT L-SHAPED ALGORITHM

The problem (4) is equivalent to:

$$\begin{aligned} & \min_x\ f(x) + \theta, \\ & s.t.\ \max_{P\in\pi_\alpha} \sum_{i=1}^{N} p_i Q(x,\omega_i) \le \theta, \\ & \qquad x \in K = K_1 \cap K_2, \end{aligned} \tag{5}$$

where

$$\begin{aligned} Q(x,\omega_i) = & \min_{y_i}\ g(y_i,\omega_i), \\ & s.t.\ W y_i = h(\omega_i) - T(\omega_i) x, \\ & \qquad y_i \ge 0, \end{aligned} \tag{6}$$

and

$$K_1 = \{x \mid Ax = b, x \ge 0\},$$

$$K_2 = \{x \mid for\ all\ \omega_i \in \Omega, \exists\, y_i \ge 0, s.t.\ W y_i = h(\omega_i) - T(\omega_i) x\}.$$

The standard L-shaped algorithm for solving above problem can be designed under outer linearization (see e.g. [9]). Suppose that $\overline{P} = (\overline{p}_1, \overline{p}_2, \cdots, \overline{p}_N)^T$ is solution of $\max_{P\in\pi_\alpha} \sum_{i=1}^{N} p_i Q(x,\omega_i)$, problem (5) can be replaced by the following (7)

$$\begin{aligned} & \min_x\ f(x) + \sum_{i=1}^{N} \overline{p}_i \theta_i \\ & s.t.\ Q(x,\omega_i) \le \theta_i, for\ all\ i \\ & \qquad x \in K = K_1 \cap K_2 \end{aligned} \tag{7}$$

because of each constrain in (5) corresponds to N constraints in (7).

The multicut L-shaped algorithm is defined as follows:

S0. Set $s = k = 0$, $t_i = 0$ for all $i = 1, 2, \cdots, N$, and x^0 is given.

S1. Set $k = k + 1$, solve the following master problem:

$$\begin{cases} \min\ f(x) + \sum_{i=1}^{N} \overline{p}_i \theta_i & & \text{(a1)} \\ s.t.\ Ax = b, x \ge 0, & & \text{(a2)} \\ D_l x \ge d_l, & l = 1, 2, \cdots, s, & \text{(a3)} \\ E_{l(i)} x + \theta_i \ge e_{l(i)}, & l(i) = 1, 2, \cdots, t_i, & \text{(a4)} \\ & i = 1, 2, \cdots, N & \end{cases} \tag{8}$$

Let $(x^k, \theta_1^k, \cdots, \theta_N^k)$ be an optimal solution of problem(8). Note that if no constraint (a4) is present for some i, θ_i^k is set equal to $-\infty$, θ_i^k and $\overline{p}_i$ are not considered in the calculation of x^k. Go to S2.

S2. For $i = 1, \cdots, N$, solve the following linear programming problems

$$\begin{cases} \min\ z_i = e^T u^+ + e^T u^- \\ s.t.\ Wy + Iu^+ - Iu^- = h(\omega_i) - T(\omega_i) x^k \quad \text{(a5)} \\ \qquad y \ge 0, u^+ \ge 0, u^- \ge 0 \end{cases} \tag{9}$$

where $e^T = (1, \cdots, 1)$, until, for some i, if the optimal value $z_i > 0$, let σ^k be optimal dual variables value, and define

$$\begin{cases} D_{s+1} = (\sigma^k)^T T(\omega_i) \\ d_{s+1} = (\sigma^k)^T h(\omega_i) \end{cases},$$

set $s = s + 1$, add the constraint $D_{s+1} x^k \ge d_{s+1}$ to the set (a3) and return to S1. If for all $i, z_i = 0$, then go to S3.

S3. For $i = 1, \cdots, N$, and a fixed x^k, solve the following convex programming problems

$$\begin{cases} \min\ g(y_i,\omega_i) \\ s.t.\ W y_i = h(\omega_i) - T(\omega_i) x^k & \text{(a6} \\ \qquad y_i \ge 0 & \text{(a7)} \end{cases} \tag{10}$$

Let $Q(x^k,\omega_i)$ be the optimal value, and y_i^k the optimal solution. Solve the problem $\max_{P\in\pi_\alpha}\sum_{i=1}^{N} p_i^k Q(x^k,\omega_i)$, and suppose $(\overline{p}_1^k,\overline{p}_2^k,\cdots,\overline{p}_N^k)^T$ is the optimal solution, then update the objective function of the master problem. Let v_i^k and u_i^k be the optimal dual variables associated with constrain (a6) and (a7) respectively. Compute

$$\begin{cases} E_{t_i+1} = -(v_i^k)^T T(\omega_i), \\ e_{t_i+1} = g(y_i,\omega_i) - (u_i^k)^T y_i^k + (v_i^k)^T [W(y_i^k) - h(\omega_i)]. \end{cases}$$

If $\theta_i^k < e_{t_i+1} - E_{t_i+1}x^k$ does not hold for any $i=1,\cdots,N$, stop, then $(x^k,\theta_1^k,\cdots,\theta_N^k)$ is an optimal solution.

Otherwise, set $t_i = t_i + 1$, add the constraint $\theta_i^k \ge e_{t_i+1} - E_{t_i+1}x^k$ to the set (a4) , return to S1.

3. Theorem of convergence for the algorithm

Proposition 1. In the algorithm, constraint set (a3) is finite.

Proof The proof of this proposition is the same to the standard L-shaped algorithm (see e.g. [2]).

Proposition 2. For any $\omega_i \in \Omega$ and on all $x \in K_{\omega_i}$, $Q(x,\omega_i)$ is either a finite convex function or is identically $-\infty$, where $K_{\omega_i} = \{x | \omega_i \in \Omega, \exists\, y_i \ge 0, s.t.\ Wy_i = h(\omega_i) - T(\omega_i)x\}$.

Proof (see e.g. [10])

Proposition 3. If $Q(x,\omega_i)$ is a finite convex function for each $\omega_i \in \Omega$, then the function $e_{t_i+1} - E_{t_i+1}x$ is linear supporting hyper planes of $Q(x,\omega_i)$.

Proof By the duality theory (see e.g. [11]), it holds that $Q(x^k,\omega_i) = g(y_i^k,\omega_i) - (u_i^k)^T y_i^k + (v_i^k)^T [W(y_i^k) - h(\omega_i) + T(\omega_i)x^k]$. We know from the convexity of $Q(x,\omega_i)$ that

$$Q(x,\omega_i) \ge g(y_i^k,\omega_i) - (u_i^k)^T y_i^k + (v_i^k)^T [W(y_i^k) - h(\omega_i)] + (v_i^k)^T T(\omega_i)x.$$

Thus

$$g(y_i^k,\omega_i) - (u_i^k)^T y_i^k + (v_i^k)^T [W(y_i^k) - h(\omega_i)] + (v_i^k)^T T(\omega_i)x = e_{t_i+1} - E_{t_i+1}x$$

is a linear support of $Q(x,\omega_i)$.

Theorem. Suppose that the algorithm generates an infinite sequence of $(x^k,\theta_1^k,\cdots,\theta_N^k)$. If $(\overline{x},\overline{\theta}_1,\cdots,\overline{\theta}_N)$ is the limit point of an arbitrary subsequence $(x^{k_j},\theta_1^{k_j},\cdots,\theta_N^{k_j})$, and for each i, $\lim_{j\to\infty}(e_{t_i+1} - E_{t_i+1}x^{k_j} - \theta_i^{k_j}) = 0$, $i=1,\cdots,N,$ then $(\overline{x},\overline{\theta}_1,\cdots,\overline{\theta}_N)$ is an optimal solution of problem(7), $\overline{x}$ is an optimal solution of problem(4).

Proof Since the number of the constraints of type (a3) is finite, we have that $x^{k_j} \in K$ for k_j sufficiently large. We also know K is closed convex set, then $\overline{x} \in K$ .By known, for each i,

$$\theta_i^{k_j} \ge Q(x^{k_j},\omega_i) = e_{t_i+1} - E_{t_i+1}x^{k_j} ,$$

and

$$\lim_{j\to\infty}(e_{t_i+1} - E_{t_i+1}x^{k_j} - \theta_i^{k_j}) = 0$$

Then

$$\overline{\theta}_i = Q(\overline{x},\omega_i)$$

for all $i=1,\cdots,N$. Thus, $(\overline{x},\overline{\theta}_1,\cdots,\overline{\theta}_N)$ is a feasible solution of problem(7).

On the other hand, if x^* is optimal solution to the minimax problem(4), but not necessarily an optimal solution in iteration k_j , then

$$f(x^{k_j}) + \max_{P\in\pi_\alpha}\sum_{i=1}^{N} p_i Q(x^{k_j},\omega_i) \le f(x^*) + \max_{P\in\pi_\alpha}\sum_{i=1}^{N} p_i Q(x^*,\omega_i).$$

By continuity of the convex function we have that

$$f(\overline{x}) + \max_{P\in\pi_\alpha}\sum_{i=1}^{N} p_i Q(\overline{x},\omega_i) \le f(x^*) + \max_{P\in\pi_\alpha}\sum_{i=1}^{N} p_i Q(x^*,\omega_i) ,$$

then, for a certain value $(\overline{p}_1,\overline{p}_2,\cdots,\overline{p}_N)^T \in \pi_\alpha$

$$f(\overline{x}) + \sum_{i=1}^{N} \overline{p}_i\overline{\theta}_i \le f(x^*) + \max_{P\in\pi_\alpha}\sum_{i=1}^{N} p_i Q(x^*,\omega_i),$$

Hence, $(\overline{x},\overline{\theta}_1,\cdots,\overline{\theta}_N)$ is an optimal solution to problem(7), and $\overline{x}$ is an optimal solution to problem(4).

4. Numerical example

Consider the following two-stage stochastic convex programming

$$\begin{cases} \min\ -x_1 + 3x_2 \\ \quad + E_\xi \min\left\{-3.5y_1 - 5y_2 + \dfrac{y_1^2}{2} - 2y_1y_2 + \dfrac{y_2^2}{2}\right\} \\ s.t.\ \ 2x_1 + x_2 \ge 5,\ y_1 \le x_1, \\ \quad x_1 + x_2 \le 3,\ y_2 \le 2x_2, \\ \quad y_1 \le \xi_1,\ y_2 \le \xi_2, \\ \quad x_1, x_2, y_1, y_2 \ge 0. \end{cases} \tag{11}$$

where ξ_1 takes the three values 3.5, 3.8 and 4.0 with probability 1/3, that ξ_2 takes the values 0.5, 1.0 and 1.5 with probability 1/3, and that ξ_1 and ξ_2 are independent of each other, then $\xi = (\xi_1,\xi_2)^T$ can take the each vector in the set $\Pi = \{(k_1,k_2)^T \mid k_1 = 3.5, 3.8, 4.0, k_2 = 0.5, 1.0, 1.5\}$ with probability 1/9.

Under fuzzy probability distribution, assume that ξ takes the each values in Π with probability around 1/9, i.e. $p_i \cong 1/9$ $(i=1,2,\cdots,9)$, it can be confirmed by(3), where $d_i = 1/12$,

$l_i = 1/12$, $\alpha_i = 1/2$ $(i = 1,2,\cdots,9)$, then we get two-stage stochastic convex programming with fuzzy probability distribution. We solve problem (11) by the proposed algorithm and take the initial value $x^0 = (1,1)^T$. Iterations procedure and outputs are as follows.

TABLE I. ITERATIONS PROCEDURE AND OUTPUTS OF MULTICUT L-SHAPED ALGORITHM

k	$obj.val$	x^k	$\overline{P}$
1	-20.167	(3.000,0.000)	(0.153,0.111,0.069,0.153,0.111, 0.069,0.153,0.111,0.069)
2	-15.947	(2.356,0.644)	(0.111,0.111,0.111,0.111,0.111, 0.111,0.111,0.111,0.111)
3	-14.454	(2.566,0.434)	(0.153,0.111,0.069,0.153,0.111, 0.069,0.153,0.111,0.069)
4	-14.020	(2.501,0.499)	(0.153,0.090,0.090,0.153,0.090, 0.090,0.153,0.090,0.090)
5	-14.007	(2.499,0.501)	(0.153,0.090,0.090,0.153,0.090, 0.090,0.153,0.090,0.090)
6	-14.005	(2.500,0.500)	(0.153,0.111,0.069,0.153,0.111, 0.069,0.153,0.111,0.069)
7	-14.005	(2.500,0.500)	(0.153,0.090,0.090,0.153,0.090, 0.090,0.153,0.090,0.090)
8	-14.005	(2.500,0.500)	(0.153,0.111,0.069,0.153,0.111, 0.069,0.153,0.111,0.069)
t	(8,8,7,7,8,5,6,8,5)		

Where obj.val refers to the objective value, $t = (t_1,\cdots,t_9)$ is the vector on the number of iterations of each scenario.

5. Conclusion

Two-stage stochastic convex programming with fuzzy probability distribution is proposed in this paper. The multicut L-shaped algorithm for solving the problem is presented, and the theorem of convergence is given. Finally, a numerical test example demonstrates the essential character and the efficiency of the algorithm.

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