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Schultz and Modified Schultz Polynomials of Some Cog-Special Graphs

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Abstract

For a connected graph *G*, the Schultz and modified Schultz polynomials are defined as: $Sc(G;x)=\sum_{u,v\in V(G)}(\delta_u+\delta_v)x^{d(u,v)}$, and $\overset{*}{Sc}(G;x)=\sum_{u,v\in V(G)}(\delta_u\delta_v)x^{d(u,v)}$, respectively, where the summations are taken over all unordered pairs of distinct vertices in $V(G)$, δ_u is the degree of vertex u, $d(u,v)$ is the distance between u and v and $V(G)$ is the vertex set of G. In this paper, we find Schultz and modified Schultz polynomials of the Cog-special graphs such as a complete graph, a star graph, a wheel graph, a path graph and a cycle graph. The Schultz index, modified Schultz index and average distance of Schultz and modified Schultz of each such Cog-special graphs are also obtained in this paper.

Subject Areas

Combinatorial Mathematics, Discrete Mathematics

Keywords

Cog-Special Graphs, Schultz and Modified Schultz Polynomials

1. Introduction

We follow the terminology of [1] [2] [3] [4]. All the graphs considered in this paper are simple and connected finite undirected without loops or multiple edges. Distance is an important concept in graph theory and it has applications to computer science, chemistry, and a variety of other fields [5] [6].

Suppose that $G=(V(G),E(G))$ is a simple undirected connected graph of order $p=p(G)=|V(G)|$ and size $q=q(G)=|E(G)|$, the distance between two vertices u and v of G is denoted by $d(u,v)$ and it is defined as the length of a shortest (u,v)-path in connected graph G. In particular, if $u=v$, then $d(u,v)=0$. The greatest distance in G is the diameter and will be denoted by D. The number of pairs of vertices of G that are distance k is denoted by $d(G,k)$. Let $D_k(G)$ be the set of all unordered pairs of vertices with distance k such that $|D_k(G)|=d(G,k)$ and $\sum_{k=1}^{D} d(G,k)=\binom{p}{2}$, where $\binom{p}{2}$ is representation of the number of unordered pairs distinct vertices in G.

The Schultz polynomial of a graph G is defined as:

$$Sc(G;x)=\sum_{u,v\in V(G)}(\delta_u+\delta_v)x^{d(u,v)},$$

and modified Schultz polynomial of a graph G is defined as:

$$\overset{*}{Sc}(G;x)=\sum_{u,v\in V(G)}(\delta_u\delta_v)x^{d(u,v)}.$$

The molecular topological index (Schultz index) was introduced by Harry P. Schultz in 1993 [7] and the modified Schultz index was defined by S. Klavžar and I. Gutman in 1997 [8].

The Schultz index is defined as:

$$Sc(G)=\sum_{u,v\in V(G)}(\delta_u+\delta_v)d(u,v),$$

and modified Schultz index is defined as:

$$\overset{*}{Sc}(G)=\sum_{u,v\in V(G)}(\delta_u\delta_v)d(u,v).$$

where the summation for all above is taken over all unordered pairs of distinct vertices in $V(G)$.

The indices of Schultz and modified Schultz can be obtained by the derivative of Schultz and modified Schultz polynomials with respect to x at $x=1$, *i.e.*:

$$Sc(G)=\frac{\mathrm{d}}{\mathrm{d}x}Sc(G;x)\Big|_{x=1}, \text{ and } \overset{*}{Sc}(G)=\frac{\mathrm{d}}{\mathrm{d}x}\overset{*}{Sc}(G;x)\Big|_{x=1} \text{ respectively.}$$

The average distance of a connected graph G with respect Schultz and modified Schultz is defined as:

$$\overline{Sc(G)}=\frac{Sc(G)}{\binom{p}{2}} \text{ and } \overline{\overset{*}{Sc}(G)}=\frac{\overset{*}{Sc}(G)}{\binom{p}{2}}.$$

Schultz and modified Schultz polynomial of two operations Gutman's and the Cog-complete bipartite Graphs founded by Ahmed and Haitham [9] [10], the Schultz and modified Schultz polynomial of some special graphs are summarized in the following theorem (See [11]).

Theorem 1.1:

1) $Sc(K_p;x)=p(p-1)^2x^1$, $\overset{*}{Sc}(K_p;x)=\{p(p-1)^3/2\}x^1$.

2) $Sc(S_{p+1};x) = p(p+1)x^1 + p(p-1)x^2$,

$\overset{*}{Sc}(S_{p+1};x) = p^2x^1 + \{p(p-1)/2\}x^2$.

3) $Sc(W_{p+1};x) = (p^2+9p+6)x^1 + 3p(p-3)x^2$,

$\overset{*}{Sc}(W_{p+1};x) = 3(p^2+3p+3)x^1 + \{9p(p-3)/2\}x^2$.

4) $Sc(P_p;x) = \sum_{k=1}^{p-1}[4(p-k)-2]x^k$, $\overset{*}{Sc}(P_p;x) = 4\sum_{k=1}^{p-1}(p-k-1)x^k + x^{p-1}$.

5) $Sc(C_p;x) = \overset{*}{Sc}(C_p;x) = 4p\sum_{k=1}^{\lceil p/2\rceil -1} x^k + \begin{cases} 2px^{p/2}, & p \text{ is even,} \\ 0, & p \text{ is odd.} \end{cases}$

2. Main Results

2.1. Definition

A cog-complete graph K_p^c is the graph constructed from a complete graph K_p, $p \geq 3$, of vertex set $\{u_1, u_2, \cdots, u_p\}$ with p additional vertices $\{v_1, v_2, \cdots, v_p\}$, and 2p edges $\{v_iu_i, v_iu_{i+1} : i = 1, 2, \cdots, p\}$, $(u_{p+1} \equiv u_1)$, as shown in Figure 1.

It is clear that $p(K_p^c) = 2p$, $q(K_p^c) = p(p+3)/2$, and $diam\, K_p^c = 3$, for $p \geq 4$.

Theorem 2.1.1: For $p \geq 4$, we have:

1) $Sc(K_p^c;x) = p(p^2+2p+5)x^1 + p(p-1)(p+2)x^2 + 2p(p-3)x^3$.

2) $\overset{*}{Sc}(K_p^c;x) = \{p(p+1)(p^2+7)/2\}x^1 + 2p^2(p-1)x^2 + 2p(p-3)x^3$.

Proof: For every vertice $y, z \in V(K_p^c)$, there is $d(y,z) = k$, $k = 1, 2, 3$, and obviously $\sum_{i=1}^{3}|D_i| = p(2p-1)$.

We will have three partitions for proof:

P1. If $d(y,z) = 1$, then $|D_1| = p(p+3)/2$ and is equal to $q(K_p^c)$, we have two subsets of it:

P1.1. $$\left|\left\{(u_i, v_j) : u_iv_j \in E(K_p^c), \delta_{u_i} + \delta_{v_j} = p+3 \,\&\, \delta_{u_i}\delta_{v_j} = 2(p+1), i = j, j+1, 1 \leq j \leq p, (u_{p+1} \equiv u_1)\right\}\right| = 2p.$$

P1.2. $$\left|\left\{(u_i, u_j) : u_iu_j \in E(K_p^c), \delta_{u_i} + \delta_{u_j} = 2(p+1) \,\&\, \delta_{u_i}\delta_{u_j} = (p+1)^2, 1 \leq i, j \leq p, i \neq j\right\}\right| = p(p-1)/2.$$

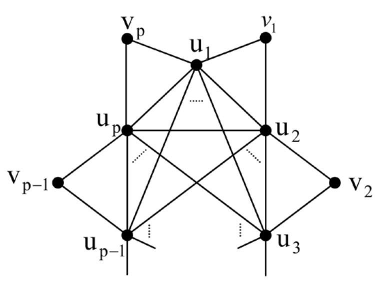

Figure 1. A cog-complete graph K_p^c.

P2. If $d(y,z)=2$, then $|D_2|=p(p-1)$, we have two subsets of it:

P2.1. $\left|\left\{(v_i,v_{i+1}):\delta_{v_i}+\delta_{v_{i+1}}=4\,\&\,\delta_{v_i}\delta_{v_{i+1}}=4, 1\le i\le p, (v_{p+1}\equiv v_1)\right\}\right|=p$.

P2.2. $\left|\left\{(v_i,u_j):v_iu_j\notin E(K_p^c), \delta_{v_i}+\delta_{u_j}=p+3\,\&\,\delta_{v_i}\delta_{u_j}=2(p+1),\right.\right.$
$\left.\left. 1\le i,j\le p, i\ne j, j+1 (u_{p+1}\equiv u_1)\right\}\right|=p(p-2)$.

P3. If $d(y,z)=3$, then $|D_3|=p(p-3)/2$, we have:

$$\left|\left\{(v_i,v_j):\delta_{v_i}+\delta_{v_j}=4\,\&\,\delta_{v_i}\delta_{v_j}=4, 1\le i,j\le p, |i-j|\ne 0,1\right\}\right|-\left|\left\{(v_1,v_p)\right\}\right|$$
$$=p(p-3)/2.$$

From **P1 - P3**, we have:

$$Sc(K_p^c;x)=p(p^2+2p+5)x^1+p(p-1)(p+2)x^2+2p(p-3)x^3.$$

$$\overset{*}{Sc}(K_p^c;x)=\left\{p(p+1)(p^2+7)/2\right\}x^1+2p^2(p-1)x^2+2p(p-3)x^3.$$

Corollary 2.1.2: For $p\ge 4$, we have:

1) $Sc(K_p^c)=p(3p^2+10p-17)$.

2) $\overset{*}{Sc}(K_p^c)=p(p^3+9p^2+11p-29)/2$.

Corollary 2.1.3: For $p\ge 4$, we have:

1) $10\frac{1}{7}\le \overline{Sc}(K_p^c)<(6p+23)/4$.

2) $15\frac{13}{14}\le \overline{\overset{*}{Sc}}(K_p^c)<(4p^2+38p+63)/16$.

Remark 2.1.4:

1) $Sc(K_3^c;x)=60x^1+30x^2$.

2) $\overset{*}{Sc}(K_3^c;x)=96x^1+36x^2$.

2.2. Definition

A cog-star graph S_{p+1}^c is the graph constructed from a star graph, S_{p+1}, $p\ge 3$, of vertex set $\{u_0,u_1,\cdots,u_{p-1},u_p\}$ with p additional vertices $\{v_1,v_2,\cdots,v_{p-1},v_p\}$, and edges $\{v_iu_i, v_iu_{i+1}: i=1,2,\cdots,p\}$, $(u_{p+1}\equiv u_1)$, as shown in Figure 2.

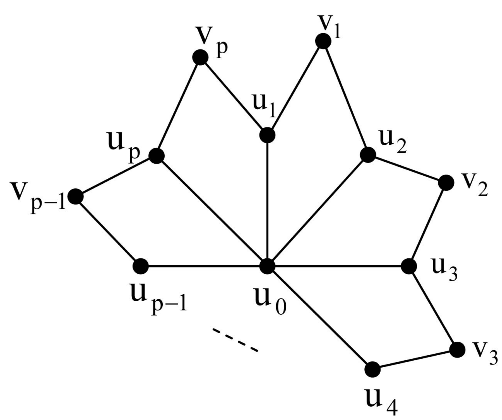

Figure 2. A cog-star graph S_{p+1}^c.

It is clear that $p\left(S_{p+1}^{c}\right)=2p+1$, $q\left(S_{p+1}^{c}\right)=3p$, $diam\,S_{p+1}^{c}=4$ for $p\geq 4$.

Theorem 2.2.1: For $p\geq 4$, we have:

1) $Sc\left(S_{p+1}^{c};x\right)=p(p+13)x^{1}+p(4p+3)x^{2}+5p(p-2)x^{3}+2p(p-3)x^{4}$.

2) $\overset{*}{Sc}\left(S_{p+1}^{c};x\right)=3p(p+4)x^{1}+\{p(13p-1)/2\}x^{2}+6p(p-2)x^{3}+2p(p-3)x^{4}$

Proof: For every vertice $y,z\in V\left(S_{p+1}^{c}\right)$, there is $d(y,z)=k$, $k=1,2,3,4$, and obviously $\sum_{i=1}^{4}|D_i|=p(2p+1)$.

We will have four partitions for proof:

P1. If $d(y,z)=1$, then $|D_1|=3p$ and is equal to $q\left(S_{p+1}^{c}\right)$, we have two subsets of it:

P1.1. $\left|\left\{(u_0,u_i):u_0u_i\in E\left(S_{p+1}^{c}\right),\delta_{u_0}+\delta_{u_i}=p+3\,\&\,\delta_{u_0}\delta_{u_i}=3p,1\leq i\leq p\right\}\right|=p.$

P1.2. $\left|\left\{(v_i,u_j):v_iu_j\in E\left(S_{p+1}^{c}\right),\delta_{v_i}+\delta_{u_j}=5\,\&\,\delta_{v_i}\delta_{u_j}=6,1\leq i\leq p,\; j=i,i+1,(u_{p+1}\equiv u_1)\right\}\right|=2p.$

P2. If $d(y,z)=2$, then $|D_2|=p(p+3)/2$, we have three subsets:

P2.1. $\left|\left\{(u_i,u_j):\delta_{u_i}+\delta_{u_j}=6\,\&\,\delta_{u_i}\delta_{u_j}=9,1\leq i,j\leq p,i\neq j\right\}\right|=p(p-1)/2.$

P2.2. $\left|\left\{(u_0,v_i):\delta_{u_0}+\delta_{v_i}=p+2\,\&\,\delta_{u_0}\delta_{v_i}=2p,1\leq i\leq p\right\}\right|=p.$

P2.3. $\left|\left\{(v_i,v_{i+1}):\delta_{v_i}+\delta_{v_{i+1}}=4\,\&\,\delta_{v_i}\delta_{v_{i+1}}=4,1\leq i\leq p,(v_{p+1}\equiv v_1)\right\}\right|=p.$

P3. If $d(y,z)=3$, then $|D_3|=p(p-2)$, we have:

$$\left|\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=5\,\&\,\delta_{u_i}\delta_{v_j}=6,1\leq i,j\leq p,i-j\neq 0,1\right\}-\left\{(u_1,v_p)\right\}\right|=p(p-2).$$

P4. If $d(y,z)=4$, then $|D_4|=p(p-3)/2$, we have:

$$\left|\left\{(v_i,v_j):\delta_{v_i}+\delta_{v_j}=4\,\&\,\delta_{v_i}\delta_{v_j}=4,1\leq i\leq p-2,i+2\leq j\leq p\right\}-\left\{(v_1,v_p)\right\}\right|=p(p-3)/2.$$

From **P1 - P4**, we have:

$$Sc\left(S_{p+1}^{c};x\right)=p(p+13)x^{1}+p(4p+3)x^{2}+5p(p-2)x^{3}+2p(p-3)x^{4}.$$

$$\overset{*}{Sc}\left(S_{p+1}^{c};x\right)=3p(p+4)x^{1}+\{p(13p-1)/2\}x^{2}+6p(p-2)x^{3}+2p(p-3)x^{4}.$$

Corollary 2.2.2: For $p\geq 4$, we have:

1) $Sc\left(S_{p+1}^{c}\right)=p(32p-35)$.

2) $\overset{*}{Sc}\left(S_{p+1}^{c}\right)=7p(6p-7)$.

Corollary 2.2.3: For $p\geq 4$, we have:

1) $10\frac{1}{3}\leq\overline{Sc}\left(S_{p+1}^{c}\right)<16$.

2) $13\frac{2}{9}\leq\overline{\overset{*}{Sc}}\left(S_{p+1}^{c}\right)<21$.

Remark 2.2.3:

1) $Sc\left(S_4^c;x\right)=48x^1+45x^2+15x^3$.

2) $\overset{*}{Sc}\left(S_4^c;x\right)=63x^1+57x^2+18x^3$.

2.3. Definition

A cog-wheel graph W_{p+1}^c is the graph constructed from a wheel W_{p+1}, $p\ge 3$, of order $p+1$, with vertex set $\{u_0,u_1,u_2,\cdots,u_p\}$ and with p additional vertices $v_1,v_2,\cdots,v_p$, and edges $\{v_iu_i,v_iu_{i+1}:i=1,2,\cdots,p\}$, $\left(u_{p+1}\equiv u_1\right)$, as shown in Figure 3.

It is clear that $p\left(W_{p+1}^c\right)=2p+1$, $q\left(W_{p+1}^c\right)=4p$, $diam W_{p+1}^c=4$ for $p\ge 6$.

Theorem 2.3.1: For $p\ge 6$, we have:

1) $Sc\left(W_{p+1}^c;x\right)=p(p+29)x^1+p(6p+5)x^2+p(7p-24)x^3+2p(p-5)x^4$.

2) $\overset{*}{Sc}\left(W_{p+1}^c;x\right)=5p(p+9)x^1+\{p(29p-27)/2\}/x^2+2p(5p-18)x^3$
$+2p(p-5)x^4$.

Proof: For every vertice $y,z\in V\left(W_{p+1}^c\right)$, there is $d(y,z)=k$, $k=1,2,3,4$, and obviously $\sum_{i=1}^{4}|D_i|=p(2p+1)$.

We will have four partitions for proof:

P1. If $d(y,z)=1$, then $|D_1|=4p$ and is equal to $q\left(W_{p+1}^c\right)$, we have three subsets of it:

P1.1. $\left|\left\{(u_0,u_i):u_0u_i\in E\left(W_{p+1}^c\right),\delta_{u_0}+\delta_{u_i}=p+5\,\&\,\delta_{u_0}\delta_{u_i}=5p,1\le i\le p\right\}\right|=p.$

P1.2. $\left|\left\{(u_i,u_{i+1}):u_iu_{i+1}\in E\left(W_{p+1}^c\right),\delta_{u_i}+\delta_{u_{i+1}}=10\right.\right.$
$\left.\left.\&\,\delta_{u_i}\delta_{u_{i+1}}=25,1\le i\le p,\left(u_{p+1}\equiv u_1\right)\right\}\right|=p.$

P1.3. $\left|\left\{(v_i,u_j):v_iu_j\in E\left(W_{p+1}^c\right),\delta_{v_i}+\delta_{u_j}=7\right.\right.$
$\left.\left.\&\,\delta_{v_i}\delta_{u_j}=10,1\le i\le p,j=i,i+1,\left(u_{p+1}\equiv u_1\right)\right\}\right|=2p.$

P2. If $d(y,z)=2$, then $|D_2|=p(p+5)/2$, we have five subsets

P2.1. $\left|\left\{(u_i,u_j):\delta_{u_i}+\delta_{u_j}=10\,\&\,\delta_{u_i}\delta_{u_j}=25,1\le i\le p-2,i+2\le j\le p\right\}\right.$
$\left.-\left\{(u_1,u_p)\right\}\right|=p(p-3)/2.$

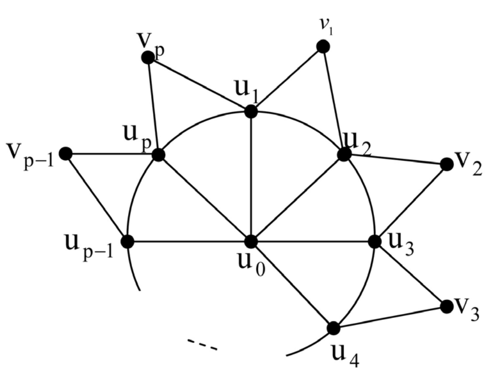

Figure 3. A cog-wheel graph W_{p+1}^c.

P2.2. $\left|\left\{(u_0,v_i):\delta_{u_0}+\delta_{v_i}=p+2\ \&\ \delta_{u_0}\delta_{v_i}=2p,1\le i\le p\right\}\right|=p$.

P2.3. $\left|\left\{(u_i,v_{i+1}):\delta_{u_i}+\delta_{v_{i+1}}=7\ \&\ \delta_{u_i}\delta_{v_{i+1}}=10,1\le i\le p,(v_{p+1}\equiv v_1)\right\}\right|=p.$

P2.4. $\left|\left\{(u_i,v_{i-2}):\delta_{u_i}+\delta_{v_{i-2}}=7\ \&\ \delta_{u_i}\delta_{v_{i-2}}=10,3\le i\le p\right\}\cup\left\{(u_1,v_{p-1}),(u_2,v_p)\right\}\right|=p.$

P2.5. $\left|\left\{(v_i,v_{i+1}):\delta_{v_i}+\delta_{v_{i+1}}=4\ \&\ \delta_{v_i}\delta_{v_{i+1}}=4,1\le i\le p,(v_{p+1}\equiv v_1)\right\}\right|=p.$

P3. If $d(y,z)=3$, then $|D_3|=p(p-3)$, we have two subsets:

P3.1. $\left|\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=7\ \&\ \delta_{u_i}\delta_{v_j}=10,3\le i\le p,1\le j\le p,j\ne i-2,i-1,i,i+1,(v_{p+1}\equiv v_1)\right\}\cup\left\{(u_i,v_j):i=1,2,i+2\le j\le p+i-3\right\}\right|$
$=p(p-4).$

P3.2.

$$\left|\left\{(v_i,v_{i+2}):\delta_{v_i}+\delta_{v_{i+2}}=4\ \&\ \delta_{v_i}\delta_{v_{i+2}}=4,1\le i\le p,(v_{p+1}\equiv v_1),(v_{p+2}\equiv v_2)\right\}\right|=p.$$

P4. If $d(y,z)=4$, then $|D_4|=p(p-5)/2$, we have:

$$\left|\left\{(v_i,v_j):\delta_{v_i}+\delta_{v_j}=4\ \&\ \delta_{v_i}\delta_{v_j}=4,3\le i\le p,1\le j\le p,j\ne i-2,i-1,i,i+1,i+2,(v_{p+1}\equiv v_1),(v_{p+2}\equiv v_2)\right\}\cup\left\{(v_i,v_j):i=1,2,i+3\le j\le p+i-3\right\}\right|$$
$$=p(p-5)/2.$$

From **P1** - **P4**, we have:

$$Sc(W_{p+1}^c;x)=p(p+29)x^1+p(6p+5)x^2+p(7p-24)x^3+2p(p-5)x^4.$$

$$\overset{*}{Sc}(W_{p+1}^c;x)=5p(p+9)x^1+\{p(29p-27)/2\}x^2+2p(5p-18)x^3+2p(p-5)x^4.$$

Corollary 2.3.2: For $p\ge 6$, we have:

1) $Sc(W_{p+1}^c)=p(42p-73)$.

2) $\overset{*}{Sc}(W_{p+1}^c)=2p(36p-65)$.

Corollary 2.3.3: For $p\ge 6$, we have:

1) $13\frac{10}{13}\le\overline{Sc}(W_{p+1}^c)<21$.

2) $23\frac{3}{13}\le\overline{\overset{*}{Sc}}(W_{p+1}^c)<36$.

Remark 2.3.4:

1) $Sc(W_6^c;x)=170x^1+175x^2+55x^3$, $Sc(W_5^c;x)=132x^1+116x^2+8x^3$, $Sc(W_4^c;x)=96x^1+48x^2$.

2) $\overset{*}{Sc}(W_6^c;x)=350x^1+295x^2+70x^3$, $\overset{*}{Sc}(W_5^c;x)=260x^1+178x^2+8x^3$, $\overset{*}{Sc}(W_4^c;x)=180x^1+60x^2$.

2.4. Definition

A saw graph P_p^c is a path of order *p*, say $\{u_1, u_2, \cdots, u_p\}$, with $p-1$ additional vertices $\{v_1, v_2, \cdots, v_{p-1}\}$ and edges $\{v_i u_i, v_i u_{i+1} : i = 1, 2, \cdots, p-1\}$ as depicted in **Figure 4**.

It is clear $p(P_p^c) = 2p-1$, $q(P_p^c) = 3(p-1)$ and $diam\, P_p^c = p-1$, for $p \ge 2$.

Theorem 2.4.1: For $p \ge 5$, we have:

1) $Sc(P_p^c; x) = 4(5p-7)x^1 + 8\sum_{k=2}^{p-3}(3p-3k-1)x^k + 40x^{p-2} + 16x^{p-1}$.

2) $\overset{*}{Sc}(P_p^c; x) = 8(4p-7)x^1 + 12\sum_{k=2}^{p-3}(3p-3k-2)x^k + 48x^{p-2} + 16x^{p-1}$.

Proof:

For every vertice $y, z \in V(P_p^c)$, there is $d(y,z) = k$, $1 \le k \le p-1$, and obviously $\sum_{i=1}^{p-1}|D_i| = (2p-1)(p-1)$.

We will have four partitions for proof:

P1. if $d(y,z) = 1$, then $|D_1| = 3(p-1)$ and is equal to $q(P_p^c)$, we have five subsets of it:

P1.1. $\left|\{(u_i, u_{i+1}) : u_i u_{i+1} \in E(P_p^c), \delta_{u_i} + \delta_{u_{i+1}} = 6, \delta_{u_i}\delta_{u_{i+1}} = 8, i = 1, p-1\}\right| = 2.$

P1.2. $\left|\{(u_1, v_1), (u_p, v_{p-1}) : u_1 v_1, u_p v_{p-1} \in E(P_p^c), \delta_{u_1(u_p)} + \delta_{v_1(v_{p-1})} = 4, \delta_{u_1(u_p)}\delta_{v_1(v_{p-1})} = 4\}\right| = 2.$

P1.3. $\left|\{(u_i, v_{i-1}) : u_i v_{i-1} \in E(P_p^c), \delta_{u_i} + \delta_{v_{i-1}} = 6, \delta_{u_i}\delta_{v_{i-1}} = 8, 2 \le i \le p-1\}\right| = p-2.$

P1.4. $\left|\{(u_i, v_i) : u_i v_i \in E(P_p^c), \delta_{u_i} + \delta_{v_i} = 6, \delta_{u_i}\delta_{v_i} = 8, 2 \le i \le p-1\}\right| = p-2.$

P1.5.

$$\left|\{(u_i, u_{i+1}) : u_i u_{i+1} \in E(P_p^c), \delta_{u_i} + \delta_{u_{i+1}} = 8, \delta_{u_i}\delta_{u_{i+1}} = 16, 2 \le i \le p-2\}\right| = p-3.$$

P2. if $d(y,z) = k$, $2 \le k \le p-3$, then $\sum_{k=2}^{p-3}|D_k| = 2(p-4)(p+1)$, we have six subsets of it:

P2.1. $\left|\{(u_1, u_{k+1}) : \delta_{u_1} + \delta_{u_{k+1}} = 6, \delta_{u_1}\delta_{u_{k+1}} = 8\} \cup \{(u_1, v_k) : \delta_{u_1} + \delta_{v_k} = 4, \delta_{u_1}\delta_{v_k} = 4\}\right| = 2.$

P2.2. $\left|\{(u_p, u_{p-k}) : \delta_{u_p} + \delta_{u_{p-k}} = 6, \delta_{u_p}\delta_{u_{p-k}} = 8\} \cup \{(u_p, v_{p-k}) : \delta_{u_p} + \delta_{v_{p-k}} = 4, \delta_{u_p}\delta_{v_{p-k}} = 4\}\right| = 2.$

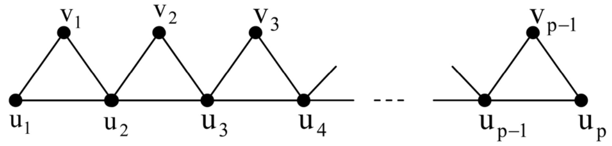

Figure 4. A saw graph P_p^c.

P2.3. $\left|\left\{\left(u_i,u_{k+i}\right):\delta_{u_i}+\delta_{u_{k+i}}=8,\delta_{u_i}\delta_{u_{k+i}}=16,2\le i\le p-k-1\right\}\right|$
$=p-k-2.$

P2.4. $\left|\left\{\left(u_i,v_{k+i-1}\right):\delta_{u_i}+\delta_{v_{k+i-1}}=6,\delta_{u_i}\delta_{v_{k+i-1}}=8,2\le i\le p-k\right\}\right|=p-k-1.$

P2.5. $\left|\left\{\left(u_{p-i+1},v_{p-k-i+1}\right):\delta_{u_{p-i+1}}+\delta_{v_{p-k-i+1}}=6,\delta_{u_{p-i+1}}\delta_{v_{p-k-i+1}}=8,2\le i\le p-k\right\}\right|$
$=p-k-1.$

P2.6. $\left|\left\{\left(v_i,v_{i+k-1}\right):\delta_{v_i}+\delta_{v_{i+k-1}}=4,\delta_{v_i}\delta_{v_{i+k-1}}=4,1\le i\le p-k\right\}\right|=p-k.$

P3. if $d(y,z)=p-2$ then $\left|D_{p-2}\right|=8$, we have six subsets of it:

P3.1. $\left|\left\{\left(u_i,u_{p+i-2}\right):\delta_{u_i}+\delta_{u_{p+i-2}}=6,\delta_{u_i}\delta_{u_{p+i-2}}=8,i=1,2\right\}\right|=2.$

P3.2. $\left|\left\{\left(u_1,v_{p-2}\right):\delta_{u_1}+\delta_{v_{p-2}}=4,\delta_{u_1}\delta_{v_{p-2}}=4\right\}\right|=1.$

P3.3. $\left|\left\{\left(u_p,v_2\right):\delta_{u_p}+\delta_{v_2}=4,\delta_{u_p}\delta_{v_2}=4\right\}\right|=1.$

P3.4. $\left|\left\{\left(u_2,v_{p-1}\right):\delta_{u_2}+\delta_{v_{p-1}}=6,\delta_{u_2}\delta_{v_{p-1}}=8\right\}\right|=1.$

P3.5. $\left|\left\{\left(u_{p-1},v_1\right):\delta_{u_{p-1}}+\delta_{v_1}=6,\delta_{u_{p-1}}\delta_{v_1}=8\right\}\right|=1.$

P3.6. $\left|\left\{\left(v_i,v_{p+i-3}\right):\delta_{v_i}+\delta_{v_{p+i-3}}=4,\delta_{v_i}\delta_{v_{p+i-3}}=4,i=1,2\right\}\right|=2.$

P4. if $d(y,z)=p-1$ then $\left|D_{p-1}\right|=4$, we have four subsets of it:

P4.1. $\left|\left\{\left(u_1,u_p\right):\delta_{u_1}+\delta_{u_p}=4,\delta_{u_1}\delta_{u_p}=4\right\}\right|=1.$

P4.2. $\left|\left\{\left(u_1,v_{p-1}\right):\delta_{u_1}+\delta_{v_{p-1}}=4,\delta_{u_1}\delta_{v_{p-1}}=4\right\}\right|=1.$

P4.3. $\left|\left\{\left(u_p,v_1\right):\delta_{u_p}+\delta_{v_1}=4,\delta_{u_p}\delta_{v_1}=4\right\}\right|=1.$

P4.4. $\left|\left\{\left(v_1,v_{p-1}\right):\delta_{v_1}+\delta_{v_{p-1}}=4,\delta_{v_1}\delta_{v_{p-1}}=4\right\}\right|=1.$

From **P1** - **P4**, we have:

$$Sc\left(P_p^{\,c};x\right)=4(5p-7)x+8\sum_{k=2}^{p-3}(3p-3k-1)x^k+40x^{p-2}+16x^{p-1}.$$

$$\overset{*}{Sc}\left(P_p^c;x\right)=8(4p-7)x+12\sum_{k=2}^{p-3}(3p-3k-2)x^k+48x^{p-2}+16x^{p-1}.$$

Corollary2.4.2: For $p\ge 5$, then:

1) $Sc\left(P_p^c\right)=4(p+1)(p-1)^2$.

2) $\overset{*}{Sc}\left(P_p^c\right)=6p(p-1)^2$.

Corollary2.4.3: For $p\ge 5$, then:

1) $10\frac{2}{3}\le\overline{Sc\left(P_p^c\right)}\le 2p+1$.

2) $13\frac{1}{3}\le\overline{\overset{*}{Sc}\left(P_p^c\right)}<3(2p-1)/2$.

Remark 2.4.4:

1) $Sc\left(P_3^c;x\right)=32x^1+16x^2$, $Sc\left(P_4^c;x\right)=52x^1+40x^2+16x^3$.

2) $\overset{*}{Sc}\left(P_3^c;x\right)=40x^1+16x^2$, $\overset{*}{Sc}\left(P_4^c;x\right)=72x^1+48x^2+16x^3$.

2.5. Definition

A Cog-Cycle is a graph $C_p^c, p \ge 3$ obtained from a cycle graph $C_p = \{u_1, u_2, \cdots, u_p, u_1\}$ with p additional vertices $\{v_1, v_2, \cdots, v_p\}$, and edges $\{v_i u_i, v_i u_{i+1} : i = 1, 2, \cdots, p, (u_1 \equiv u_{p+1})\}$ as shown in Figure 5.

It's clear that $p(C_p^c) = 2p, q(C_p^c) = 3p$, and

$$diam\, C_p^c = \begin{cases} (p/2)+1, & p \text{ is even } p \ge 4, \\ (p+1)/2, & p \text{ is odd } p \ge 3. \end{cases}$$

Theorem 2.5.1: For $p \ge 6$, then:

1) $Sc(C_p^C; x) = 20px + 24p \sum_{k=2}^{\lfloor \frac{p}{2} \rfloor - 1} x^k + 2p \begin{cases} 10x^{\frac{p}{2}} + x^{\frac{p}{2}+1}, & p \text{ is even}, \\ 12x^{\frac{p-1}{2}} + 5x^{\frac{p+1}{2}}, & p \text{ is odd}. \end{cases}$

2) $\overset{*}{Sc}(C_p^c; x) = 32px + 36p \sum_{k=2}^{\lfloor \frac{p}{2} \rfloor - 1} x^k + 2p \begin{cases} 14x^{\frac{p}{2}} + x^{\frac{p}{2}+1}, & p \text{ is even}, \\ 18x^{\frac{p-1}{2}} + 6x^{\frac{p+1}{2}}, & p \text{ is odd}. \end{cases}$

Proof: For every vertice $y, z \in V(C_p^c)$, there is $d(y,z) = k$, $1 \le k \le \left\lfloor \frac{p}{2} \right\rfloor + 1$, and obviously $\sum_{i=1}^{\lfloor \frac{p}{2} \rfloor + 1} |D_i| = p(2p-1)$. We will four partitions for proof:

P1. if $d(y,z) = 1$, then $|D_i| = 3p$ and is equal to $q(C_p^c)$. We have three subsets of it:

P1.1.

$$\left|(u_i, u_{i+1}) : u_i u_{i+1} \in E(C_p^c), \delta_{u_i} + \delta_{u_{i+1}} = 8, \delta_{u_i} \delta_{u_{i+1}} = 16, 1 \le i \le p, (u_{p+1} \equiv u_1)\right| = p.$$

P1.2. $\left|\{(u_i, v_i) : u_i v_i \in E(C_p^c), \delta_{u_i} + \delta_{v_i} = 6, \delta_{u_i} \delta_{v_i} = 8, 1 \le i \le p\}\right| = p.$

P1.3.

$$\left|\{(u_{i+1}, v_i) : u_{i+1} v_i \in E(C_p^c), \delta_{u_{i+1}} + \delta_{v_i} = 6, \delta_{u_{i+1}} \delta_{v_i} = 8, 1 \le i \le p, (u_{p+1} \equiv u_1)\}\right| = p$$

P2. If $d(y,z) = k$, $2 \le k \le \left\lfloor \frac{p}{2} \right\rfloor - 1$, then $\sum_{k=2}^{\lfloor \frac{p}{2} \rfloor - 1} |D_k| = 4p$. We have four subsets of it:

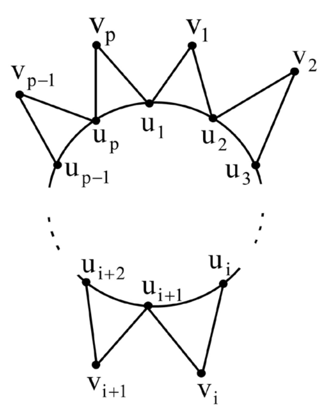

Figure 5. A Cog-Cycle Graph C_p^c.

when u_i moving to v_j clockwise.

P2.1. $\left|\left\{(u_i,u_j):\delta_{u_i}+\delta_{u_j}=8,\delta_{u_i}\delta_{u_j}=16,1\le i,j\le p,|i-j|=k,p-k\right\}\right|=p.$

P2.2. $\left|\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=6,\delta_{u_i}\delta_{v_j}=8,1\le i,j\le p,|i-j|=k-1,p-k+1\right\}\right|=p.$

when u_i moving to v_j reversed clockwise.

P2.3. $\left|\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=6,\delta_{u_i}\delta_{v_j}=8,1\le i,j\le p,|i-j|=k,p-k\right\}\right|=p.$

P2.4. $\left|\left\{(v_i,v_j):\delta_{v_i}+\delta_{v_j}=4,\delta_{v_i}\delta_{v_j}=4,1\le i,j\le p,|i-j|=k-1,p-k+1\right\}\right|=p.$

P3. If $d(y,z)=\lfloor p/2\rfloor$, when p is even, then $\left|D_{p/2}\right|=7p/2$, we have four subsets of it:

P3.1. $\left|\left\{(u_i,u_{i+p/2}):\delta_{u_i}+\delta_{u_{p/2}}=8,\delta_{u_i}\delta_{u_{p/2}}=16,1\le i\le p/2\right\}\right|=p/2.$

when u_i moving to v_j clockwise.

P3.2. $$\left|\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=6,\delta_{u_i}\delta_{v_j}=8,1\le i\le 1+(p/2)+1,j=(p/2)+i-1\right\}\right.$$
$$\left.\cup\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=6,\delta_{u_i}\delta_{v_j}=8,(p/2)+2\le i\le p,j=i-(p/2)-1\right\}\right|$$
$$=p.$$

when u_i moving to v_j reversed clockwise.

P3.3. $$\left|\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=6,\delta_{u_i}\delta_{v_j}=8,1\le i\le p/2,j=(p/2)+i\right\}\right.$$
$$\left.\cup\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=6,\delta_{u_i}\delta_{v_j}=8,1+(p/2)\le i\le p,j=i-(p/2)\right\}\right|$$
$$=p.$$

P3.4. $$\left|\left\{(v_i,v_j):\delta_{v_i}+\delta_{v_j}=4,\delta_{v_i}\delta_{v_j}=4,1\le i\le (p/2)+1,j=i+(p/2)-1\right\}\right.$$
$$\left.\cup\left\{(v_i,v_j):\delta_{v_i}+\delta_{v_j}=4,\delta_{v_i}\delta_{v_j}=4,2+(p/2)\le i\le p,j=i-(p/2)-1\right\}\right|$$
$$=p.$$

when p is odd, then $\left|D_{(p-1)/2}\right|=4p$, we have four subsets of it:

P'3.1. $\left|\left\{(u_i,u_j):\delta_{u_i}+\delta_{u_j}=8,\delta_{u_i}\delta_{u_j}=16,1\le i,j\le p,|i-j|=\left\lfloor\frac{p}{2}\right\rfloor,\left\lceil\frac{p}{2}\right\rceil\right\}\right|=p.$

when u_i moving to v_j clockwise.

P'3.2. $$\left|\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=6,\delta_{u_i}\delta_{v_j}=8,1\le i\le (p+3)/2,j=i+(p-3)/2\right\}\right.$$
$$\left.\cup\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=6,\delta_{u_i}\delta_{v_j}=8,(p+5)/2\le i\le p,j=i-(p+3)/2\right\}\right|$$
$$=p.$$

when u_i moving to v_j reversed clockwise.

P'3.3. $$\left|\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=6,\delta_{u_i}\delta_{v_j}=8,1\le i\le (p-1)/2,j=i+(p+1)/2\right\}\right.$$
$$\left.\cup\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=6,\delta_{u_i}\delta_{v_j}=8,(p+1)/2\le i\le p,j=i-(p-1)/2\right\}\right|$$
$$=p.$$

P'3.4. $$\left|\left\{(v_i,v_j):\delta_{v_i}+\delta_{v_j}=4,\delta_{v_i}\delta_{v_j}=4,1\le i\le (p+3)/2,j=i+(p-3)/2\right\}\right.$$
$$\left.\cup\left\{(v_i,v_j):\delta_{v_i}+\delta_{v_j}=4,\delta_{v_i}\delta_{v_j}=4,(p+5)/2\le i\le p,j=i-(p-3)/2\right\}\right|$$
$$=p.$$

P4. If $d(y,z)=\lfloor p/2\rfloor+1$, when p is even then $|D_{1+p/2}|=p/2$, we have:

$$\left|\left\{(v_i,v_{i+p/2}):\delta_{v_i}+\delta_{v_{i+p/2}}=4,\delta_{v_i}\delta_{v_{i+p/2}}=4,1\le i\le p/2\right\}\right|=p/2.$$

when p is odd then $|D_{(p+1)/2}|=2p$, we have two subsets of it:

P4.1. $$\left|\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=6,\delta_{u_i}\delta_{v_j}=8,1\le i\le (p+1)/2,\ j=i+(p-1)/2\right\}\right.$$
$$\left.\cup\left\{(u_i,v_j):\delta_{u_i}+\delta_{v_j}=6,\delta_{u_i}\delta_{v_j}=8,(p+3)/2\le i\le p,\ j=i-(p+1)/2\right\}\right|=p.$$

P4.2. $$\left|\left\{(v_i,v_j):\delta_{v_i}+\delta_{v_j}=4,\delta_{v_i}\delta_{v_j}=4,1\le i\le (p+1)/2,\ j=i+(p-1)/2\right\}\right.$$
$$\left.\cup\left\{(v_i,v_j):\delta_{v_i}+\delta_{v_j}=4,\delta_{v_i}\delta_{v_j}=4,(p+3)/2\le i\le p,\ j=i-(p+1)/2\right\}\right|=p.$$

From **P1 - P4**, we have:

$$Sc(C_p^c;x)=20px+24p\sum_{k=2}^{\lfloor p/2\rfloor-1}x^k+2p\begin{cases}10x^{\frac{p}{2}}+x^{\frac{p}{2}+1}; & p\text{ is even},\\ 12x^{\frac{p-1}{2}}+5x^{\frac{p+1}{2}}; & p\text{ is odd}.\end{cases}$$

$$\overset{*}{Sc}(C_p^c;x)=32px+36p\sum_{k=2}^{\lfloor p/2\rfloor-1}x^k+2p\begin{cases}14x^{\frac{p}{2}}+x^{\frac{p}{2}+1}; & p\text{ is even},\\ 18x^{\frac{p-1}{2}}+6x^{\frac{p+1}{2}}; & p\text{ is odd}.\end{cases}$$

Corollary2.5.2: For $p\ge 6$, then:

1) $Sc(C_p^c)=p(3p^2+5p-2)$.

2) $\overset{*}{Sc}(C_p^c)=\frac{p}{2}\begin{cases}9p^2+12p-4; & p\text{ is even},\\ 9p^2+12p-5; & p\text{ is odd}.\end{cases}$

Corollary2.5.3: For $p\ge 6$, then:

1) $12\frac{4}{11}\le\overline{Sc}(C_p^c)<(3p+7)/2$.

2) $17.7<\overline{\overset{*}{Sc}}(C_p^c)<(18p+53)/8$.

Remark 2.5.4:

1) $Sc(C_4^c;x)=80x^1+80x^2+8x^3$, $Sc(C_5^c;x)=100x^1+120x^2+50x^3$.

2) $\overset{*}{Sc}(C_4^c;x)=128x^1+112x^2+8x^3$, $\overset{*}{Sc}(C_5^c;x)=160x^1+180x^2+60x^3$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Buckley, F. and Harary, F. (1990) Distance in Graphs. Addison-Wesley, New York.

[2] Chartrand, G. and Lesniak, L. (1986) Graphs and Digraphs. 2nd Edition, Wadsworth and Brooks/Cole, Pacific Grove.

[3] Devillers, J. and Balaban, A.T. (1999) Topological Indices and Related Descriptors in QSAR and QSPR. Gordon & Breach, Amsterdam.

[4] Diestel, R. (2000) Graph Theory, Electronic. Springer-Verlag, New York.

[5] Diudea, M.V. (2001) QSPR/QSAR Studies by Molecular Descriptors. Nova, Huntington, New York.

[6] Diudea, M.V., Gutman, I. and Jantschi, L. (2001) Molecular Topology. Nova, Huntington, New York.

[7] Schultz, H.P. (1989) Topological Organic Chemistry 1. Graph Theory and Topological Indices of Alkanes. *The Journal for Chemical Information and Computer scientists*, **29**, 227-228. https://doi.org/10.1021/ci00063a012

[8] Klavžar, S. and Gutman, I. (1997) Wiener Number of Vertex-Weighted Graphs and a Chemical Application. *Discrete Applied Mathematics*, **80**, 73-81. https://doi.org/10.1016/S0166-218X(97)00070-X

[9] Ahmed, M. and Haitham, N. (2017) Schultz and Modified Schultz Polynomials of Two Operations Gutman's. *International Journal of Enhanced Research in Science, Technology & Engineering*, **6**, 68-74. https://doi.org/10.11648/j.acm.20170606.14

[10] Ahmed, M. and Haitham, N. (2017) Schultz and Modified Schultz Polynomials of Cog-Complete Bipartite Graphs. *Applied and Computational Mathematics*, **6**, 259-264. https://doi.org/10.11648/j.acm.20170606.14

[11] Behmaram, A., Yousefi-Azari, H. and Ashrafi, A.R. (2011) Some New Results on Distance-Based Polynomials. *MATCH Communications in Mathematical and in Computer Chemistry*, **65**, 39-50.