Open Access Library Journal
2019, Volume 6, e5408
ISSN Online: 2333-9721
ISSN Print: 2333-9705

On Almost β-Topological Vector Spaces

Shallu Sharma, Sahil Billawria, Madhu Ram, Tsering Landol

Department of Mathematics, University of Jammu, Jammu and Kashmir, India
Email: shallujamwal09@gmail.com, sahilbillawria2@gmail.com, madhuram0502@gmail.com, tseringlandol09@gmail.com

How to cite this paper: Sharma, S., Billawria, S., Ram, M. and Landol, T. (2019) On Almost β-Topological Vector Spaces *Open Access Library Journal*, **6**: e5408.
https://doi.org/10.4236/oalib.1105408

Received: April 16, 2019
Accepted: May 26, 2019
Published: May 29, 2019

Abstract

In this paper, we have introduced a new generalized form of topological vector spaces, namely, almost β-topological vector spaces by using the concept of β-open sets. We have also presented some examples and counterexamples of almost β-topological vector spaces and determined its relationship with topological vector spaces. Some properties of β-topological vector spaces are also characterized.

Subject Areas

Mathematical Analysis

Keywords

β-Open Sets, δ-Open Sets, Regular-Open Sets, Almost β-Topological Vector Spaces

1. Introduction

The concept of topological vector spaces was introduced by Kolmogroff [1] in 1934. Its properties were further studied by different mathematicians. Due to its large number of exciting properties, it has been used in different advanced branches of mathematics like fixed point theory, operator theory, differential calculus etc. In 1963, N. Levine introduced the notion of semi-open sets and semi-continuity [2]. Nowadays there are several other weaker and stronger forms of open sets and continuities like pre-open sets [3], precontinuous and weak precontinuous mappings [3], β-open sets and β-continuous mappings [4], δ-open sets [5], etc. These weaker and stronger forms of open sets and continuities are used for extending the concept of topological vector spaces to several new notions like s-topological vector spaces [6] by M. Khan *et al.* in 2015, irresolute topological vector spaces [7] by M. Khan and M. Iqbal in 2016, β-topological vector spaces [8] by S. Sharma and M. Ram in 2018, almosts-topological vector spaces [9] by M. Ram *et al.* in 2018, etc. The aim of this paper is to introduce the class of almost β-topological vector spaces and present some examples of it. Further, some general properties of almost β-topological vector spaces are also investigated.

2. Preliminaries

Throughout this paper, (X,τ) (or simply *X*) and (Y,σ) (or simply *Y*) mean topological spaces. For a subset $A \subseteq X$, $Cl(A)$ denotes the closure of *A* and $Int(A)$ denote the interior of *A*. The notation $\mathbb{F}$ denotes the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$ with usual topology and ε, η represent the negligibly small positive numbers.

Definition 2.1 *A subset A of a topological space X is said to be*:

1) *regular open if* $A = Int(Cl(A))$.

2) *β-open* [4] *if* $A \subseteq Cl(Int(Cl(A)))$.

Definition 2.2 *A subset A of a topological space X is said to be δ-open* [5] *if for each* $x \in A$, *there exists a regular open set U in X such that* $x \in U \subseteq A$.

The union of all β-open (resp. δ-open) sets in *X* that are contained in $A \subseteq X$ is called β-interior [10] (resp. δ-interior) of *A* and is denoted by $\beta Int(A)$ (resp. $Int_\delta(A)$). A point *x* is called a β-interior point of $A \subseteq X$ if there exists a β-open *V* in *X* such that $x \in V \subseteq A$. The set of all β-interior points of *A* is equal to $\beta Int(A)$. It is well known fact that a subset $A \subseteq X$ is β-open (resp. δ-open) if and only if $A = \beta Int(A)$ (resp. $A = Int_\delta(A)$). The complement of β-open (resp. δ-open, regular open) set is called β-closed (resp. δ-closed [5], regular closed). The intersection of all β-closed (resp. δ-closed) sets in *X* containing a subset $A \subseteq X$ is called β-closure [10] (resp. δ-closure) of *A* and is denoted by $\beta Cl(A)$ (resp. $Cl_\delta(A)$). It is also known that a subset *A* of *X* is β-closed (resp. δ-closed) if and only if $A = \beta Cl(A)$ (resp. $A = Cl_\delta(A)$). A point $x \in \beta Cl(A)$ if and only if $A \cap V \neq \phi$ for each β-open set *V* in *X* containing *x*. A point $x \in Cl_\delta(A)$ if $A \cap Int(Cl(O)) \neq \phi$ for each open set *O* in *X* containing *x*.

The family of all β-open (resp. β-closed, regular open) sets in *X* is denoted by $\beta O(X)$ (resp. $\beta C(X)$, $RO(X)$). If $A \in \beta O(X)$, $B \in \beta O(Y)$, then $A \times B \in \beta O(X \times Y)$ (with respect to the product topology). The family of all β-open sets in *X* containing *x* is denoted by $\beta O(X,x)$.

Definition 2.3 [11] *A function* $f: X \to Y$ *from a topological space X to a topological space Y is called almost β-continuous at* $x \in X$ *if for each open set O of Y containing* $f(x)$, *there exists* $V \in \beta O(X,x)$ *such that* $f(V) \subset Int(Cl(O))$.

Also we recall some definitions that will be used later.

Definition 2.4 [12] *Let T be a vector space over the field* $\mathbb{F}$. *Let* τ *be a topology on T such that*

1) *For each* $x, y \in T$ *and each open neighborhood O of* $x+y$ *in T, there exist open neighborhoods* O_1 *and* O_2 *of x and y respectively in T such that* $O_1 + O_2 \subseteq O$, *and*

2) *For each* $\lambda \in \mathbb{F}$, $x \in T$ *and each open neighborhood O of* λx *in T, there exists open neighborhoods* O_1 *of* λ *in* $\mathbb{F}$ *and* O_2 *of x in T such that* $O_1 O_2 \subseteq O$.

Then the pair $\left(T_{(\mathbb{F})}, \tau\right)$ *is called topological vector space.*

Definition 2.5 [8] *Let T be a vector space over the field* $\mathbb{F}$. *Let* τ *be a topology on T such that*

1) *For each* $x, y \in T$ *and each open neighborhood O of* $x + y$ *in T, there exist β-open sets* V_1 *and* V_2 *in T containing x and y respectively such that* $V_1 + V_2 \subseteq O$, *and*

2) *For each* $\lambda \in \mathbb{F}$, $x \in T$ *and each open neighborhood O of* λx *in T, there exist β-open sets* V_1 *containing* λ *in* $\mathbb{F}$ *and* V_2 *containing x in T such that* $V_1 \cdot V_2 \subseteq O$.

Then the pair $\left(T_{(\mathbb{F})}, \tau\right)$ *is called β-topological vector space.*

Definition 2.6 [13] *Let T be a vector space over the field* $\mathbb{F}$. *Let* τ *be a topology on T such that*

1) *For each* $x, y \in T$ *and each regular open set* $U \subseteq T$ *containing* $x + y$, *there exist pre-open sets* P_1 *and* P_2 *in T containing x and y respectively such that* $P_1 + P_2 \subseteq U$, *and*

2) *For each* $\lambda \in \mathbb{F}$, $x \in T$ *and each regular open set* $U \subseteq T$ *containing* λx, *there exist pre-open sets* P_1 *in* $\mathbb{F}$ *containing* λ *and* P_2 *containing x in T such that* $P_1 \cdot P_2 \subseteq U$.

Then the pair $\left(T_{(\mathbb{F})}, \tau\right)$ *is called an almost pretopological vector space.*

Definition 2.7 [9] *Let T be a vector space over the field* $\mathbb{F}$. *Let* τ *be a topology on T such that*

1) *For each* $x, y \in T$ *and each regular open set* $U \subseteq T$ *containing* $x + y$, *there exist semi-open sets* S_1 *and* S_2 *in T containing x and y respectively such that* $S_1 + S_2 \subseteq U$, *and*

2) *For each* $\lambda \in \mathbb{F}$, $x \in T$ *and each regular open set* $U \subseteq T$ *containing* λx, *there exist semi-open sets* S_1 *in* $\mathbb{F}$ *containing* λ *and* S_2 *containing x in T such that* $S_1 \cdot S_2 \subseteq U$.

Then the pair $\left(T_{(\mathbb{F})}, \tau\right)$ *is called an almost s-topological vector space.*

3. Almost β-Topological Vector Spaces

In this section, we define β-topological vector spaces and present some examples of it.

Definition 3.1 *Let Z be a vector space over the field* $\mathbb{F}$ ($\mathbb{R}$ *or* $\mathbb{C}$ *with standard topology). Let* τ *be a topology on Z such that*

1) *For each* $x, y \in Z$ *and each regular open set* $U \subseteq Z$ *containing* $x + y$, *there exist β-open sets* V_1 *and* V_2 *in Z containing x and y respectively such that* $V_1 + V_2 \subseteq U$, *and*

2) *For each* $\lambda \in \mathbb{F}$, $x \in Z$ *and each regular open set* $U \subseteq Z$ *containing* λx, *there exist β-open sets* V_1 *in* $\mathbb{F}$ *containing* λ *and* V_2 *containing x in Z such that* $V_1 V_2 \subseteq U$.

Then the pair $\left(Z_{(\mathbb{F})},\tau\right)$ *is called an almost β-topological vector space.*

Some examples of almost β-topological vector space are given below:

Example 3.1 Let $Z=\mathbb{R}$ be the real vector space over the field $\mathbb{F}$, where $\mathbb{F}=\mathbb{R}$ with the standard topology and τ be the usual topology endowed on *Z*, that is, τ is generated by the base $\mathcal{B}=\{(a,b):a,b\in\mathbb{R}\}$. Then $\left(Z_{(\mathbb{F})},\tau\right)$ is an almost β-topological vector space. For proving this, we have to verify the following two conditions:

1) Let $x,y\in Z$. Consider any regular open set $U=(x+y-\epsilon,x+y+\epsilon)$ in Z containing $x+y$. Then we can opt for β-open sets $V_1=(x-\eta,x+\eta)$ and $V_2=(y-\eta,y+\eta)$ in *Z* containing *x* and *y* respectively, such that $V_1+V_2\subseteq U$ for each $\eta<\frac{\epsilon}{2}$. Thus first condition of the definition of almost β-topological vector space is satisfied.

2) Let $\lambda\in\mathbb{F}=\mathbb{R}$ and $x\in Z$. Consider a regular open set $U=(\lambda x-\epsilon,\lambda x+\epsilon)$ in $Z=\mathbb{R}$ containing λx. Then we have the following cases:

Case (I). If $\lambda>0$ and $x>0$, then $\lambda x>0$. We can choose β-open sets $V_1=(\lambda-\eta,\lambda+\eta)$ in $\mathbb{F}$ containing λ and $V_2=(y-\eta,y+\eta)$ in Z containing *x*, such that $V_1\cdot V_2\subseteq U$ for each $\eta<\frac{\epsilon}{\lambda+x+1}$.

Case (II). If $\lambda<0$ and $x<0$, then $\lambda x>0$. We can choose β-open sets $V_1=(\lambda-\eta,\lambda+\eta)$ in $\mathbb{F}$ containing λ and $V_2=(x-\eta,x+\eta)$ in Z containing *x*, such that $V_1\cdot V_2\subseteq U$ for each $\eta<\frac{\epsilon}{1-\lambda-x}$.

Case (III). If $\lambda>0$ and $x<0$ (resp. $\lambda<0$ and $x>0$), then $\lambda x<0$. We can choose β-open sets $V_1=(\lambda-\eta,\lambda+\eta)$ in $\mathbb{F}$ containing λ and $V_2=(x-\eta,x+\eta)$ in *Z* containing *x*, such that $V_1\cdot V_2\subseteq U$ for each $\eta<\frac{\epsilon}{1+\lambda-x}$ (resp. $\eta<\frac{\epsilon}{1-\lambda+x}$).

Case (IV). If $\lambda=0$ and $x>0$ (resp. $\lambda>0$ and $x=0$), then $\lambda x=0$. We can select β-open neighborhoods $V_1=(-\eta,\eta)$ (resp. $V_1=(\lambda-\eta,\lambda+\eta)$) in $\mathbb{F}$ containing λ and $V_2=(x-\eta,x+\eta)$ (resp. $V_2=(-\eta,\eta)$ in Z containing *x*, such that $V_1\cdot V_2\subseteq U$ for each $\eta<\frac{\epsilon}{x+1}$ (resp. $\eta<\frac{\epsilon}{\lambda+1}$).

Case (V). If $\lambda=0$ and $x<0$ (resp. $\lambda<0$ and $x=0$), then $\lambda x=0$. We can select β-open neighborhoods $V_1=(-\eta,\eta)$ (resp. $V_1=(\lambda-\eta,\lambda+\eta)$) in $\mathbb{F}$ containing λ and $V_2=(x-\eta,x+\eta)$ (resp. $V_2=(-\eta,\eta)$ in Z containing *x*, such that $V_1\cdot V_2\subseteq U$ for each $\eta<\frac{\epsilon}{1-x}$ (resp. $\eta<\frac{\epsilon}{1-\lambda}$).

Case (VI). If $\lambda=0$ and $x=0$, then $\lambda x=0$. Then for β-open neighborhoods $V_1=(-\eta,\eta)$ of λ in $\mathbb{F}$ and $V_2=(-\eta,\eta)$ of x in *Z*, we have $V_1\cdot V_2\subseteq U$ for each $\eta<\sqrt{\epsilon}$.

This verifies the second condition of the definition of almost β-topological vector space.

Example 3.2 Let $Z = \mathbb{R}$ be the real vector space over the field $\mathbb{F}$ with the topology τ generated by the base $\mathcal{B} = \{(a,b): a,b \in \mathbb{R}\} \cup \{(c,d) \cap \mathbb{Q}^c : c,d \in \mathbb{R}\}$, where $\mathbb{Q}^c$ denotes the set of irrational numbers. Then $(Z_{(\mathbb{R})}, \tau)$ is an almost β-topological vector space.

Example 3.3 Consider the field $\mathbb{F} = \mathbb{R}$ with standard topology. Let $Z = \mathbb{R}$ be the real vector space over the field $\mathbb{F}$ endowed with topology $\tau = \{\phi, \{0\}, \mathbb{R}\}$. Then $(Z_{(\mathbb{R})}, \tau)$ is an almost β-topological vector space.

Example 3.4 Let τ be the topology induced by open intervals (a,b) and the sets $[c,d)$ where $a,b,c,d \in \mathbb{R}$ with $0 < c < d$. Let $Z = \mathbb{R}$ be the real vector space over the field $\mathbb{F}$ endowed with topology τ, where $\mathbb{F} = \mathbb{R}$ with the standard topology. Then $(Z_{(\mathbb{F})}, \tau)$ is an almost β-topological vector space.

The above four examples are examples of almost β-topological vector spaces, we now present an example which don't lie in the class of almost β-topological vector spaces.

Example 3.5 Let τ be the topology generated by the base $\mathcal{B} = \{[a,b): a,b \in \mathbb{R}\}$ and let this topology τ is imposed on the real vector space $Z = \mathbb{R}$ over the topological field $\mathbb{F} = \mathbb{R}$ with standard topology. Then $(Z_{(\mathbb{F})}, \tau)$ fails to be an almost β-topological vector space. For, $U = [0,1)$ is regular open set in Z containing $0 = -1.0$ ($-1 \in \mathbb{F} = \mathbb{R}$ and $0 \in Z$) but there do not exist β-open sets V_1 in $\mathbb{F}$ containing -1 and V_2 in Z containing 0 such that $V_1 \cdot V_2 \subseteq U$.

Remark 3.1 By definitions, it is clear that, every topological vector space is an almost β-topological vector space. But converse need not be true in general. For, examples 3.2 and 3.3 are almost β-topological vector spaces which fails to be topological vector spaces.

Remark 3.2 The class of almost pretopological vector spaces and almost s-topological vector spaces lie completely inside the class of almost β-topological vector spaces.

4. Characterizations

Throughout this section, an almost β-topological vector space $(Z_{(\mathbb{F})}, \tau)$ over the topological field $\mathbb{F}$ will be simply written by Z and by a scalar, we mean an element from the topological field $\mathbb{F}$.

Theorem 4.1 *Let A be any δ-open set in an almost β-topological vector space Z. Then* $x + A, \lambda A \in \beta O(Z)$, *for each* $x \in Z$ *and each non-zero scalar* λ.

Proof. Let $y \in x + A$. Then $y = x + a$ for some $a \in A$. Since A is δ-open, there exists a regular open set U in Z such that $a \in U \subseteq A$. $\Rightarrow -x + y \in U$. Since Z is an almost β-topological vector space, there exist β-open sets V_1 and V_2 in Z such that $-x \in V_1, y \in V_2$ such that $V_1 + V_2 \subseteq U$. Now $-x + y \in -x + V_2 \subseteq U \subseteq A \quad \Rightarrow V_2 \subseteq x + A$. Since V_2 is β-open, $y \in \beta Int(x + A)$. This shows that $x + A = \beta Int(x + A)$. Hence $x + A \in \beta O(Z)$.

Further, let $x \in \lambda A$ be arbitrary. Since A is δ-open, there exists a regular open set U in Z such that $\lambda^{-1} x \in U \subseteq A$. Since Z is an almost β-topological vector space, there exist β-open sets V_1 in the topological field $\mathbb{F}$ containing λ^{-1} and V_2 in Z containing x such that $V_1V_2 \subseteq U$. Now $\lambda^{-1}x \in \lambda^{-1}V_2 \subseteq A$ $\Rightarrow V_2 \subseteq \lambda A$ $\Rightarrow x \in \beta Int(\lambda A)$ and hence $\lambda A = \beta Int(\lambda A)$. Thus λA is β-open in *Z*; *i.e.*, $\lambda A \in \beta O(Z)$.

Theorem 4.2 *Let B be any δ-closed set in an almost β-topological vector space Z. Then* $x+B, \lambda B \in \beta Cl(Z)$ *for each* $x \in Z$ *and each non-zero scalar* λ.

Proof. We need to show that $x+B = \beta Cl(x+B)$. For, let $y \in \beta Cl(x+B)$ be arbitrary and let W be any δ-open set in Z containing $-x+y$. By definition of δ-open sets, there is a regular open set U in Z such that $-x+y \in U \subseteq W$. Then there exist β-open sets V_1 and V_2 in Z such that $-x \in V_1, y \in V_2$ and $V_1+V_2 \subseteq U$. Since $y \in \beta Cl(x+B)$, then by definition, $(x+B)\bigcap V_2 \neq \phi$ $\Rightarrow$ there is some $a \in (x+B)\bigcap V_2$
$\Rightarrow -x+a \in B\bigcap(-x+V_2) \subseteq B\bigcap(V_1+V_2) \subseteq B\cap U \subseteq B\bigcap W$ $\Rightarrow B\bigcap W \neq \phi$. Thus $-x+y \in Cl_\delta(B)$. Since B is δ-closed set, we have, $-x+y \in B \Rightarrow y \in x+B$. Therefore $x+B = \beta Cl(x+B)$. Hence $x+B \in \beta Cl(Z)$.

Next, we have to prove that $\lambda B = \beta Cl(\lambda B)$. For, let $x \in \beta Cl(\lambda B)$ be arbitrary and let W be any δ-open set in Z containing $\lambda^{-1}x$. By definition, there is a regular open set U in Z such that $\lambda^{-1}x \in U \subseteq W$. Then there exist β-open sets V_1 containing λ^{-1} in topological field $\mathbb{F}$ and V_2 containing x in Z such that $V_1 \cdot V_2 \subseteq U$. Since $x \in \beta Cl(\lambda B)$, then there is some $a \in (\lambda B)\bigcap V_2$. Now $\lambda^{-1}a \in B\bigcap(\lambda^{-1}\cdot V_2) \subseteq B\bigcap(V_1\cdot V_2) \subseteq B\bigcap U \subseteq B\bigcap W$ $\Rightarrow B\bigcap W \neq \phi$. Thus $\lambda^{-1}x \in Cl_\delta(B) = B \Rightarrow x \in \lambda B$. Therefore $\lambda B = \beta Cl(\lambda B)$. Hence $\lambda B \in \beta Cl(Z)$.

Theorem 4.3 *For any subset A of an almost β-topological vector space Z, the following assertions hold*:

1) $x+\beta Cl(A) \subseteq Cl_\delta(x+A)$ *for each* $x \in Z$.

2) $\lambda\beta Cl(A) \subseteq Cl_\delta(\lambda A)$ *for each non zero scalar* λ.

Proof. 1) Let $z \in x+\beta Cl(A)$. Then $z = x+y$ for some $y \in \beta Cl(A)$. Let O be an open set in Z containing *z*, then $z \in O \subseteq Int(Cl(O))$. Since Z is an almost β-topological vector space, then there exist $V_1, V_2 \in \beta O(Z)$ containing *x* and *y* respectively such that $V_1+V_2 \subseteq Int(Cl(O))$. Since $y \in \beta Cl(A)$, then there is some $a \in A\bigcap V_2$. As a result, $x+a \in (x+A)\bigcap(V_1+V_2) \subseteq (x+A)\bigcap Int(Cl(O))$ $\Rightarrow (x+A)\bigcap Int(Cl(O)) \neq \phi$. Thus $z \in Cl_\delta(x+A)$. Therefore $x+\beta Cl(A) \subseteq Cl_\delta(x+A)$.

2) Let $x \in \beta Cl(A)$ and let W be an open set in Z containing λx. Then $\lambda x \in O \subseteq Int(Cl(O))$, so there exist β-open sets V_1 containing λ in topological field $\mathbb{F}$ and V_2 containing *x* in Z such that $V_1\cdot V_2 \subseteq Int(Cl(O))$. Since $x \in \beta Cl(A)$, then there is some $b \in A\bigcap V_2$. Now $\lambda b \in (\lambda A)\bigcap(\lambda V_2) \subseteq (\lambda A)\bigcap(V_1\cdot V_2) \subseteq (\lambda A)\bigcap Int(Cl(O))$ and hence $\lambda x \in Cl_\delta(\lambda A)$. Therefore $\lambda\beta Cl(A) \subseteq Cl_\delta(\lambda A)$.

Theorem 4.4 *For any subset A of an almost β-topological vector space X, the following hold*:

1) $\beta Cl(x+A) \subseteq x+Cl_\delta(A)$ *for each* $x \in X$.

2) $\beta Cl(\lambda A) \subseteq \lambda Cl_{\delta}(A)$ *for each non-zero scalar* λ.

Proof. 1) Let $y \in \beta Cl(x+A)$ and let O be an open set in Z containing $-x+y$. Since Z is an almost β-topological vector space, there exist β-open sets V_1 and V_2 in Z such that $-x \in V_1$, $y \in V_2$ and $V_1 + V_2 \subseteq Int(Cl(O))$. Since $y \in \beta Cl(x+A)$, there is some $a \in (x+A) \cap V_2$ and hence $-x+a \in A \cap (V_1+V_2) \subseteq A \cap Int(Cl(O)) \Rightarrow -x+y \in Cl_{\delta}(A) \Rightarrow y \in x + Cl_{\delta}(A)$. Hence $\beta Cl(x+A) \subseteq x + Cl_{\delta}(A)$.

2) Let $x \in \beta Cl(\lambda A)$ and O be an open set in Z containing $\lambda^{-1}x$. So there exist β-open sets V_1 in topological field $\mathbb{F}$ containing λ^{-1} and V_2 in Z containing x such that $V_1 \cdot V_2 \subseteq Int(Cl(O))$. As $x \in \beta Cl(\lambda A)$, $(\lambda A) \cap V_2 \neq \phi$ and as a result, $A \cap Int(Cl(O)) \neq \phi$. Therefore $\lambda^{-1}x \in Cl_{\delta}(A)$. Hence $\beta Cl(\lambda A) \subseteq \lambda Cl_{\delta}(A)$.

Theorem 4.5 *Let A be an open set in an almost β-topological vector space Z, then*:

1) $\beta Cl(x+A) \subseteq x + Cl(A)$ *for each* $x \in E$.

2) $\beta Cl(\lambda A) \subseteq \lambda Cl(A)$ *for each non zero scalar* λ.

Proof. 1) Let $y \in \beta Cl(x+A)$ and O be any open set in Z containing $-x+y$. Then there exist $V_1, V_2 \in \beta O(Z)$ such that $-x \in V_1$, $y \in V_2$ and $V_1 + V_2 \subseteq Int(Cl(O))$. Since $y \in \beta Cl(x+A)$, there is some $a \in (x+A) \cap V_2$. Now $-x+a \in A \cap (V_1+V_2) \subseteq A \cap Int(Cl(O)) \Rightarrow A \cap Int(Cl(O)) \neq \phi$. Since A is open, $A \cap O \neq \phi$. Thus $-x+y \in Cl(A)$; that is, $y \in x + Cl(A)$. Hence $\beta Cl(x+A) \subseteq x + Cl(A)$.

2) Let $x \in \beta Cl(\lambda A)$ and O be any open set in Z containing $\lambda^{-1}y$. Then there exist β-open sets V_1 in topological field $\mathbb{F}$ containing λ^{-1} and V_2 in Z containing x such that $V_1 \cdot V_2 \subseteq Int(Cl(O))$. As $x \in \beta Cl(\lambda A)$, there is some $b \in (\lambda A) \cap V_2$. Thus $\lambda^{-1}b \in A \cap Int(Cl(O)) \Rightarrow A \cap Int(Cl(O)) \neq \phi$. Since A is open, $A \cap O \neq \phi$. Thus $\lambda^{-1}y \in Cl(A)$; that is, $y \in \lambda Cl(A)$. Hence $\beta Cl(\lambda A) \subseteq \lambda Cl(A)$.

Theorem 4.6 *Let A and B be subsets of an almost β-topological vector space Z. Then* $\beta Cl(A) + \beta Cl(B) \subseteq Cl_{\delta}(A+B)$.

Proof. Let $x \in \beta Cl(A)$ and $y \in \beta Cl(B)$ and let O be an open neighborhood of $x+y$ in Z. Since $O \subseteq Int(Cl(O))$ and $Int(Cl(O))$ is regular open, there exist $V_1, V_2 \in \beta O(Z)$ such that $x \in V_1$, $y \in V_2$ and $V_1 + V_2 \subseteq Int(Cl(O))$. Since $x \in \beta Cl(A)$ and $y \in \beta Cl(B)$, there are $a \in A \cap V_1$ and $b \in B \cap V_2$. Then $a+b \in (A+B) \cap (V_1+V_2) \subseteq (A+B) \cap Int(Cl(O)) \Rightarrow (A+B) \cap Int(Cl(O)) \neq \phi$. Thus $x+y \in Cl_{\delta}(A+B)$; that is, $\beta Cl(A) + \beta Cl(B) \subseteq Cl_{\delta}(A+B)$.

Theorem 4.7 *For any subset A of an almost β-topological vector space Z, the following are true*:

1) $Int_{\delta}(x+A) \subseteq x + \beta Int(A)$, *and*

2) $x + Int_{\delta}(A) \subseteq \beta Int(x+A)$, *for each* $x \in Z$.

Proof. 1) We need to show that for each $y \in Int_{\delta}(x+A)$, $-x+y \in \beta Int(A)$. We know $Int_{\delta}(x+A)$ is δ-open. Then for each $y \in Int_{\delta}(x+A)$, there exists a regular open set U in Z such that $y \in U \subseteq Int_\delta(x+A)$. Since $y \in Int_\delta(x+A)$, $y = x + a$ for some $a \in A$. Since Z is almost β-topological vector space, then there exist β-open sets V_1 and V_2 in Z containing x and a respectively and $V_1 + V_2 \subseteq U$. Thus $x + V_2 \subseteq U \Rightarrow V_2 \subseteq -x + U \subseteq -x + (x+A) = A$. Since V_2 is β-open, then $V_2 \subseteq \beta Int(A)$ and therefore $a \in \beta Int(A) \Rightarrow -x + y \in \beta Int(A)$ $\Rightarrow y \in x + \beta Int(A)$. Hence the assertion follows.

2) Let $y \in x + Int_\delta(A)$. Then there exists a regular open set U in Z such that $-x + y \in U \subseteq Int_\delta(A) \subseteq A$. By definition of almost β-topological vector spaces, we have β-open sets V_1 and V_2 in Z containing $-x$ and y respectively, such that $V_1 + V_2 \subseteq U$. Thus $V_2 \subseteq x + U \subseteq x + A \Rightarrow y \in \beta Int(x+A)$. Hence $x + Int_\delta(A) \subseteq \beta Int(x+A)$.

Theorem 4.8 *For any subset A of an almost β-topological vector space Z, the following are true:*

1) $Int_\delta(\lambda A) \subseteq \lambda \beta Int(A)$, *and*

2) $\lambda Int_\delta(A) \subseteq \beta Int(\lambda A)$, *for each non zero scalar* λ.

Proof. Follows from the proof of above theorem by using second axiom of an almost β-topological vector space.

Theorem 4.9 *Let Z be an almost β-topological vector space. Then*

1) *the translation mapping* $T_x : Z \to Z$ *defined by* $T_x(y) = x + y, \forall x, y \in Z$, *is almost β-continuous.*

2) t*he multiplication mapping* $M_\lambda : Z \to Z$ *defined by* $M_\lambda(x) = \lambda x, \forall x \in Z$, *is almost β-continuous, where* λ *be non-zero scalar in* $\mathbb{F}$.

Proof. 1) Let $y \in X$ be an arbitrary. Let O be any open set in Z containing $T_x(y)$. As $O \subseteq Int(Cl(O))$, we have $T_x(y) \in Int(Cl(O))$ Since Z is an almost β-topological vector space, there exist β-open sets V_1 and V_2 in Z containing x and y respectively such that $V_1 + V_2 \subseteq Int(Cl(O))$. Thus $x + V_2 \subseteq Int(Cl(O))$ $\Rightarrow T_x(V_2) \subseteq Int(Cl(O))$. This proves that T_x is almost β-continuous at y. Since $y \in Z$ was arbitrary, it follows that T_x is almost β-continuous.

2) Let $x \in Z$ and O be any open set in Z containing $M_\lambda(x)$. Then there exist β-open sets V_1 in the topological field $\mathbb{F}$ containing λ and V_2 in Z containing x such that $V_1 \cdot V_2 \subseteq Int(Cl(O))$. Thus $\lambda V_2 \subseteq Int(Cl(O))$ $\Rightarrow M_\lambda(V_2) \subseteq Int(Cl(O))$. This shows that M_λ is almost β-continuous at x and hence M_λ is almost β-continuous everywhere in Z.

Theorem 4.10 *For an almost β-topological vector space Z, the mapping* $\phi : Z \times Z \to Z$ *defined by* $\phi(x, y) = x + y$, $\forall (x, y) \in Z \times Z$, *is almost β-continuous.*

Proof. Let $(x, y) \in Z \times Z$ and let U be regular open set in Z containing $x + y$. Then, there exist β-open sets V_1 and V_2 in Z such that $x \in V_1$, $y \in V_2$ and $V_1 + V_2 \subseteq U$. Since $V_1 \times V_2$ is β-open in $Z \times Z$ (with respect to product topology) such that $(x, y) \in V_1 \times V_2$ and $\phi(V_1 \times V_2) = V_1 + V_2 \subseteq U$. It follows that ϕ is almost β-continuous at (x, y). Since $(x, y) \in Z \times Z$ is arbitrary, ϕ is almost β-continuous.

Theorem 4.11 *For an almost β-topological vector space Z, the mapping*

$\psi:\mathbb{F}\times Z\to Z$ *defined by* $\phi(\lambda,x)=\lambda x$, $\forall(\lambda,x)\in\mathbb{F}\times Z$, *is almost* β*-continuous.*

Proof. Follows from the proof of theorem 4.10 by using the second axiom of almost β-topological vector space.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Kolmogroff, A. (1934) Zur Normierbarkeit eines topologischen linearen Raumes. *Studia Mathematica*, **5**, 29-33. https://doi.org/10.4064/sm-5-1-29-33

[2] Levine, N. (1963) Semi-Open Sets and Semi-Continuity in Topological Spaces. *American Mathematical Monthly*, **70**, 36-41. https://doi.org/10.2307/2312781

[3] Mashhour, A.S., Abd El-Monsef, M.E. and El-Deeb, S.N. (1982) On Precontinuous and Weak Precontinuous Mappings. *Proceedings of the Mathematical and Physical Society of Egypt*, **53**, 47-53.

[4] Abd El-Monsef, M.E., El-Deeb, S.N. and Mahmoud, R.A. (1983) β-Open Sets and β-Continuous Mappings. *Bulletin of the Faculty of Science. Assiut University*, **12**, 77-90.

[5] Veliko, N.V. (1968) H-Closed Topological Spaces. *American Mathematical Society Translations*, **78**, 103-118.

[6] Khan, M.D., Azam, S. and Bosan, M.S. (2015) s-Topological Vector Spaces. *Journal of Linear and Topological Algebra*, **4**, 153-158.

[7] Khan, M. and Iqbal, M.I. (2016) On Irresolute Topological Vector Spaces. *Advances in Pure Mathematics*, **6**, 105-112. https://doi.org/10.4236/apm.2016.62009

[8] Sharma, S. and Ram, M. (2019) On β-Topological Vector Spaces. *Journal of Linear and Topological Algebra*, **8**, 63-70.

[9] Ram, M., Sharma, S., Billawria, S. and Hussain, A. (2018) On Almost s-Topological Vector Spaces. *Journal of Advanced Studies in Topology*, **9**, 139-146. https://doi.org/10.20454/jast.2018.1490

[10] Abd El-Monsef, M.E., Mahmoud, R.A. and Lashin, E.R. (1986) β-Closure and β-Interior. *Ain Shams Engineering Journal*, **10**, 235-245.

[11] Noiri, T. and Popa, V. (1998) On Almost β-Continuous Functions. *Acta Mathematica Hungarica*, **79**, 329-339. https://doi.org/10.1023/a:1006519213848

[12] Grothendieck, A. (1973) Topological Vector Spaces. Gordon and Breach Science Publishers, NewYork.

[13] Sharma, S., Ram, M. and Billawria, S. (2018) On Almost Pretopological Vector Spaces. *Open Access Library Journal*, **5**, e4937. https://doi.org/10.4236/oalib.1104937