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The uniform convergence of upwind schemes on layer-adapted meshes for a singularly perturbed Robin BVP

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Abstract—In this paper, we discuss the uniform convergence of the simple upwind scheme on the Shishkin mesh and the Bakhvalov-Shishkin mesh for solving a singularly perturbed Robin boundary value problem, and investigate the midpoint upwind scheme on the Shishkin mesh and the Bakhvalov-Shishkin mesh to achieve better uniform convergence. The elaborate ε-uniform pointwise estimates are proved by using the comparison principle and barrier functions. The numerical experiments support the theoretical results for the schemes on the meshes.

Keywords—Singularly perturbed Robin BVP; simple upwind scheme; midpoint upwind scheme; layer-adapted mesh; uniform convergence

1. Introduction

Let us consider a singularly perturbed convection-diffusion Robin boundary value problem:

$$\begin{cases} Lu \equiv -\varepsilon u'' - b(x)u' + c(x)u = f(x), x \in (0,1) \\ B_0 u \equiv u(0) = A, B_1 u \equiv u(1) + u'(1) = B, \end{cases} \tag{1}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter, A and B are given constants, and function $b(x), c(x)$ and $f(x)$ are sufficiently smooth with $b(x) \ge \beta_0 > \beta > 0$ and $c(x) \ge 0$. Under these conditions, the singularly perturbed problem (1) has a unique solution with a boundary layer at $x = 0$. Singularly perturbed problems arise in many branches of science and engineering such as modeling fluid flows and simulating semiconductor devices (see [1-4]). A wide variety of numerical methods, including the simple upwind scheme and the midpoint upwind scheme on layer-adapted meshes, were constructed to solve the problems in the past few decades (see [5-9]).

In this paper, the properties of the exact solution and the Shishkin mesh are introduced in section 2. In section 3, we discuss the simple upwind scheme on the Shishkin mesh for solving the singularly perturbed Robin BVP (1) and prove its ε-uniform pointwise convergence of order $O(N^{-1})$ on the nodes in coarse part and $O(N^{-1}\ln N)$ on the nodes in fine part. In section 4, the simple upwind scheme on the Bakhvalov-Shishkin mesh, and the midpoint upwind scheme on the Shishkin mesh and the Bakhvalov-Shishkin mesh are studied to reach higher orders of uniform convergence. In section 5, several numerical examples support the elaborate error estimates.

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2. The Solution and the Mesh

Lemma 1 (see [5]) For any positive integer $q > 0$, if $u(x)$ is the solution of problem (1) with sufficiently smooth data then $u(x)$ can be decomposed as $u = S + E$, where the smooth part S satisfies

$$LS(x) = f(x) \text{ and } \left|S^{(i)}(x)\right| \le C, \quad 0 \le i \le q,$$

while the part E satisfies $LE(x) = 0$,

and $E^{(i)}(x) \le C\varepsilon^{-1}\exp(-\frac{\beta x}{\varepsilon})$, $0 \le i \le q$.

Let $\tau = \min\{\frac{1}{2}, \frac{2\varepsilon \ln N}{\beta}\}$, N be an even positive number, and τ be the transition point, where $\varepsilon \le N^{-1}$ as generally in practice. We have the Shishkin mesh:

$$x(i) = \begin{cases} \dfrac{2\tau}{N} i, 0 \le i \le \dfrac{N}{2}, \\ \tau + \dfrac{2(1-\tau)(i - \frac{N}{2})}{N}, \dfrac{N}{2} \le i \le N, \end{cases} \tag{2}$$

which is simply piecewise equidistant. Denoting $h_i = x_i - x_{i-1}$, we have

Lemma 2. $h_i \le \dfrac{4\varepsilon \ln N}{\beta N}$, $N^{-1} \le h_{N/2+i} \le 2N^{-1}$, $i = 1, 2, ..., N/2$.

Throughout the paper, C is a generic positive constant that is independent of ε and h_i, and note that C can take different values at each occurrence, even in the same argument.

3. The Scheme and Its Estimate

For the simple upwind scheme:

$$\begin{cases} L^N u_i^N \equiv -\varepsilon D^+ D^- u_i^N - b_i D^+ u_i^N + c_i u_i^N = f_i \\ \qquad\qquad\qquad i = 1,2,\ldots,N-1, \\ B_0^N u_0^N \equiv u_0^N = A, B_1^N u_N^N \equiv u_N^N + D^- u_N^N = B, \end{cases} \quad (3)$$

where

$$D^+ u_i^N = \frac{u_{i+1}^N - u_i^N}{h_{i+1}}, D^- u_i^N = \frac{u_i^N - u_{i-1}^N}{h_i}, \quad D^+ D^- u_i^N = \frac{2(D^+ u_i^N - D^- u_i^N)}{h_{i+1} + h_i},$$

we have

$$\begin{aligned} L^N u_i^N &= -\frac{2\varepsilon}{(h_{i+1}+h_i)h_i} u_{i-1}^N + (\frac{2\varepsilon}{h_{i+1}h_i} + \frac{b_i}{h_{i+1}} + c_i) u_i^N - (\frac{2\varepsilon}{(h_{i+1}+h_i)h_{i+1}} + \frac{b_i}{h_{i+1}}) u_{i+1}^N \\ &= -r_i^- u_{i-1}^N + r_i^c u_i^N - r_i^+ u_{i+1}^N, \end{aligned}$$

and $r_i^-, r_i^c, r_i^+ > 0, \quad -r_i^- + r_i^c - r_i^+ = c_i \geq 0, \quad i = 1,2,\ldots,N-1.$

Lemma 3. If grid function $v_i(x)$ and $w_i(x)$ satisfy $B_0^N v_0 \leq B_0^N w_0$, $B_1^N v_N \leq B_1^N w_N$ and $L^N v_i \leq L^N w_i$, $i = 1,2,\ldots,N-1$, then $v_i(x) \leq w_i(x)$, $i = 0,1,2,\ldots,N$, and the equation (3) has a unique solution.

Proof. It is proved by that the coefficient matrix associated with L^N is an M-matrix.

By direct computation and Taylor formulas as usual, we have the following two lemmas.

Lemma 4. If $Z_0 = 1, Z_i = \prod_{j=1}^{i} (1 + \frac{\beta h_j}{2\varepsilon})$,

then $L^N Z_i \geq \frac{C Z_i}{\max\{\varepsilon, h_{i+1}\}}$, $i = 1,2,\ldots,N-1$.

Lemma 5. $\left| L^N (u_i - u_i^N) \right| \leq C[\varepsilon \int_{x_{i-1}}^{x_{i+1}} |u'''(t)| dt + \int_{x_{i-1}}^{x_i} |u''(t)| dt]$.

As in the continuous case, decompose the numerical solution into the smooth part and the layer part by $u_i^N = S_i^N + E_i^N$, we have $L^N S_i^N = f_i$, $i = 0,1,2,\ldots,N-1$, $B_0^N S_0^N = S(0)$,

$$B_1^N S_N^N \equiv S_N^N + \frac{S_N^N - S_{N-1}^N}{h_N} = S(1) + S'(1),$$

and $L^N E_i^N = 0$, $i = 0,1,\ldots,N-1$, $B_0^N E_0^N = E(0)$,

$$B_1^N E_N^N \equiv E_N^N + \frac{E_N^N - E_{N-1}^N}{h_N} = E(1) + E'(1).$$

Therefore, the error can be estimated by

$$\left| u_i - u_i^N \right| \leq \left| S_i - S_i^N \right| + \left| E_i - E_i^N \right|.$$

For the smooth part, we have $\left| B_0^N (S_0 - S_0^N) \right| = 0$,

$$\left| B_1^N (S_N - S_N^N) \right| = \left| B_1^N S_N - (B_1 S)_N \right| \leq CN^{-1},$$

$$\left| L_1^N (S_i - S_i^N) \right| = \left| L_1^N S_i - (L_1 S)_i \right| \leq CN^{-1},$$

for $i = 1,2,\ldots,N-1$, by Lemma 1 and Lemma 5. Setting $\omega_i = CN^{-1}(3 - x_i)$ for all i, we have $L^N \omega_i \geq CN^{-1} \geq \left| L^N (S_i - S_i^N) \right|$. By the discrete comparison principle, we get

$$\left| S_i - S_i^N \right| \leq \omega_i \leq CN^{-1}, \quad i = 0,1,2,\ldots,N. \quad (4)$$

For the layer part, we have

Lemma 6. There exists a constant C such that

$$\left| E_i - E_i^N \right| \leq CN^{-1}, \quad i = N/2,\ldots,N.$$

Proof. By Lemma 1, we have $|B_0^N E_0^N| = |E(0)| \leq C$ and

$$\begin{aligned} \left| B_1^N E_N^N \right| &\leq |E(1)| + |E'(1)| \leq C(\varepsilon^{-1} + 1) e^{\frac{-\beta}{\varepsilon}} \leq C e^{\frac{-\beta}{2\varepsilon}} \\ &= C \prod_{j=1}^{N} e^{\frac{-\beta h_j}{2\varepsilon}} \leq C \prod_{j=1}^{N} (1 + \frac{\beta h_j}{2\varepsilon})^{-1}. \end{aligned}$$

Let $Y_i = C_0 Z_i + C_0 \frac{\beta}{2\varepsilon} Z_N$ and C_0 to be sufficiently large, then $B_0^N Y_0 \geq C_0 \geq \left| B_0^N E_0^N \right|$, $B_1^N Y_N = C_0 Z_N \geq \left| B_1^N E_N^N \right|$ and $L^N Y_i \geq 0 = \left| L^N E_i^N \right|$, $i = 1,2,\ldots,N-1$. So, Y_i is a discrete barrier function for E_i^N, and noting

$$\begin{aligned} \ln \prod_{j=1}^{N/2} (1 + \frac{\beta h_j}{2\varepsilon}) &\geq \sum_{j=1}^{N/2} (\frac{\beta h_j}{2\varepsilon} - \frac{1}{2} (\frac{\beta h_j}{2\varepsilon})^2) \\ &\geq \frac{\beta x_{N/2}}{2\varepsilon} - \frac{2 \ln^2 N}{N} \geq \frac{\beta x_{N/2}}{2\varepsilon} - C, \end{aligned}$$

$$\prod_{j=1}^{N/2} (1 + \frac{\beta h_j}{2\varepsilon})^{-1} \leq C e^{-\frac{\beta x_{N/2}}{2\varepsilon}} \leq CN^{-1},$$

$$\frac{\beta}{2\varepsilon} \prod_{j=N/2+1}^{N} (1 + \frac{\beta h_j}{2\varepsilon})^{-1} \leq \frac{\beta}{2\varepsilon} \prod_{j=N/2+1}^{N} (1 + \frac{\beta}{2\varepsilon N})^{-1} \leq \frac{\frac{\beta}{2\varepsilon}}{1 + \frac{\beta}{4\varepsilon}} \leq 2,$$

$$\frac{\beta}{2\varepsilon} \prod_{j=1}^{N} (1 + \frac{\beta h_j}{2\varepsilon})^{-1} \leq 2 \prod_{j=1}^{N/2} (1 + \frac{\beta h_j}{2\varepsilon})^{-1} \leq CN^{-1},$$

we have

$$E_i^N \leq Y_i \leq Y_{N/2} \leq CN^{-1}, \quad i = N/2,\ldots,N.$$

From Lemma 1, we have

$$|E_i| \leq C e^{-\frac{\beta\tau}{\varepsilon}} \leq CN^{-1}, \quad i = N/2,\ldots,N.$$

Thus, the proof is complete.

Lemma 7. There exists a constant C such that

$$\left| E_i - E_i^N \right| \leq CN^{-1} \ln N, \quad i = 1,\ldots,N/2.$$

Proof. By Lemma 5, Lemma 1, the mesh generating function (2) and noting that

$$\sinh(\frac{\beta h_i}{2\varepsilon}) = \sinh(2N^{-1} \ln N) \leq CN^{-1} \ln N, \quad i = 1,\ldots,N/2,$$

we have

$$\left|L^N(E_i - E_i^N)\right| \le C[\varepsilon\int_{x_{i-1}}^{x_{i+1}}\left|E'''(x)\right|dx + \int_{x_{i-1}}^{x_i}\left|E''(x)\right|dx]$$

$$\le C\int_{x_{i-1}}^{x_{i+1}}\varepsilon^{-2}\exp(\frac{\beta x}{2\varepsilon})dx = C\varepsilon^{-1}\exp(\frac{-\beta x_i}{2\varepsilon})\sinh(\frac{\beta h_i}{2\varepsilon}),$$

$$\le C\varepsilon^{-1}N^{-1}\ln N\prod_{j=1}^{i}\exp(\frac{-\beta h_j}{2\varepsilon}) \le C\varepsilon^{-1}N^{-1}\ln N\prod_{j=1}^{i}(1+\frac{\beta h_j}{2\varepsilon})^{-1}.$$

Let $\phi_i = C_0 N^{-1}\ln N(1+Z_i)$, from Lemma 4 and 6, we have

$$L^N\phi_i \ge C_0\varepsilon^{-1}N^{-1}\ln NZ_i \ge \left|L^N(E_i - E_i^N)\right|,$$

$$\phi_0 \ge 0 = \left|E_0 - E_0^N\right|, \quad \phi_{N/2} \ge C_0 N^{-1} \ge \left|E_{N/2} - E_{N/2}^N\right|,$$

provided that the constant C_0 is chosen sufficiently large. So, $\left|E_i - E_i^N\right| \le \phi_i \le CN^{-1}\ln N$ by a discrete comparison principle.

Theorem 1. The simple upwind scheme on the Shishkin mesh for the singularly perturbed Robin boundary value problem (1) satisfies:

$$\left|u_i - u_i^N\right| \le \begin{cases} CN^{-1}\ln N, 0 \le i \le \frac{N}{2}, \\ CN^{-1}, \frac{N}{2} < i \le N. \end{cases} \tag{5}$$

Proof. It is proved by (4), Lemma 6 and 7.

I. Further Results

On the Bakhvalov-Shishkin mesh (see [8]):

$$x_i = x(t_i), t_i = \frac{i}{N}, i = 0,1,2,\ldots,N,$$

where the mesh generating function is as follows:

$$x(t) = \begin{cases} -\frac{2\varepsilon}{\beta}\ln(1-2(1-\frac{1}{N})t), 0 \le t \le \frac{1}{2}, \\ \frac{2\varepsilon\ln N}{\beta} + 2(1-\frac{2\varepsilon\ln N}{\beta})(t-\frac{1}{2}), \frac{1}{2} \le t \le 1, \end{cases} \tag{6}$$

the simple upwind scheme for solving the singularly perturbed Robin BVP was proved to be uniform first-order convergence (see [4]):

$$\left|u_i - u_i^N\right| \le CN^{-1}, 0 \le i \le N. \tag{7}$$

Further, we consider the midpoint upwind scheme for Dirichlet BVP in [9] to be modified for the Robin BVP (1) as follows:

$$\begin{cases} L^N u_i^N \equiv -\varepsilon D^+D^-u_i^N - b_{i+\frac{1}{2}}D^+u_i^N + c_{i+\frac{1}{2}}\frac{u_{i+1}^N + u_i^N}{2} = f_{i+\frac{1}{2}}, i = 1,2,\ldots,N-1, \\ L^N u_N^N \equiv (-\frac{\varepsilon}{h_N^2} + \frac{b_N}{2h_N})u_{N-1}^N + (\frac{2\varepsilon}{h_N^2} + c_N)u_N^N + (-\frac{\varepsilon}{h_N^2} - \frac{b_N}{2h_N})u_{N+1}^N = f_N, \\ B_0^N u_0^N \equiv u_0^N = A, B_1^N u_N^N \equiv u_N^N + Du_N^N = B, \end{cases} \tag{8}$$

where $Du_N^N = (u_{N+1}^N - u_{N-1}^N)/(2h_N)$. For $i = 1,2,\ldots,N-1$, we have

$$L^N u_i^N = -\frac{2\varepsilon}{(h_{i+1}+h_i)h_i}u_{i-1}^N + (\frac{2\varepsilon}{h_{i+1}h_i} + \frac{b_{i+1/2}}{h_{i+1}} + \frac{c_{i+1/2}}{2})u_i^N - (\frac{2\varepsilon}{(h_{i+1}+h_i)h_{i+1}} + \frac{b_{i+1/2}}{h_{i+1}} - \frac{c_{i+1/2}}{2})u_{i+1}^N$$
$$= -r_i^- u_{i-1}^N + r_i^c u_i^N - r_i^+ u_{i+1}^N,$$

Supposed that $h_i < 2\beta_0/\|c\|_\infty$, we have $r_i^-, r_i^c, r_i^+ > 0$,

$$-r_i^- + r_i^c - r_i^+ = c_{i+1/2} \ge 0, \quad i = 1,2,\ldots,N-1.$$

For $i = N$, from (8), we have

$$L^N u_N^N \equiv -\frac{2\varepsilon}{h_N^2}u_{N-1}^N + (\frac{2\varepsilon}{h_N^2} + c_N + \frac{2\varepsilon}{h_N} + b_N)u_N^N = f_N + (\frac{2\varepsilon}{h_N} + b_N)B.$$

The coefficient matrix associated with this L^N is also an M-matrix and a discrete comparison principle holds. By using barrier functions, we can obtain the same error estimate on the Shishkin mesh for Robin BVP as that for Dirichlet BVP in the following:

$$\left|u_i - u_i^N\right| \le \begin{cases} CN^{-1}\ln N, 0 \le i \le \frac{N}{2}, \\ CN^{-2}, \frac{N}{2} \le i \le N. \end{cases} \tag{9}$$

Moreover, we can prove that the midpoint upwind scheme on the Bakhvalov-Shishkin mesh has the uniform convergence:

$$\left|u_i - u_i^N\right| \le \begin{cases} CN^{-1}, 0 \le i \le \frac{N}{2}, \\ CN^{-2}, \frac{N}{2} \le i \le N. \end{cases} \tag{10}$$

4. Numerical Examples

The numerical results in tables 1 and 2 agree with the error estimates for the simple upwind scheme on the Shishkin mesh and the Bakhvalov-Shishkin mesh, denoted by S-S and S-BS, and the midpoint upwind scheme on the Shishkin mesh and the Bakhvalov-Shishkin mesh, denoted by M-S and M-BS. The numerical convergence rates are computed by

$\log_2(\max\left|u_i - u_i^N\right|/\max\left|u_i - u_i^{2N}\right|)$ on the coarse part and the fine part, respectively. Denoting the error estimate by $|u_i - u_i^N| \le C\sigma(N)$, the constant is computed by $\max\left|u_i - u_i^N\right|/\sigma(N)$.

Problem 1. $\begin{cases} -\varepsilon y'' - y' + y = 0, 0 < x < 1, \\ y(0) = 0, y(1) + y'(1) = 1. \end{cases}$

The exact solution of this problem is

$y(x) = (e^{m_1 x} - e^{m_2 x})/[(1+m_1)e^{m_1} - (1+m_2)e^{m_2}]$, where

$m_1, m_2 = (-1 \pm \sqrt{(1+4\varepsilon)})/(2\varepsilon)$.

Table 1. The errors for Problem 1 with $\varepsilon = 10^{-6}$ and $\beta = 0.75$

N	$i \le N/2$	rate	const	$i \ge N/2$	rate	const
S-S:						
64	0.0069	0.59	.107	0.0077	0.98	.492
128	0.0046	0.67	.122	0.0039	1.04	.496
256	0.0029	0.77	.133	0.0019	0.96	.498
512	0.0017	0.79	.140	9.746e-4	1.00	.499
1024	9.851e-4	0.82	.146	4.878e-4	1.00	.500
2048	5.565e-4	0.85	.149	2.440e-4	1.00	.500
S-BS:						
64	0.0055	0.97	.353	0.0077	0.98	.492
128	0.0028	1.00	.362	0.0039	1.04	.496
256	0.0014	1.00	.366	0.0019	0.96	.498
512	7.170e-4	0.97	.367	9.746e-4	1.00	.499
1024	3.590e-4	1.00	.368	4.878e-4	1.00	.500
2048	1.796e-4	1.00	.368	2.440e-4	1.00	.500
M-S:						
64	0.0103	0.71	.660	1.497e-5	2.00	.613e-1
128	0.0063	0.77	.809	3.744e-6	2.00	.613e-1
256	0.0037	0.82	.956	9.366e-7	2.00	.111e-1
512	0.0021	0.81	1.10	2.344e-7	2.00	.985e-2
1024	0.0012	0.85	1.23	5.874e-8	1.99	.889e-2
2048	6.663e-4	0.87	1.36	1.475e-8	1.99	.812e-2
M-BS:						
64	0.0055	0.97	.350	1.497e-5	2.00	.613e-1
128	0.0028	1.00	.354	3.746e-6	2.00	.614e-1
256	0.0014	1.00	.356	9.373e-7	2.00	.614e-1
512	6.981e-4	1.00	.357	2.347e-7	2.00	.615e-1
1024	3.496e-4	1.00	.358	5.882e-8	1.99	.616e-1
2048	1.749e-4	1.00	.358	1.478e-8	1.99	.619e-1

Problem 2. $\begin{cases} -\varepsilon y'' - y' = 1 + 2x, 0 < x < 1, \\ y(0) = 1, y(1) + y'(1) = 0. \end{cases}$

Its exact solution is given by

$$y(x) = [(5-4\varepsilon) - (1-\frac{1}{\varepsilon})e^{\frac{-1}{\varepsilon}} + 4(\varepsilon - 1)e^{\frac{-x}{\varepsilon}}] / [1 - (1-\frac{1}{\varepsilon})e^{\frac{-1}{\varepsilon}}] - (x + x^2 - 2\varepsilon x).$$

TABLE 2. THE ERRORS FOR PROBLEM 2 WITH $\varepsilon = 10^{-6}$ AND $\beta = 0.5$

N	$i \le N/2$	rate	const	$i \ge N/2$	rate	const
S-S:						
16	0.8961	0.64	5.170	.3594	0.97	5.75
32	0.5749	0.63	5.310	.1836	0.98	5.87
64	0.3706	0.70	5.700	.0928	0.99	5.94
128	0.2280	0.77	6.010	.0466	0.99	5.97
256	0.1336	0.80	6.170	.0234	1.00	5.98
512	0.0764	0.84	6.270	.0117	0.99	5.99
S-BS:						
16	0.7088	0.90	11.30	.3594	0.97	5.75
32	0.3796	0.93	12.10	.1836	0.98	5.87
64	0.1986	0.97	12.70	.0928	0.99	5.94
128	0.1017	0.98	13.00	.0466	0.99	5.97
256	0.0515	0.99	13.20	.0234	1.00	5.98
512	0.0259	0.99	13.30	.0117	0.99	5.99
M-S:						
16	0.6784	0.56	10.9	6.081e-8	3.37	.561e-5
32	0.4614	0.54	14.8	5.902e-9	3.92	.174e-5
64	0.3174	0.69	20.3	3.895e-10	4.13	.384e-6
128	0.1974	0.73	25.3	2.231e-11	4.18	.753e-7
256	0.1191	0.79	30.5	1.228e-12	2.25	.145e-7
512	0.0689	0.82	35.3	2 .576e-13	2.00	.108e-7
M-BS:						
16	0.4455	0.84	7.13	2.070e-7	2.16	.530e-4
32	0.2488	0.91	7.96	4.646e-8	2.52	.476e-4
64	0.1320	0.95	8.45	8.078e-9	2.74	.331e-4
128	0.0681	0.98	8.71	1.205e-9	2.87	.197e-4
256	0.0346	0.99	8.85	1.649e-10	2.92	.108e-4
512	0.0174	0.98	8.93	2.175e-11	2.98	.570e-5

Form the theoretical analysis and the numerical results, we conclude that S-S, S-BS, M-S and M-BS are robust, efficient and ε-uniform convergent.

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