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The Equitable Total Chromatic Number of Some Join graphs

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***Abstract*—A proper total-coloring of graph G is said to be equitable if the number of elements (vertices and edges) in any two color classes differ by at most one, which the required minimum number of colors is called the equitable total chromatic number. In this paper, we prove some theorems on equitable total coloring and derive the equitable total chromatic numbers of $P_m \vee S_n$, $P_m \vee F_n$ and $P_m \vee W_n$.**

Keywords-join graph; equitable total coloring; equitable total chromatic numbers

I. Introduction

The coloring problem is one of the most important problems in the graph theory. As an extension of proper vertex coloring, edge coloring and total coloring[1−5], the concept and some conjectures on the equitable total coloring[6−8] is developed. It is a very difficult problem to obtain the equitable total chromatic number, which meaningful results are rare.

The adjacent vertex distinguishing-equitable total chromatic numbers of some double graphs are research in references [9]. Zhang et al.(2008) introduced the vertex distinguishing equitable edge coloring in references [10], and the vertex distinguishing equitable edge chromatic numbers of the join-graphs between path and path, path and cycle, cycle and cycle with equivalent order are obtained in references [11].

In this paper, we study the equitable total coloring of some join graphs and get some results. Some terms and marks aren't described in this paper, please refer them to [1-3].

II. Definition and Lemma

Definition 2.1[2] For a simple graph $G(V,E)$, let f be a proper k-edge coloring of G, and

$$||E_i| - |E_j|| \leq 1,\ i,j = 1,2,\cdots,k.$$

The partition $\{E_i|1 \leq i \leq k\}$ is called a k-equitable edge coloring (k-PEEC of G in brief), and

$$\chi'_e(G) = \min\{k|k - \text{PEEC of } G\}$$

is called the equitable edge chromatic number of G, where $\forall e \in E_i,\ f(e) = i,\ i = 1,2,\cdots,k$.

Definition 2.2[6−8] For a simple graph $G(V,E)$, let f be a proper k-total coloring of G, and

$$||T_i| - |T_j|| \leq 1,\ i,j = 1,2,\cdots,k.$$

The partition $\{T_i\} = \{V_i \cup E_i|1 \leq i \leq k\}$ is called a k-equitable total coloring (k-ETC of G in brief), and

$$\chi_{et}(G) = \min\{k|k - \text{ETC of } G\}$$

is called the equitable total chromatic number of G, where $\forall x \in T_i = V_i \cup E_i,\ f(x) = i,\ i = 1,2,\cdots,k$.

Conjecture 2.1[6−8] For any simple graph $G(V,E)$,

$$\chi_{et}(G) \leq \Delta(G) + 2 \text{ and } \chi_{et}(G) = \chi_t(G),$$

where $\chi_t(G)$ is the total chromatic number of G.

Definition 2.3[2] For graph G and $H(V(G) \cap V(H) = \phi, E(G) \cap E(H) = \phi)$, a new graph, denoted by $G \vee H$, is called the join of G and H if

$$V(G \vee H) = V(G) \cup V(H),$$

$$E(G \vee H) = E(G) \cup E(H) \cup \{uv|u \in V(G), v \in V(H)\}.$$

Lemma 2.1[6−8] For any simple graph $G(V,E)$,

$$\chi_{et}(G) \geq \Delta(G) + 1.$$

Lemma 2.2[2] For any simple graph $G(V,E)$,

$$\chi'_e(G) \geq \Delta(G).$$

For any simple graph G and H, $\chi'_e(G) = \chi'(G)$[6], and if $H \subseteq G$, then $\chi'(H) \leq \chi'(G)$[1,2], where $\chi'(G)$ is the proper edge chromatic number of G. So Lemma 2.3 and Lemma 2.4 are obtained.

Lemma 2.3 For any simple graph G and H, if H is a subgraph of G, then

$$\chi'_e(H) \leq \chi'_e(G).$$

Lemma 2.4 For complete graph K_p with order p,

$$\chi'_e(K_p) = \begin{cases} p, & p \equiv 1 (\text{mod } 2), \\ p-1, & p \equiv 0 (\text{mod } 2). \end{cases}$$

Lemma 2.5[2,5] Let G be a simple graph, if $G[V_\Delta]$ does not contain cycle, then

$$\chi'_e(G) = \Delta(G).$$

Where $V(G[V_\Delta]) = V_\Delta = \{v|d(v) = \Delta(G), v \in V(G)\}$, $E(G[V_\Delta]) = \{uv|u,v \in V_\Delta, uv \in E(G)\}$.

Lemma 2.6[6−8] For complete graph K_p with order p,

$$\chi_{et}(K_p) = \begin{cases} p, & p \equiv 1(\text{mod } 2), \\ p+1, & p \equiv 0(\text{mod } 2). \end{cases}$$

Lemma 2.7 Suppose P_m is a Path with order m, S_n, F_n and W_n are Star, Fan and Wheel with order $n+1$, respectively. Then

$$\Delta(P_m \vee S_n) = \Delta(P_m \vee F_n) = \Delta(P_m \vee W_n) = m + n.$$

III. Main Results

For some simple graphs, we obtain Theorem 3.1 and Theorem 3.2 as following.

Theorem 3.1 Let G be a simple graph, if $\Delta(G) = \mid V(G) \mid -1$ and G only has a vertex with maximum degree, then

$$\chi_{et}(G) = \Delta(G) + 1.$$

Proof By Lemma 2.1, we only prove that G has an f of $(\Delta(G)+1)$-ETC. Suppose $w \notin V(G)$, $G^* = G \vee \{w\}$, then $G^*[V_\Delta] = P_2$, so $\chi'_e(G^*) = \Delta(G^*) = \Delta(G)+1$ by Lemma 2.5.

Let f^* be a $(\Delta(G)+1)$-PEEC of G,

$$\forall u \in V(G),\ f(u) = f^*(wu);$$

$$\forall uv \in E(G),\ f(uv) = f^*(uv).$$

Obviously, f is a $(\Delta(G)+1)$-ETC of G, so the Theorem 3.1 is true.

Theorem 3.2 Let G be a simple graph, if $\Delta(G) = \mid V(G) \mid -1$ and $\mid V(G) \mid \equiv 1(\text{mod } 2)$, then

$$\chi_{et}(G) = \Delta(G) + 1.$$

Proof By Lemma 2.1, we only prove that G has an f of n-ETC, where $n = |V(G)|$. Suppose $w \notin V(G)$, $G^* = G \vee \{w\}$, obviously $\Delta(G^*) = n$, $G^* \subseteq K_{n+1}$ and $(n+1) \equiv 0(\text{mod } 2)$, so $\chi'_e(G^*) = n$ by Lemma 2.2, Lemma 2.3 and Lemma 2.4.

Let f^* be an n-PEEC of G,

$$\forall u \in V(G),\ f(u) = f^*(wu);$$

$$\forall uv \in E(G),\ f(uv) = f^*(uv).$$

Obviously, f is an n-ETC of G, so the Theorem 3.2 is true.

In the following discussion, let

$P_m = u_1 u_2 \cdots u_m$;

$V(S_n) = \{v_i \mid i = 0,1,2,\cdots,n\}$, $E(S_n) = \{v_0 v_i \mid i = 1,2,\cdots,n\}$;

$V(F_n) = \{v_i \mid i = 0,1,2,\cdots,n\}$, $E(F_n) = \{v_0 v_i \mid i = 1,2,\cdots,n\} \cup \{v_i v_{i+1} \mid i = 1,2,\cdots,n-1\}$;

$V(W_n) = \{v_i \mid i = 0,1,2,\cdots,n\}$, $E(W_n) = \{v_0 v_i \mid i = 1,2,\cdots,n\} \cup \{v_i v_{i+1} \mid i = 1,2,\cdots,n-1\} \cup \{v_n v_1\}$.

Theorem 3.3 When $m \geq 2$, then

$$\chi_{et}(P_m \vee S_n) = \begin{cases} 5, & m=2, n=1, \\ m+n+1, & \text{otherwise.} \end{cases}$$

Proof There are seven cases to be considered.

Case 1 When $m = 2$ and $n = 1$, obviously $P_2 \vee S_1 = K_4$. By Lemma 2.6, it's clear that the result is true.

Case 2 When $m = 3, n = 1$ or $m = n = 2$, $\chi_{et}(P_3 \vee S_1) = \chi_{et}(P_2 \vee S_2) = 5$ by Lemma 2.7 and Theorem 3.2, so clearly the result is true.

Case 3 When $m = 2$ and $n = 3$, $\chi_{et}(P_3 \vee S_3) \geq 6$ by Lemma 2.1 and Lemma 2.7. We only need to prove that $P_2 \vee S_3$ has an f of 6-ETC. Define f by

$f(v_0v_1) = f(v_2u_1) = f(u_2) = 1;$
$f(v_0v_2) = f(v_3u_2) = f(u_1) = 2;$
$f(v_1u_2) = f(v_3u_1) = f(v_0) = 3;$
$f(v_0u_1) = f(v_2u_2) = f(v_3) = 4;$
$f(v_0u_2) = f(v_1u_1) = f(v_2) = 5;$
$f(v_0v_3) = f(u_1u_2) = f(v_1) = 6.$

Obviously, the f is a 6-ETC of $P_2 \vee S_3$, so the result is true.

Case 4 When $m = 2$ and $n \geq 4$, $\chi_{et}(P_2 \vee S_n) \geq n+3$ by Lemma 2.1 and Lemma 2.7. We only need to prove that $P_2 \vee S_n$ has an f of $(n+3)$-ETC. Let matching $M = \{v_0v_1, u_1u_2\}$. Suppose $w \notin V(P_2 \vee S_n)$, denote G^* by

$V(G^*) = V(P_2 \vee S_n) \cup \{w\}$,

$E(G^*) = E(P_2 \vee S_n \backslash M) \cup \{wu_1, wu_2\} \cup \{wv_i \mid i = 0,1,\cdots,n-2\}$.

Hence $G^*[V_\Delta] = u_1v_0u_2$ is a Path with order 3, so $\chi'_e(G^*) = \Delta(G^*) = n+2$ by Lemma 2.5.

Let f_1 be an $(n+2)$-PEEC of G^*, let f_2 be a mapping of $P_2 \vee S_n$ based on f_1, that is,

$f_2(u_i) = f_1(wu_i),\ i = 1,2;$
$f_2(v_i) = f_1(wv_i),\ i = 0,1,\cdots,n-2.$

Define mapping f_3 by

$f_3(v_0v_1) = f_3(u_1u_2) = f_3(u_{n-1}) = f_3(u_n) = n+3.$

Put $f = f_1 \cup f_2 \cup f_3$. So, for $P_2 \vee S_n$, we

$$\forall i \in \{1,2,\cdots,n+3\},\ \ \mid T_i \mid = 3 \text{ or } 4.$$

Obviously, f is an $(n+3)$-ETC of $P_2 \vee S_n$, hence the result is true.

Case 5 When $m \geq 4$ and $n = 1$, $\chi_{et}(P_m \vee S_1) \geq m+2$ by Lemma 2.1 and Lemma 2.7. We only need to prove that $P_m \vee S_1$ has an f of $(m+2)$-ETC. Let matching $M = \{v_0v_1, u_2u_3\}$. Suppose $w \notin V(P_m \vee S_1)$, denote G^* by

$V(G^*) = V(P_m \vee S_1) \cup \{w\}$,

$E(G^*) = E(P_m \vee S_1 \backslash M) \cup \{wv_0, wv_1\} \cup \{wu_i \mid i = 2,3,\cdots,m\}$.

Hence $G^*[V_\Delta] = v_0wv_1$ is a Path with order 3, so $\chi'_e(G^*) = \Delta(G^*) = m+1$ by Lemma 2.5.

Let f_1 be an $(m+1)$-PEEC of G^*, let f_2 be a mapping of $P_m \vee S_1$ based on f_1, that is,

$f_2(v_i) = f_1(wv_i),\ i = 0,1;$
$f_2(u_i) = f_1(wu_i),\ i = 2,3,\cdots,m.$

Define mapping f_3 by

$f_3(v_0v_1) = f_3(u_2u_3) = f_3(u_1) = m+2.$

Put $f = f_1 \cup f_2 \cup f_3$. So, for $P_m \vee S_1$, we have

$$\forall i \in \{1,2,\cdots,m+2\},\ \ \mid T_i \mid = \begin{cases} 3, & m=4, \\ 3 \text{ or } 4, & m \geq 5. \end{cases}$$

Obviously, f is an $(m+2)$-ETC of $P_m \vee S_1$, hence the result is true.

Case 6 When $m = 3$ and $n \geq 2$, $\chi_{et}(P_3 \vee S_n) \geq n+4$ by Lemma 2.1 and Lemma 2.7. We only need to prove that $P_3 \vee$

S_n has an f of $(n+4)$-ETC. Let matching $M = \{v_0v_1, v_2u_2\}$. Suppose $w \notin V(P_3 \vee S_n)$, denote G^* by

$V(G^*) = V(P_3 \vee S_n) \cup \{w\}$,

$E(G^*) = E(P_3 \vee S_n \backslash M) \cup \{wu_2\} \cup \{wv_i \mid i = 0, 1, \cdots, n\}$.

Hence $G^*[V_\Delta] = v_0u_2$ is a Path with order 2, so $\chi'_e(G^*) = \Delta(G^*) = n+3$ by Lemma 2.5.

Let f_1 be an $(n+3)$-PEEC of G^*, let f_2 be a mapping of $P_3 \vee S_n$ based on f_1, that is,

$f_2(u_2) = f_1(wu_2)$;

$f_2(v_i) = f_1(wv_i),\ i = 0, 1, \cdots, n$.

Define mapping f_3 by

$f_3(v_0v_1) = f_3(v_2u_2) = f_3(u_1) = f_3(u_3) = n+4$.

Put $f = f_1 \cup f_2 \cup f_3$. So, for $P_3 \vee S_n$, we have

$$\forall i \in \{1, 2, \cdots, n+4\},\quad |T_i| = \begin{cases} 3 \text{ or } 4, & 2 \le n \le 6, \\ 4, & n = 7, \\ 4 \text{ or } 5, & n \ge 8. \end{cases}$$

Obviously, f is an $(n+4)$-ETC of $P_3 \vee S_n$, hence the result is true.

Case 7 When $m \ge 4$ and $n \ge 2$, $P_m \vee S_n$ only has a vertex v_0 with maximum degree and $d(v_0) = m+n = |V(P_m \vee S_n)| - 1$, so clearly the result is true by Theorem 3.1.

From what stated above, the proof is completed.

Theorem 3.4 When $m \ge 2$ and $n \ge 2$, then

$$\chi_{et}(P_m \vee F_n) = \begin{cases} 7, & m = 2, n = 3 \text{ or } m = 3, n = 2, \\ m+n+1, & \text{otherwise.} \end{cases}$$

Proof Since $P_m \vee F_n \cong P_n \vee F_m$, so we only prove that the result is true when $m \ge n \ge 2$. There are six cases to be considered.

Case 1 When $m = n = 2$, obviously $P_2 \vee F_2 = K_5$. By Lemma 2.6, it's clear that the result is true.

Case 2 When $m = 3$ and $n = 2$, $\chi_{et}(P_3 \vee F_2) \ge 6$ by Lemma 2.1 and Lemma 2.7, obviously $P_3 \vee F_2 = K_6 - u_1u_3$. Suppose $\chi_{et}(K_6 - u_1u_3) = 6$, only the color contains at most 4 elements which colored u_1 and u_3, each color of the left contains 3 elements, so 6 colors colored at most 19 elements, but $|V(K_6 - u_1u_3)| + |E(K_6 - u_1u_3)| = 20$. Hence, 6-ETC is impossible. Moreover, 7-ETC of $P_3 \vee F_2$ is getatable, denote f by

$f(u_iv_j) = i + j,\ i = 1, 2, 3,\ j = 0, 1, 2$;

$f(u_1u_2) = f(v_1) = 5;\ f(u_3) = 2;\ f(v_0) = 4$;

$f(u_2u_3) = f(v_0v_1) = f(v_2) = 6$;

$f(v_0v_2) = f(u_1) = 7;\ f(v_1v_2) = f(u_2) = 1$.

Obviously, f is a 7-ETC of $P_3 \vee F_2$, hence the result is true.

Case 3 When $m \ge 4$ and $n = 2$, $\chi_{et}(P_m \vee F_2) \ge m+3$ by Lemma 2.1 and Lemma 2.7. We only prove that $P_m \vee F_2$ has an f of $(m+3)$-ETC. Let matching $M = \{v_0u_2, v_1v_2\}$. Suppose $w \notin V(P_m \vee F_2)$, denote G^* by

$V(G^*) = V(P_m \vee F_2) \cup \{w\}$,

$E(G^*) = E(P_m \vee F_2 \backslash M) \cup \{wv_0, wv_1, wv_2, wu_2\} \cup \{wu_i \mid i = 4, 5, \cdots, m\}$.

Hence $G^*[V_\Delta] = v_1v_0v_2$ is a Path with order 3, so $\chi'_e(G^*) = \Delta(G^*) = m+2$ by Lemma 2.5.

Let f_1 be an $(m+2)$-PEEC of G^*, let f_2 be a mapping of $P_m \vee F_2$ based on f_1, that is,

$f_2(v_i) = f_1(wv_i),\ i = 0, 1, 2;\ f_2(u_2) = f_1(wu_2)$;

$f_2(u_i) = f_1(wu_i),\ i = 4, 5, \cdots, m$.

Define mapping f_3 by

$f_3(v_1v_2) = f_3(v_0u_2) = f_3(u_1) = f_3(u_3) = m+3$.

Put $f = f_1 \cup f_2 \cup f_3$. So, for $P_m \vee F_2$, we have

$$\forall i \in \{1, 2, \cdots, m+3\},\quad |T_i| = \begin{cases} 3 \text{ or } 4, & m = 4, 5, 6, \\ 4, & m = 7, \\ 4 \text{ or } 5, & m \ge 8. \end{cases}$$

Obviously, f is an $(m+3)$-ETC of $P_m \vee F_2$, hence the result is true.

Case 4 When $m = n = 3$, $\chi_{et}(P_3 \vee F_3) = 7$ by Lemma 2.7 and Theorem 3.2, so clearly the result is true.

Case 5 When $m > n = 3$, $\chi_{et}(P_m \vee F_3) \ge m+4$ by Lemma 2.1 and Lemma 2.7. We only prove that $P_m \vee F_3$ has an f of $(m+4)$-ETC. Let matching $M = \{v_1u_2, v_0v_2, v_3u_3\}$. Suppose $w \notin V(P_m \vee F_3)$, denote G^* by

$V(G^*) = V(P_m \vee F_3) \cup \{w\}$,

$E(G^*) = E(P_m \vee F_3 \backslash M) \cup \{wu \mid u \in V(P_m \vee F_3), \text{and } u \ne u_1, u_m\}$.

Hence $G^*[V_\Delta] = v_0v_2$ is a Path with order 2, so $\chi'_e(G^*) = \Delta(G^*) = m+3$ by Lemma 2.5.

Let f_1 be an $(m+3)$-PEEC of G^*, let f_2 be a mapping of $P_m \vee F_3$ based on f_1, that is,

$f_2(u) = f_1(wu),\ u \in V(P_m \vee F_3), \text{and } u \ne u_1, u_m$.

Define mapping f_3 by

$f_3(v_1u_2) = f_3(v_0v_2) = f_3(v_3u_3) = f_3(u_1) = f_3(u_m) = m+4$.

Put $f = f_1 \cup f_2 \cup f_3$. So, for $P_m \vee F_3$, we have

$$\forall i \in \{1, 2, \cdots, m+4\},\quad |T_i| = \begin{cases} 4, & m = 4, \\ 4 \text{ or } 5, & 5 \le m \le 11, \\ 5, & m = 12, \\ 5 \text{ or } 6, & m \ge 13. \end{cases}$$

Obviously, f is an $(m+4)$-ETC of $P_m \vee F_3$, hence the result is true.

Case 6 When $m \ge n \ge 4$, $P_m \vee F_n$ only has a vertex v_0 with maximum degree and $d(v_0) = m+n = |V(P_m \vee F_n)| - 1$, so clearly the result is true by Theorem 3.1.

From what stated above, the proof is completed.

Theorem 3.5 When $m \ge 2$ and $n \ge 3$, then

$$\chi_{et}(P_m \vee W_n) = \begin{cases} 7, & m = 2, n = 3, \\ m+n+1, & \text{otherwise.} \end{cases}$$

Proof There are six cases to be considered.

Case 1 When $m = 2$ and $n = 3$, obviously $P_2 \vee W_3 = K_6$. By Lemma 2.6, it's clear that the result is true.

Case 2 When $m = 2, n = 4$ or $m = n = 3$, $\chi_{et}(P_2 \vee W_4) = \chi_{et}(P_3 \vee W_3) = 7$ by Lemma 2.7 and Theorem 3.2, so clearly the result is true.

Case 3 When $m = 2$ and $n \ge 5$, $\chi_{et}(P_2 \vee W_n) \ge n+3$ by Lemma 2.1 and Lemma 2.7. We only prove that $P_2 \vee W_n$ has an f of $(n+3)$-ETC. Define f by

$f(v_iu_j) = i + j,\ i = 0, 1, \cdots, n,\ j = 1, 2$;

$f(v_0v_i) = i + 3,\ i = 1, 2, \cdots, n;\ f(v_0) = 3;$
$f(u_1u_2) = f(v_1) = f(v_3) = n + 3;\ f(v_1v_2) = n + 1;$
$f(v_2v_3) = f(u_1) = n + 2;\ f(v_2) = 2;$
$f(v_iv_{i+1}) = i - 2,\ i = 3, 4, \cdots, n - 1;$
$f(v_mv_1) = f(u_2) = 1;\ f(v_i) = i,\ i = 4, 5, \cdots, n.$
We have

$$\forall i \in \{1, 2, \cdots, m + 3\},\quad |\ T_i\ | = \begin{cases} 3 \text{ or } 4, & n = 5, \\ 4, & n = 6, \\ 4 \text{ or } 5, & n \geq 7. \end{cases}$$

Obviously, the f is an $(n + 3)$-ETC of $P_2 \vee W_n$, hence the result is true.

Case 4 When $m = 3$ and $n \geq 4$, $\chi_{et}(P_3 \vee W_n) \geq n+4$ by Lemma 2.1 and Lemma 2.7. We only prove that $P_3 \vee W_n$ has an f of $(n+4)$-ETC. Let matching $M = \{v_0u_2, v_1v_2, v_3v_4\}$. Suppose $w \notin V(P_3 \vee W_n)$, denote G^* by
$V(G^*) = V(P_3 \vee W_n) \cup \{w\}$,
$E(G^*) = E(P_3 \vee W_n \backslash M) \cup \{wu_2\} \cup \{wv_i \mid i = 0, 1, \cdots, n\}$.
Hence the edge set of $G^*[V_\Delta]$ is empty, so $\chi'_e(G^*) = \Delta(G^*) = n + 3$ by Lemma 2.5.

Let f_1 be an $(n + 3)$-PEEC of G^*, let f_2 be a mapping of $P_3 \vee W_n$ based on f_1, that is,
$f_2(u_2) = f_1(wu_2);\ f_2(v_i) = f_1(wv_i),\ i = 0, 1, \cdots, n.$
Define mapping f_3 by
$f_3(v_0u_2) = f_3(v_1v_2) = f_3(v_3v_4) = f_3(u_1) = f_3(u_3) = n + 4.$

Put $f = f_1 \cup f_2 \cup f_3$. So, for $P_3 \vee W_n$, we have

$$\forall i \in \{1, 2, \cdots, n + 4\},\quad |\ T_i\ | = \begin{cases} 4 \text{ or } 5, & 4 \leq n \leq 10, \\ 5, & n = 11, \\ 5 \text{ or } 6, & n \geq 12. \end{cases}$$

Obviously, f is an $(n + 4)$-ETC of $P_3 \vee W_n$, hence the result is true.

Case 5 When $m \geq 4$ and $n = 3$, $\chi_{et}(P_m \vee W_3) \geq m + 4$ by Lemma 2.1 and Lemma 2.7. We only prove that $P_m \vee W_3$ has an f of $(m + 4)$-ETC. Let edge set $M = \{v_0v_1, v_2v_3, u_2u_3;\ v_0v_3, v_1v_2, u_1u_2\}$. Suppose $w \notin V(P_m \vee W_3)$, denote G^* by
$V(G^*) = V(P_m \vee W_3) \cup \{w\}$,
$E(G^*) = E(P_m \vee W_3 \backslash M) \cup \{wv_i \mid i = 0, 1, 2, 3\} \cup \{wu_2\} \cup \{wu_i \mid i = 6, 7, \cdots, m,\ m \geq 6\}$.
Hence the edge set of $G^*[V_\Delta]$ is $\{v_0v_2, v_1v_3\}$, $\chi'_e(G^*) = \Delta(G^*) = m + 2$ by Lemma 2.5.

Let f_1 be an $(m + 2)$-PEEC of G^*, let f_2 be a mapping of $P_m \vee W_3$ based on f_1, that is,
$f_2(u_2) = f_1(wu_2);\ f_2(v_i) = f_1(wv_i),\ i = 0, 1, 2, 3;$
$f_2(u_i) = f_1(wu_i),\ i = 6, 7, \cdots, m,\ (m \geq 6).$
Define mapping f_3 by
$f_3(v_0v_1) = f_3(v_2v_3) = f_3(u_2u_3) = f_3(u_1) = f_3(u_4) = m + 3;$
$f_3(v_0v_3) = f_3(v_1v_2) = f_3(u_1u_2) = f_3(u_3) = f_3(u_5) = m + 4,$ (only if $m \geq 5$, has it vertex u_5).

Put $f = f_1 \cup f_2 \cup f_3$. So, for $P_m \vee W_3$, we have

$$\forall i \in \{1, 2, \cdots, m + 4\},\quad |\ T_i\ | = \begin{cases} 4 \text{ or } 5, & 4 \leq m \leq 10, \\ 5, & m = 11, \\ 5 \text{ or } 6, & m \geq 12. \end{cases}$$

Obviously, f is an $(m+4)$-ETC of $P_m \vee W_3$, hence the result is true.

Case 6 When $m \geq 4$ and $n \geq 4$, $P_m \vee W_n$ only has a vertex v_0 with maximum degree and $d(v_0) = m + n = |\ V(P_m \vee W_n)\ | -1$, so clearly the result is true by Theorem 3.1.

From what stated above, the proof is completed.

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