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On 3-Dimensional Pseudo-Quasi-Conformal Curvature Tensor on $(LCS)_n$-Manifolds

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Abstract

The purpose of the present paper is to sudy the pseudo-quasi-conformally flat, $\tilde{C} \cdot S = 0$, pseudo-quasi-conformal ϕ-symmetric and pseudo-quasi-conformal ϕ-recurrent 3-dimensional $(LCS)_n$ maifolds.

Subject Areas

Geometry

Keywords

Lorentzian Metric, $(LCS)_n$-Manifolds, ϕ-Symmetric, Three-Dimensional ϕ-Recurrent, Einstein Manifold

1. Introduction

The notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifold) was introduced by [1], investigated the application to the general theory of relativity and cosmology with an example, which generalizes the notion of *LP*-Sasakian manifold introduced by Matsumoto [2]. The notion of Riemannian manifold has been weakened by many authors in a different extent [3]-[8].

Shaikh and Jana in 2005 [9] introduced and studied a tensor field, called Pseudo-Quasi-Conformal curvature tensor $\tilde{C}$ on a Riemannian manifold of dimension ($n \geq 3$). This includes the Projective, Quasi-conformal, Weyl conformal and Concircular curvature tensor as special cases. Recently Kundu and others [10] [11] studied pseudo-quasi-conformal curvature tensor on *P*-Sasakian manifolds.

In this paper, we consider a $(LCS)_n$-manifold satisfying certain conditions on the 3-dimensional pseudo-quasi-conformal curvature tensor. In section 2, we have the preliminaries. In Section 3, we studied a 3-dimensional pseudo-quasi-conformally flat $(LCS)_n$-manifold and proved that the manifold is η-Einstein and it is not a conformal curvature tensor. In section 4, we proved a 3-dimensional pseudo-quasi-conformal $(LCS)_n$-manifold satisfies $\tilde{C}\cdot S=0$; this reduces to η-Einstein and it is not a conformal curvature tensor. In section 5, we studied 3-dimensional pseudo-quasi-conformal ϕ-symmetric $(LCS)_n$-manifold with constant scalar curvature and obtained the manifold is Einstein (provided $p\neq 0$). In section 6, we studied a pseudo-quasi-conformal ϕ-recurrent $(LCS)_n$-manifold with constant scalar curvature, which generalizes the notion of ϕ-symmetric $(LCS)_n$-manifold.

2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p\in M$, the tensor $g_p : T_pM\times T_pM\to\Re$ is a non-degenerate inner product of signature $(-,+,\cdots,+)$, where T_pM denotes the tangent vector space of M at p and R is the real number space. A non zero vector $v\in T_pM$ is said to be timelike (resp., non-spacelike, null, space like) if it satisfies $g_p(v,v)<0$ [12].

Definition 2.1. *In a Lorentzian manifold* (M,g) *a vector field P defined by*

$$g(X,P)=A(X),$$

for any $X\in\chi(M)$ *is said to be a concircular vector field if*

$$(\nabla_X A)(Y)=\alpha g(X,Y)+w(X)A(Y),$$

where α *is a non-zero scalar and w is a closed* 1*-form.*

Let M be a Loretzian manifold admitting a unit timelike concircular vector field ξ is called the characteristic vector field of the manifold. Then we have

$$g(\xi,\xi)=-1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exist a non-zero 1-form η such that for

$$g(X,\xi)=\eta(X), \tag{2.2}$$

the equation of the following form holds

$$(\nabla_X\eta)(Y)=\alpha\{g(X,Y)+\eta(X)\eta(Y)\}\ (\alpha\neq 0), \tag{2.3}$$

for all vector field X,Y, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X\alpha=(X\alpha)=d\alpha(X)=\rho\eta(X), \tag{2.4}$$

ρ being a certain scalar function given by $\rho=-(\xi\alpha)$. Let us put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.5}$$

then from (2.3) and (2.5), we have

$$\phi X = X = \eta(X)\xi, \tag{2.6}$$

which tell us that ϕ is a symmetric $(1,1)$ tensor. thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and $(1,1)$-type tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) [1]. Especially, we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [2]. In a $(LCS)_n$-manifold, the following relation hold [1].

$$\eta(\xi) = -1, \phi\xi = 0, \eta(\phi X) = 0, \tag{2.7}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.8}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{2.9}$$

In a three dimensional $(LCS)_n$-manifolds, the following relation holds [13].

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ & -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.10}$$

$$S(X,\xi) = 2(\alpha^2 - \rho)\eta(X), \tag{2.11}$$

$$Q\xi = 2(\alpha^2 - \rho)\xi, \tag{2.12}$$

$$QX = \left[-(\alpha^2 - \rho) + \frac{r}{2}\right]X + \left[-3(\alpha^2 - \rho) + \frac{r}{2}\right]\eta(X)\xi. \tag{2.13}$$

The pseudo-quasi-conformal curvatur tesor $\tilde{C}$ is defined by [14].

$$\begin{aligned} \tilde{C}(X,Y)Z = {} & (p+d)R(X,Y)Z + \left(q - \frac{d}{n-1}\right)[S(Y,Z)X - S(X,Z)Y] \\ & + q[g(Y,Z)QX - g(X,Z)QY] \\ & - \frac{r}{n(n-1)}\{p + 2(n-1)q\}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.14}$$

where $X,Y,Z \in \chi(M)$, R,S,Q and r are the curvature tensor, the Ricci tensor, the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S and the scalar curvature, *i.e*, $g(QX,Y) = S(X,Y)$ and p,q,d are real constants such that $p^2 + q^2 + d^2 > 0$.

In particular, if (1) $p = q = 0, d = 1$; (2) $p \neq 0, q \neq 0, d = 0$; (3) $p = 1, q = -\frac{1}{n-2}, d = 0$; (4) $p = 1, q = d = 0$; then $\tilde{C}$ reduces to the projective curvature tensor; quasi-conformal curvature tensor; conformal curvature tensor and concircular curvature tensor, respectively.

3. 3-Dimensional Pseudo-quasi-conformally Flat $(LCS)_n$-Manifold

Definition 3.2. *An n-dimensional* ($n \geq 3$) $(LCS)_n$*-manifold M is called a* *pseudo-quasi-conformally flat, if the condition* $\tilde{C}(X,Y)Z=0$, *for all* $X,Y,Z\in T_pM$.

Let us consider the three dimensional $(LCS)_n$-manifold M is a pseudo-quasi-conformally flat, then from (2.6), (2.7) and (2.8) relation to (2.10) that

$$\begin{aligned}&(p+d)R(X,Y)Z+\left(q-\frac{d}{2}\right)\left[S(Y,Z)X-S(X,Z)Y\right]\\&+q\left[g(Y,Z)QX-g(X,Z)QY\right]-\frac{r}{6}\{p+4q\}\left[g(Y,Z)X-g(X,Z)Y\right]=0,\end{aligned}\tag{3.1}$$

Putting $Z=\xi$ in (3.1) and by using (2.10), (2.11), we get

$$\begin{aligned}&(p+d+q)\left[\eta(Y)QX-\eta(X)QY\right]\\&+\left[2\left(\alpha^2-\rho\right)\left(p+\frac{d}{2}+q\right)-(2p+3d+4q)\frac{r}{2}\right]\eta(Y)X-\eta(X)Y=0,\end{aligned}\tag{3.2}$$

again plugging $Y=\xi$ in (3.2) by using (2.12) and taking inner product with respect to W, we get

$$S(X,W)=\frac{1}{p+d+q}\left\{Ag(X,W)+B\eta(X)\eta(W)\right\}.\tag{3.3}$$

where

$$A=\left\{p\left[2\left(\alpha^2-\rho\right)-\frac{r}{3}\right]+d\left[\left(\alpha^2-\rho\right)-\frac{r}{2}\right]+2q\left[\left(\alpha^2-\rho\right)-\frac{r}{3}\right]\right\},$$

and $B=\left\{p\left[4\left(\alpha^2-\rho\right)-r\right]+3d\left[\left(\alpha^2-\rho\right)-\frac{r}{2}\right]+2q\left[4\left(\alpha^2-\rho\right)-r\right]\right\}.$

Hence we can state the following theorem.

Theorem 3.1. *Let* M *be a* 3-*dimensional pseudo-quasi-conformally flat* $(LCS)_n$*-manifold is an* η*-Einstein manifold, provided pseudo-quasi-conformal curvature tensor is not a conformal curvature tensor* [15] ($p=1$, $q=-1$ *and* $d=0$).

4. 3-Dimensional Pseudo-quasi-conformal $(LCS)_n$-Manifold Satisfies $\tilde{C}\cdot S=0$

Let us consider a 3-dimensional Riemannian manifold which satisfies the condition

$$\tilde{C}(X,Y)\cdot S=0.\tag{4.1}$$

Then we have

$$S\left(\tilde{C}(X,Y)U,V\right)+S\left(U,\tilde{C}(X,Y)V\right)=0.\tag{4.2}$$

Put $X=\xi$ in (4.2) by using (2.10), (2.11), (2.12) and (2.14) and also on plugging $U=\xi$, we get

$$\begin{aligned}&4(p+d+q)\left(\alpha^2-\rho\right)^2\eta(V)\eta(Y)+(p+q+d)S(QY,V)\\&-\frac{r}{6}(4p+3d+4q)S(Y,V)-2\left(\alpha^2-\rho\right)(p+d+q)\eta(V)\eta(QY)\\&+\frac{r}{3}\left(\alpha^2-\rho\right)(4p+3d+4q)g(Y,V)=0,\end{aligned}\tag{4.3}$$

by using (2.13) in (4.3), we get

$$S(V,Y) = Ag(V,Y) + B\eta(V)\eta(Y). \tag{4.4}$$

where

$$A = \frac{2r(\alpha^2 - \rho)(4p + 3d + 4q)}{6(\alpha^2 - \rho)(p + d + q) + r(p + q)},$$

and $B = \dfrac{6(\alpha^2 - \rho)\left[r - 6(\alpha^2 - \rho)\right](p + d + q)}{6(\alpha^2 - \rho)(p + d + q) + r(p + q)}$.

Hence we can state the following theorem.

Theorem 4.2. *Let a 3-dimensional pseudo-quasi-conformal $(LCS)_n$-manifold satisfying $\tilde{C} \cdot S = 0$ is an η-Einstein manifold.*

5. On 3-Dimensional Pseudo-quasi-conformal ϕ-Symmetric $(LCS)_n$-Manifold

Definition 5.3. *An $(LCS)_n$-manifold is said to be pseudo-quasi-conformal ϕ-symmetric if the condition*

$$\phi^2\left(\left(\nabla_W \tilde{C}\right)(X,Y)Z\right) = 0, \tag{5.1}$$

for any vector field $X, Y, Z, W \in TpM$.

Let us consider 3-dimensional $(LCS)_n$-manifold of a pseudo-quasi-conformal curvature tensor has the following from (2.6), we get

$$\left(\nabla_W \tilde{C}\right)(X,Y)Z + \eta\left(\left(\nabla_W \tilde{C}\right)(X,Y)Z\right)\xi = 0. \tag{5.2}$$

which follows that

$$g\left(\left(\nabla_W \tilde{C}\right)(X,Y)Z, U\right) + \eta\left(\left(\nabla_W \tilde{C}\right)(X,Y)Z\right)\eta(U) = 0. \tag{5.3}$$

By virtue of (2.10), (2.11) and (2.14) and contracting we get

$$\begin{aligned}&(q - d)(\nabla_W S)(Y,Z) + \frac{drW}{6}\left[(2p + 2q + 3d)g(Y,Z)(4p + 3d + 4q)\eta(Y)\eta(Z)\right]\\&+(-(p + d) + q)\eta(Z)(\nabla_W S)(Y,\xi) + \left(-p - \frac{3d}{2} + q\right)\eta(Y)(\nabla_W S)(Z,\xi) = 0.\end{aligned} \tag{5.4}$$

On plugging $Z = \xi$ in (5.4), gives

$$S(Y,W) = 2(\alpha^2 - \rho)g(Y,W) - \frac{1}{p\alpha}(p + q)\eta(Y)\frac{drW}{3}. \tag{5.5}$$

If the manifold has a constant scalar curvature r, then $drW = 0$.

Hence the Equation (5.5) turns into

$$S(Y,W) = 2(\alpha^2 - \rho)g(Y,W). \tag{5.6}$$

Hence we can state the following:

Theorem 5.3. *Let M be a 3-dimensional pseudo-quasi-conformal ϕ-symmetric $(LCS)_n$-manifold with constant scalar curvature, then the manifold is reduces to* *a Einstein manifold.*

6. 3-Dimensional Pseudo-quasi-conformal ϕ-Recurrent on $(LCS)_n$-Manifold

Definition 6.4. *An $(LCS)_n$-manifold is said to be pseudo-quasi-conformal ϕ-recurrent if*

$$\phi^2\left(\left(\nabla_W \tilde{C}\right)(X,Y)Z\right) = A(W)\tilde{C}(X,Y)Z, \tag{6.1}$$

for any vector field $X,Y,Z,W \in TpM$. If $A(W)=0$ then pseudo-quasi-conformal ϕ-recurrent reduces to ϕ-symmetric.

Let us consider a 3-dimensional pseudo-quasi-conformal ϕ-recurrent $(LCS)_n$-manifold. Then by virtue of (2.6) and (6.1), we have

$$\left(\nabla_W \tilde{C}\right)(X,Y)Z + \eta\left(\left(\nabla_W \tilde{C}\right)(X,Y)Z\right)\xi = A(W)\tilde{C}(X,Y)Z, \tag{6.2}$$

from which it follows that

$$g\left(\left(\nabla_W \tilde{C}\right)(X,Y)Z,U\right) + \eta\left(\left(\nabla_W \tilde{C}\right)(X,Y)Z\right)\eta(U) = A(W)g\left(\tilde{C}(X,Y)Z,U\right). \tag{6.3}$$

By virtue of (2.10), (2.11) and (2.14) and contracting, also plugging $Z=\xi$, we get

$$\begin{aligned} &S(Y,W) \\ &= 2\left(\alpha^2-\rho\right)g(Y,W) - \frac{2}{3}\left[\frac{r(2p+3d-q)+6(p+q)\left(\alpha^2-\rho\right)}{(2p+d+2q)\alpha}\right]\eta(Y)A(W) \\ &\quad + drW\eta(Y)\frac{2(p+d)}{\alpha(2p+d+2q)}, \end{aligned} \tag{6.4}$$

again putting $Y=\xi$ in (6.4), we get

$$A(W) = \frac{3(p+d)}{r(q-2p-3d)-6(p+q)\left(\alpha^2-\rho\right)}drW. \tag{6.5}$$

If the manifold has a constant scalar curvature r, then $drW=0$. Hence the Equation (6.5) turns into

$$A(W)=0. \tag{6.6}$$

Using (6.6) in (6.1), we get

$$\phi^2\left(\left(\nabla_W \tilde{C}\right)(X,Y)Z\right) = 0. \tag{6.7}$$

Hence we can state the following:

Theorem 6.4. *If M is a 3-dimensional pseudo-quasi-conformal ϕ-recurrent $(LCS)_n$-manifold with constant scalar curvature, then it is ϕ-symmetric.*

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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