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Against Phase Velocities of Elastic Waves in Thin Transversely Isotropic Cylindrical Shell

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ABSTRACT

This paper receives the characteristic equation for the determine of wave numbers of phase velocities of elastic waves, in the thin cylindrical shell with the help of the dynamic theory of the elasticity for the transversely isotropic medium and of the hypothesis of thin shells.

Keywords: Theory of Elasticity; Phase Velocity; Transversely Isotropic Medium; Characteristic Equation

1. Introduction

Based on the use of the dynamic theory of the elasticity for the anisotropic medium and with the help of the hypothesis of thin shells, this paper is determined by the characteristic equation for wave numbers of elastic waves in the thin transversely isotropic cylindrical shell.

2. The Dynamic Theory of the Elasticity for the Transversely Isotropic Medium

Let's consider the infinite thin transversely isotropic cylindrical shell. The elastic wave is spread along the axis Z that orthogonal of the plane of the isotropy. The transversely isotropic elastic medium is characterized by five elastic moduluses [1]: $A_{11}, A_{12}, A_{13}, A_{33}, A_{44}$ or by technical moduluses $E_1, E_3, \mu_1, \mu_3, \nu_1, \nu_3$. In the chosen orientation of the axis $Z - E_1$ is the Joung's modulus, μ_1 is the shear modulus, ν_1 is the Poisson's ratio in the plane of the isotropy. E_3, μ_3 and ν_3 are the same values in the transverse plane. These moduluses connected with each other by the relationship [1-4]:

$$A_{11} = \frac{E_1}{(1+\nu_1)} \cdot \left(1 - \nu_3^2 \cdot \frac{E_1}{E_3}\right); \quad A_{33} = \frac{E_3}{m} \cdot (1-\nu_3);$$

$$A_{44} = \mu_3; \quad A_{12} = \frac{E_1}{(1+\nu_1)\cdot m} \cdot \left(\nu_1 + \nu_3^2 \cdot \frac{E_1}{E_3}\right); \quad (1)$$

$$A_{13} = \frac{E_1 \cdot \nu_3}{m}; \quad \frac{A_{11} - A_{12}}{2} = \mu_1; \quad m = 1 - \nu_1 - 2 \cdot \nu_3^2 \cdot \frac{E_1}{E_3}.$$

The Hooke's law for the transversely isotropic elastic medium is written in the next form [1]:

$$\sigma_r = A_{11} \cdot \varepsilon_r + A_{12} \cdot \varepsilon_\varphi + A_{13} \cdot \varepsilon_z;$$

$$\sigma_\varphi = A_{12} \cdot \varepsilon_r + A_{11} \cdot \varepsilon_\varphi + A_{13} \cdot \varepsilon_z;$$

$$\sigma_z = A_{13} \cdot (\varepsilon_r + \varepsilon_\varphi) + A_{33} \cdot \varepsilon_z; \quad (2)$$

$$\tau_{rz} = A_{44} \cdot \gamma_{rz}; \quad \tau_{\varphi z} = A_{44} \cdot \gamma_{\varphi z}; \quad \tau_{r\varphi} = 0.5 \cdot (A_{11} - A_{12}) \cdot \gamma,$$

where $\varepsilon_r, \varepsilon_\varphi, \varepsilon_z, \gamma_{rz}, \gamma_{\varphi z}, \gamma_{r\varphi}$ are components of the tensor of deformations, which are equals [1]:

$$\varepsilon_r = \frac{\partial U_r}{\partial r}; \quad \varepsilon_\varphi = \frac{1}{r} \cdot \frac{\partial U_\varphi}{\partial \varphi} + \frac{U_r}{r};$$

$$\varepsilon_z = \frac{\partial U_z}{\partial z}; \quad \gamma_{r\varphi} = \frac{\partial U_\varphi}{\partial r} - \frac{U_\varphi}{r} + \frac{1}{r} \cdot \frac{\partial U_r}{\partial \varphi}; \quad (3)$$

$$\gamma_{rz} = \frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r}; \quad \gamma_{\varphi z} = \frac{1}{r} \cdot \frac{\partial U_z}{\partial \varphi} + \frac{\partial U_\varphi}{\partial z},$$

where U_r, U_φ, U_z are components of the displacement vector U.

Equations of the dynamic balance in the circular cylindrical system of coordinates [with the harmonic dependence from the time $\exp(i\omega t)$] have the following appearance [1-4]:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \cdot \frac{\partial \tau_{r\varphi}}{\partial \varphi} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r} \cdot (\sigma_r - \sigma_\varphi) + \rho \cdot \omega^2 \cdot U_r = 0;$$

$$\frac{\partial \tau_{r\varphi}}{\partial r} + \frac{1}{r} \cdot \frac{\partial \sigma_\varphi}{\partial \varphi} + \frac{\partial \tau_{\varphi z}}{\partial z} + \frac{1}{r} \cdot 2 \cdot \tau_{r\varphi} + \rho \cdot \omega^2 \cdot U_\varphi = 0; \quad (4)$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \cdot \frac{\partial \tau_{\varphi z}}{\partial \varphi} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r} \cdot \tau_{rz} + \rho \cdot \omega^2 \cdot U_z = 0,$$

where

$$\sigma_r = A_{11} \cdot \frac{\partial U_r}{\partial r} + \frac{A_{12}}{r} \cdot \frac{\partial U_\varphi}{\partial \varphi} + \frac{A_{12}}{r} \cdot U_r + A_{13} \cdot \frac{\partial U_z}{\partial z};$$

$$\sigma_\varphi = A_{12} \cdot \frac{\partial U_r}{\partial r} + \frac{A_{11}}{r} \cdot \frac{\partial U_\varphi}{\partial \varphi} + \frac{A_{11}}{r} \cdot U_r + A_{13} \cdot \frac{\partial U_z}{\partial z};$$

$$\sigma_z = A_{13} \cdot \frac{\partial U_r}{\partial r} + \frac{A_{13}}{r} \cdot \frac{\partial U_\varphi}{\partial \varphi} + \frac{A_{13}}{r} \cdot U_r + A_{33} \cdot \frac{\partial U_z}{\partial z};$$

$$\tau_{r\varphi} = \frac{A_{11}}{2} \cdot \frac{\partial U_\varphi}{\partial r} - \frac{A_{11}}{2 \cdot r} \cdot U_\varphi + \frac{A_{11}}{2 \cdot r} \cdot \frac{\partial U_r}{\partial \varphi} \qquad (5)$$
$$- \frac{A_{12}}{2} \cdot \frac{\partial U_\varphi}{\partial r} + \frac{A_{12}}{2 \cdot r} \cdot U_\varphi - \frac{A_{12}}{2 \cdot r} \cdot \frac{\partial U_r}{\partial \varphi}.$$

$$\tau_{rz} = A_{44} \cdot \frac{\partial U_r}{\partial z} + A_{44} \cdot \frac{\partial U_z}{\partial r};$$

$$\tau_{\varphi z} = \frac{A_{44}}{r} \cdot \frac{\partial U_z}{\partial \varphi} + A_{44} \cdot \frac{\partial U_\varphi}{\partial z}.$$

Components of the displacement vector U_r, U_φ, U_z can be presented in the series form [2-4]:

$$U_r = \exp(i \cdot k \cdot z) \cdot \sum_{m=0}^{\infty} \cos(m \cdot \varphi) \cdot U_m(r);$$

$$U_\varphi = \exp(i \cdot k \cdot z) \cdot \sum_{m=1}^{\infty} \sin(m \cdot \varphi) \cdot V_m(r); \qquad (6)$$

$$U_z = \exp(i \cdot k \cdot z) \cdot \sum_{m=0}^{\infty} \cos(m \cdot \varphi) \cdot W_m(r),$$

where k is the wave number of the elastic wave.

Then we substituted (4) in (5), we receive equations of the dynamic balance in displacements [2-4]:

$$\frac{\partial^2 U_r}{\partial r^2} - \frac{a_1}{r^2} \cdot \frac{\partial U_\varphi}{\partial \varphi} + \frac{a_2}{r} \cdot \frac{\partial^2 U_\varphi}{\partial r \partial \varphi} - \frac{U_r}{r^2}$$
$$+ \frac{1}{r} \cdot \frac{\partial U_r}{\partial r} + a_3 \cdot \frac{\partial^2 U_z}{\partial r \partial z} + \frac{a_1}{r^2} \cdot \frac{\partial^2 U_r}{\partial \varphi^2} \qquad (7)$$
$$+ a_4 \cdot \frac{\partial^2 U_r}{\partial z^2} + a_4 \cdot \frac{\partial^2 U_z}{\partial r \partial z} - \frac{1}{r^2} \cdot \frac{\partial U_\varphi}{\partial \varphi} + a_5 \cdot U_r = 0;$$

$$\frac{1}{r^2} \cdot \frac{\partial^2 U_\varphi}{\partial \varphi^2} + a_1 \cdot \frac{\partial^2 U_\varphi}{\partial r^2} - \frac{a_1}{r^2} \cdot U_\varphi + \frac{a_1}{r} \cdot \frac{\partial U_\varphi}{\partial r}$$
$$+ \frac{1 + a_1}{r^2} \cdot \frac{\partial U_r}{\partial \varphi} + a_2 \cdot \frac{\partial^2 U_r}{\partial r \partial \varphi} + \frac{a_3}{r} \cdot \frac{\partial^2 U_z}{\partial z \partial \varphi} \qquad (8)$$
$$+ a_4 \cdot \frac{\partial^2 U_\varphi}{\partial z^2} + a_5 \cdot U_\varphi = 0;$$

$$\frac{\partial^2 U_z}{\partial r^2} + a_6 \cdot \frac{\partial^2 U_z}{\partial z^2} + a_7 \cdot \frac{\partial^2 U_r}{\partial r \partial z} + \frac{a_7}{r} \cdot \frac{\partial^2 U_\varphi}{\partial z \partial \varphi}$$
$$+ \frac{1}{r^2} \cdot \frac{\partial^2 U_z}{\partial \varphi^2} + \frac{a_7}{r} \cdot \frac{\partial U_r}{\partial z} + \frac{1}{r} \cdot \frac{\partial U_z}{\partial r} + a_8 \cdot U_z = 0, \qquad (9)$$

where

$$a_1 = \frac{A_{11} - A_{12}}{2 \cdot A_{11}}; \quad a_2 = \frac{A_{11} + A_{12}}{2 \cdot A_{11}}; \quad a_3 = \frac{A_{13} + A_{44}}{A_{11}};$$

$$a_4 = \frac{A_{44}}{A_{11}}; \quad a_5 = \frac{\rho \cdot \omega^2}{A_{11}}; \quad a_6 = \frac{A_{33}}{A_{44}}; \quad a_7 = \frac{A_{13}}{A_{44}} + 1;$$

$$a_8 = \frac{\rho \cdot \omega^2}{A_{44}}.$$

Now if components of the displacement vector U_r, U_φ, U_z taken from (6) substitute in (7)-(9), then we receive following equations for radial functions $U_m(r), V_m(r), W_m(r)$ [2-4]:

$$\frac{\partial^2 U_m}{\partial r^2} - \frac{a_1}{r^2} \cdot m \cdot V_m + \frac{a_2}{r} \cdot m \cdot \frac{\partial V_m}{\partial r}$$
$$- \frac{U_m}{r^2} + \frac{1}{r} \cdot \frac{\partial U_m}{\partial r} + a_3 \cdot i \cdot k \cdot \frac{\partial W_m}{\partial r} \qquad (10)$$
$$- \frac{a_1}{r^2} \cdot m^2 \cdot U_m - a_4 \cdot k^2 \cdot U_m + a_4 \cdot i \cdot k \cdot W_m$$
$$- \frac{m}{r^2} \cdot V_m + a_5 \cdot U_m = 0;$$

$$\frac{m^2}{r^2} \cdot V_m + a_1 \cdot \frac{\partial^2 V_m}{\partial r^2} - \frac{a_1}{r^2} \cdot V_m$$
$$+ \frac{a_1}{r} \cdot \frac{\partial V_m}{\partial r} - \frac{a_1 + 1}{r^2} \cdot m \cdot U_m \qquad (11)$$
$$- a_2 \cdot m \cdot \frac{\partial U_m}{\partial r} - \frac{a_3}{r} \cdot m \cdot i \cdot k \cdot W_m$$
$$- a_4 \cdot k^2 \cdot V_m + a_5 \cdot V_m = 0;$$

$$\frac{\partial^2 W_m}{\partial r^2} - a_6 \cdot k^2 \cdot W_m + a_7 \cdot i \cdot k \cdot \frac{\partial U_m}{\partial r} + \frac{a_7}{r} \cdot m \cdot i \cdot k \cdot V_m$$
$$- \frac{m^2}{r^2} \cdot W_m + \frac{a_7}{r} \cdot i \cdot k \cdot U_m + \frac{1}{r} \cdot \frac{\partial W_m}{\partial r} + a_5 \cdot W_m = 0. \qquad (12)$$

Boundary conditions: normal (σ_r) and tangent $(\tau_{r\varphi}, \tau_{rz})$ stresses are equal zero at external $(r = a)$ and internal $(r = b)$ surfaces of the elastic shell are added to Equations (10)-(12) [2-4]:

$$\frac{\partial U_m}{\partial r} + \frac{a_9}{r} \cdot m \cdot V_m + \frac{a_9}{r} \cdot U_m \qquad (13)$$
$$+ a_{10} \cdot i \cdot k \cdot W_m = 0; \; [r = a; r = b]$$

$$\frac{\partial V_m}{\partial r} - \frac{1}{r} \cdot V_m - \frac{m}{r} \cdot U_m = 0; \; [r = a; r = b] \qquad (14)$$

$$i \cdot k \cdot U_m + \frac{\partial W_m}{\partial r} = 0, \; [r = a; r = b] \qquad (15)$$

where

$$a_9 = \frac{A_{12}}{A_{11}}; \quad a_{10} = \frac{A_{13}}{A_{11}}.$$

3. Hypothesis of Thin Shells

The fellow parameter

$$\xi = \frac{z}{R_0}$$

can be used for thin shells, where

$$R_0 = \frac{a+b}{2}$$

is middle radius and $z = r - R_0$ is the coordinate taking from the middle surface [2-5]:

$$U_m(r) = \sum_{n=0}^{N_1} x_n \cdot \xi^n;$$

$$V_m(r) = \sum_{n=0}^{N_1} y_n \cdot \xi^n; \qquad (16)$$

$$W_m(r) = \sum_{n=0}^{N_1} z_n \cdot \xi^n.$$

We substitute decompositions in boundary Conditions (13)-(15) and 6 equations relative to $3(N_1 + 1)$ unknown coefficients x_n, y_n, z_n [2-4]:

$$R_0^{-1} \cdot \sum_{n=0}^{N_1} x_n \cdot n \cdot (\xi_1)^{n-1}$$
$$+ a_9 \cdot m \cdot (R_0 + 0.5 \cdot h)^{-1} \cdot \sum_{n=0}^{N_1} y_n \cdot (\xi_1)^n$$
$$+ a_9 \cdot (R_0 + 0.5 \cdot h)^{-1} \cdot \sum_{n=0}^{N_1} x_n \cdot (\xi_n)^n \qquad (17)$$
$$+ a_{10} \cdot i \cdot k \cdot \sum_{n=0}^{N_1} z_n \cdot (\xi_1)^n = 0;$$

$$R_0^{-1} \cdot \sum_{n=0}^{N_1} x_n \cdot n \cdot (-\xi_1)^{n-1}$$
$$+ a_9 \cdot m \cdot (R_0 - 0.5 \cdot h)^{-1} \cdot \sum_{n=0}^{N_1} y_n \cdot (-\xi_1)^n$$
$$+ a_9 \cdot (R_0 - 0.5 \cdot h)^{-1} \cdot \sum_{n=0}^{N_1} x_n \cdot (-\xi_1)^n \qquad (18)$$
$$+ a_{10} \cdot i \cdot k \cdot \sum_{n=0}^{N_1} z_n \cdot (-\xi_1)^n = 0;$$

$$R_0^{-1} \cdot \sum_{n=0}^{N_1} y_n \cdot n \cdot (\xi_1)^{n-1} - (R_0 + 0.5 \cdot h)^{-1} \cdot \sum_{n=0}^{N_1} y_n \cdot (\xi_1)^n$$
$$- m \cdot (R_0 + 0.5 \cdot h)^{-1} \cdot \sum_{n=0}^{N_1} x_n \cdot (\xi_1)^n = 0 \qquad (19)$$

$$R_0^{-1} \cdot \sum_{n=0}^{N_1} y_n \cdot (-\xi_1)^{n-1} - (R_0 - 0.5 \cdot h)^{-1} \square \sum_{n=0}^{N_1} y_n (-\xi_1)^n$$
$$- m \cdot (R_0 - 0.5 \cdot h)^{-1} \cdot \sum_{n=0}^{N_1} x_n \cdot (-\xi_1)^n = 0; \qquad (20)$$

$$i \cdot k \cdot \sum_{n=0}^{N_1} x_n \cdot (\xi_1)^n + R_0^{-1} \cdot \sum_{n=0}^{N_1} z_n (\xi_1)^{n-1} = 0; \qquad (21)$$

$$i \cdot k \cdot \sum_{n=0}^{N_1} x_n \cdot (-\xi_1)^n + R_0^{-1} \cdot \sum_{n=0}^{N_1} z_n \cdot (-\xi_1)^{n-1} = 0. \qquad (22)$$

The rest of equations can be received, by substitution of decompositions (16) in Equations (10)-(12) and by equated of coefficients at identical powers ξ_1 [2-4]:

$$x_{n+2} \cdot (n+2) \cdot (n+1) + x_{n+1} \cdot (n+1) \cdot (2 \cdot n + 1)$$
$$+ x_n \cdot (n^2 - 1 - a_1 \cdot m^2 - a_4 \cdot k^2 \cdot R_0^2 + a_5 \cdot R_0^2)$$
$$+ x_{n-1} \cdot 2 \cdot R_0^2 \cdot (a_5 - a_4 \cdot k^2) + x_{n-2} \cdot R_0^2 \cdot (a_5 - a_4 \cdot k^2)$$
$$+ y_{n+1} \cdot (n+1) \cdot a_2 \cdot m + y_n \cdot [a_2 \cdot m \cdot n - a_1 \cdot (m+1)] \qquad (23)$$
$$+ z_{n+1} \cdot i \cdot k \cdot R_0 \cdot (n+1) \cdot (a_3 + a_4)$$
$$+ z_n \cdot 2 \cdot i \cdot k \cdot R_0 \cdot n \cdot (a_3 + a_4)$$
$$+ z_{n+1} \cdot i \cdot k \cdot R_0 \cdot (n-1) \cdot (a_3 + a_4) = 0;$$

$$-x_{n+1} \cdot a_2 \cdot m \cdot R_0 \cdot (n+1) - x_n \cdot m \cdot (1 + a_1 + 2 \cdot a_2 \cdot R_0 \cdot n)$$
$$- x_{n-1} \cdot a_2 \cdot m \cdot R_0 \cdot (n-1) + y_{n+2} \cdot a_1 \cdot (n+2) \cdot (n+1)$$
$$+ y_{n+1} \cdot (n+1) \cdot a_1 \cdot (1 + 2 \cdot n)$$
$$- y_n \cdot [a_1 \cdot (n^2 - 1) - m^2 - a_4 \cdot k^2 \cdot R_0^2 + a_5 \cdot R_0^2] \qquad (24)$$
$$- y_{n-1} \cdot 2 \cdot R_0^2 \cdot (a_4 \cdot k^2 - a_5)$$
$$- y_{n-2} \cdot R_0^2 \cdot (a_4 \cdot k^2 - a_5) - z_n \cdot a_3 \cdot m \cdot i \cdot k \cdot R_0$$
$$- z_{n-1} \cdot a_3 \cdot m \cdot i \cdot k \cdot R_0 = 0;$$

$$x_{n+1} \cdot a_7 \cdot i \cdot k \cdot R_0 \cdot (n+1) + x_n \cdot i \cdot k \cdot R_0 \cdot a_7 \cdot (1 + 2 \cdot n)$$
$$+ x_{n-1} \cdot a_7 \cdot i \cdot k \cdot R_0 \cdot n + y_n \cdot a_7 \cdot m \cdot i \cdot k \cdot R_0$$
$$+ y_{n-1} \cdot a_7 \cdot m \cdot i \cdot k \cdot R_0 + z_{n+2} \cdot (n+1) \cdot (n+2)$$
$$+ z_{n+1} \cdot (n+1) \cdot (2 \cdot n + 1) + z_n \cdot (n^2 - m^2 - a_6 \cdot k^2 \cdot R_0^2)$$
$$- z_{n-1} \cdot 2 \cdot R_0^2 \cdot (a_6 \cdot k^2 - a_8) - z_{n-2} \cdot R_0^2 \cdot (a_6 \cdot k^2 - a_8) = 0, \qquad (25)$$

where $n = 0, 1, 2, \cdots$.

It is necessary to use $3 \cdot (N+1) - 6$ of Equations (23)-(25) and for $n = 0$ and $n = 1$ coefficients with negative indexes are equal to zero. Then in common with the Equations (17)-(22) the homogeneous system of $3 \cdot (N_1 + 1)$ equations relative to coefficients x_n, y_n, z_n is formed. Afterwards, we expand the determinant of this system and let this determinant is equal zero we receive the characteristic equation for wave numbers k of elastic waves of the mode m in the transversely isotropic cylindrical shell.

Now we sell pay attention to elastic waves, which have axial symmetry: the dependence from the angle φ

disappears. If vector of the shell displacement $\boldsymbol{U}$ has not of the component U_φ, then we have waves with the vertical polarization. In thin case components of strains $\gamma_{r\varphi}$, $\gamma_{\varphi z}$ and tangent stresses $\tau_{r\varphi}, \tau_{\varphi z}$ are equal to zero, but stresses $\sigma_r, \sigma_\varphi, \sigma_z$ and τ_{rz} are equal [2-4]:

$$\sigma_r = A_{11r} \cdot \frac{\partial U_r}{\partial r} + \frac{A_{12}}{r} \cdot U_r + A_{13} \cdot \frac{\partial U_z}{\partial z}; \qquad (26)$$

$$\sigma_\varphi = A_{12} \cdot \frac{\partial U_r}{\partial r} + \frac{A_{11}}{r} \cdot U_r + A_{13} \cdot \frac{\partial U_z}{\partial z}; \qquad (27)$$

$$\sigma_z = A_{12} \cdot \frac{\partial U_r}{\partial r} + \frac{A_{13}}{r} \cdot U_r + A_{33} \cdot \frac{\partial U_z}{\partial z}; \qquad (28)$$

$$\tau_{rz} = A_{44} \cdot \frac{\partial U_r}{\partial z} + A_{44} \cdot \frac{\partial U_z}{\partial r}. \qquad (29)$$

Equations of the dynamic balance (their only 2) have the following form [2-4]:

$$A_{11} \cdot \left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial U_r}{\partial r} - \frac{U_r}{r^2} \right) + A_{44} \cdot \left(\frac{\partial^2 U_r}{\partial z^2} + \frac{\partial^2 U_z}{\partial r^2} \right)$$
$$+ A_{13} \cdot \frac{\partial^2 U_z}{\partial z \partial r} + \rho \cdot \omega^2 \cdot U_r = 0; \qquad (30)$$

$$A_{44} \cdot \left(\frac{\partial^2 U_r}{\partial r \partial z} + \frac{\partial^2 U_z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial U_r}{\partial z} + \frac{1}{r} \cdot \frac{\partial U_z}{\partial r} \right)$$
$$+ A_{13} \cdot \left(\frac{\partial^2 U_r}{\partial r \partial z} + \frac{1}{r} \cdot \frac{\partial U_r}{\partial z} \right) + A_{33} \cdot \frac{\partial^2 U_z}{\partial z^2} + \rho \cdot \omega^2 \cdot U_z = 0. \qquad (31)$$

Displacements U_r and U_z can be taken in the form [2-4]:

$$U_r = \exp(i \cdot k \cdot z) \cdot U(r); \qquad (32)$$

$$U_z = \exp(i \cdot k \cdot z) \cdot W(r). \qquad (33)$$

For the thin shell $U(r)$ and $W(r)$ can be expanded in serieses:

$$U(r) = U = \sum_{n=0}^{N_1} x_n \cdot \xi_n; \qquad (34)$$

$$W(r) = W = \sum_{n=0}^{N_1} z_n \cdot \xi^n. \qquad (35)$$

Boundary conditions (their only 2) can be expressed as [2-4]:

$$\frac{\partial U}{\partial r} + \frac{a_9}{r} \cdot U + a_{10} \cdot i \cdot k \cdot W = 0; \quad [r = a; r = b] \quad (36)$$

$$i \cdot k \cdot U + \frac{\partial W}{\partial r} = 0. \quad [r = a; r = b] \qquad (37)$$

The substitution (32), (33) and (34), (35) into boundary conditions (36), (37) and into equations of the dynamic balance (30), (31) results in the system of $2 \cdot (N_1 + 1)$ equations to calculate unknown coefficients x_n, z_n. The characteristic equation for wave numbers k of elastic axisymmetrical waves in the transversely isotropic cylindrical shell we receive by expanding the determinant, which is equals zero. The axisymmetrical wave of the horizontal polarization (torsional wave) has only one component U_φ of the displacement vector $\boldsymbol{U}$. The problem in this case has the analytic solution. Components of strains $\varepsilon_r, \varepsilon_\varphi, \varepsilon_z, \gamma_{rz}$ are equal to zero, but components of strains $\gamma_{\varphi z}$ and $\gamma_{r\varphi}$ are equal to:

$$\gamma_{\varphi z} = \frac{\partial U_\varphi}{\partial z}; \gamma_{r\varphi} = \frac{\partial U_\varphi}{\partial r} - \frac{U_\varphi}{r}.$$

The equation of the dynamic balance has the following form:

$$\frac{\partial \tau_{r\varphi}}{\partial r} + \frac{\partial \tau_{\varphi z}}{\partial z} + \frac{2 \cdot \tau_{r\varphi}}{r} + \rho \cdot \omega^2 \cdot U_\varphi = 0. \qquad (38)$$

Used (2) and (3), we can describe (38) in the form:

$$\mu_1 \cdot \frac{\partial^2 U_\varphi}{\partial r^2} + \mu_3 \cdot \frac{\partial^2 U_\varphi}{\partial z^2} - \frac{\mu_1}{r^2} \cdot U_\varphi$$
$$+ \frac{\mu_1}{r} \cdot \frac{\partial U_\varphi}{\partial r} + \rho \cdot \omega^2 \cdot U_\varphi = 0. \qquad (39)$$

The component U_φ can be presented as:

$$U_\varphi = V(r) \cdot \exp(k_t \cdot z - \omega \cdot t), \qquad (40)$$

where k_t is the torsional wave number.

We substitute (40) in (39) and have:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial V}{\partial r} + \left(\frac{\rho \cdot \omega^2}{\mu_1} - k_t^2 \cdot \frac{\mu_3}{\mu_1} - \frac{1}{r^2} \right) \cdot V = 0. \quad (41)$$

The Equation (41) is the Bessel's equation for Bessel's $J_1(\chi_1 \cdot r)$ and Neiman $N_1(\chi_1 \cdot r)$ functions of the first order:

$$V(r) = B \cdot J_1(\chi_1 \cdot r) + C \cdot N_1(\chi_1 \cdot r), \qquad (42)$$

where B and C are arbitrary constants;

$$\chi_1 = \left(\chi^2 - k_t^2 \cdot \frac{\mu_3}{\mu_1} \right)^{1/2}; \chi = \left(\frac{\mu_1}{\rho} \right)^{-1/2} \cdot \omega.$$

From the boundary condition $\tau_{r\varphi} = 0 \left[r = a; r = b \right]$, we receive the characteristic equation for torsional wave numbers k_t:

$$\left[J_1'(\chi_1 \cdot a) - \frac{1}{a} \cdot J_1(\chi_1 \cdot a) \right] \cdot \left[N_1(\chi_1 \cdot b) - \frac{1}{b} \cdot N_1(\chi_1 \cdot b) \right] - \left[J_1(\chi_1 \cdot b) - \frac{1}{b} \cdot J_1(\chi_1 \cdot b) \right] \cdot \left[N_1'(\chi_1 \cdot a) - \frac{1}{a} \cdot N_1(\chi_1 \cdot a) \right] = 0,$$

where

$$J_1'(\chi_1 \cdot a) = \frac{\partial J_1(\chi_1 \cdot r)}{\partial r}[r = a].$$

4. Conclusions

In the paper, we found the characteristic equation for wave numbers of elastic waves in thin transversely isotropic cylindrical shell with the help of the dynamic theory of the elasticity for the orthotropic medium and of the hypothesis of thin shells both for three—dimensional and axially symmetric problems.

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