On Landsberg and Berwald Spaces of Two Dimensional Finslerian Space with Special (α, β)-Metric

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Abstract

In the present paper, we find a condition for the two-dimensional Finsler space with a special (α, β)-metric $L(\alpha, \beta) = \alpha + \dfrac{\alpha^2}{\beta}$ to be a Berwald space.

Also we have proved that, if the two-dimensional Finsler space with above metric is a Landsberg space, then it is a Berwald space.

Subject Areas

Mathematical Analysis

Keywords

Finsler Space, Berwald Space, Cartan Connection, Landsberg Space, Main Scalar

1. Introduction

In the carton connection $C\Gamma$, if the covariant derivative $C_{hij|k}$ satisfies $C_{hij|k}(x, y) y^k = 0$ then the Finsler space is known as Landsberg space. L. Berwald introduced a class of Finslerian spaces which are known as Berwald spaces in which local coefficients of the Berwald connection depend only on position coordinates. If Landsberg space satisfy some conditions, then it is Berwald space [1]. On the other hand, in two-dimensional case, the main scalar of a general Finsler space $I(x, y)$ satisfies $I_{|i} y^i = 0$ if and only if general Finslerian space is a Landsberg space [2].

The purpose of the present paper is to find a two-dimensional Landsberg space

with a special (α,β)-metric $L(\alpha,\beta) = \alpha + \dfrac{\alpha^2}{\beta}$ satisfying some conditions. First we find the condition for a Finsler space with a special (α,β)-metric to be a Berwald space. Next, we determine the difference vector and the main scalar of F^2 with the aforesaid metric.

Finally, we derive the condition for a two-dimensional Finsler space F^2 with a special (α,β)-metric $L(\alpha,\beta) = \alpha + \dfrac{\alpha^2}{\beta}$ to be a Landsberg space and we have shown that if F^2 with the mentioned metric is a Landsberg space, then it is a Berwald space.

2. Preliminaries

Let an n-dimensional Finsler space $F^n = \left(M^n, L(\alpha,\beta)\right)$ with (α,β)-metric and the associated Riemannian space $R^n = \left(M^n, \alpha\right)$ where $\alpha^2 = a_{ij}(x)y^i y^j, \beta = b_i(x)y^i$. We put $a^{ij} = \left(a_{ij}\right)^{-1}$ since a_{ij} is invertible. In the following, we restrict our discussions to a domain of (x,y) where β does not vanish by taking Riemannian metric α is not supposed to be positive definite. The semi-colon denotes the covariant differentiation in the Levi-Civita connection $\left(\gamma^i_{jk}(x)\right)$ of R^n.

We have the following symbols

$$b^i = a^{ir}b_r, b^2 = a^{rs}b_r b_s,$$

$$2r_{ij} = b_{i;j} + b_{j;i}, 2s_{ij} = b_{i;j} - b_{j;i},$$

$$r^i_j = a^{ir}r_{rj}, s^i_j = a^{ir}s_{rj}, r_i = b_r r^r_i, s_i = b_r s^r_i,$$

$$L_\alpha = \partial L/\partial \alpha, L_\beta = \partial L/\partial \beta, L_{\alpha\alpha} = \partial^2 L/\partial \alpha^2, L_{\beta\beta} = \partial^2 L/\partial \beta^2.$$

Here $B\Gamma = \left(G^i_{jk}, G^i_j, 0\right)$ of F^n plays an important role. Denote by B^i_{jk} the difference tensor of Matsumoto [3] of G^i_{jk} from γ^i_{jk}:

$$G^i_{jk}(x,y) = \gamma^i_{jk}(x,y) + B^i_{jk}(x,y). \tag{2.1}$$

Transvecting above by y^i and then by y^j, we have

$$\left.\begin{array}{l} G^i_j = \gamma^i_{0j} + B^i_j, \\ 2G^i = \gamma^i_{00} + 2B^i. \end{array}\right\} \tag{2.2}$$

Then $B^i_j = \dot{\partial}_j B^i$ and $B^i_{jk} = \dot{\partial}_k B^i_j$.

On account Matsumoto [3], the components of B^i_{jk} is determined by

$$L_\alpha B^k_{ji} y^i y_k = \alpha L_\beta \left(b_{j;i} - B^k_{ji}b_k\right)y^j. \tag{2.3}$$

According to Matsumoto [3], $B^i(x,y)$ is called the difference vector if

$$\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha} \neq 0,$$

where $\gamma^2 = b^2\alpha^2 - \beta^2$.

Then B^i is written as follows

$$B^i = \frac{E}{\alpha}y^i + \frac{\alpha L_\beta}{L_\alpha}s^i_0 - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}C\left(\frac{1}{\alpha}y^i - \frac{\alpha}{\beta}b^i\right), \tag{2.4}$$

where

$$E = \left(\frac{\beta L_\beta}{L}\right)C, C = \frac{\alpha\beta\left(r_{00}L_\alpha - 2\alpha s_0 L_\beta\right)}{2\left(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}\right)}.$$

Further, by means of M. Hashiguchi, S. Hojo and M. Matsumoto [4], we have

$$\left.\begin{array}{l} \alpha_{|i} = -\dfrac{L_\beta}{L_\alpha}\beta_{|i}, \\[2mm] \beta_{|i}y^i = r_{00} - 2b_r B^r, \\[2mm] b_{|i}^2 y^i = 2\left(r_0 + s_0\right), \\[2mm] \gamma_{|i}^2 y^i = 2\left(r_0 + s_0\right)\alpha^2 - 2\left(\dfrac{L_\beta}{L_\alpha}b^2\alpha + \beta\right)\left(r_{00} - 2b_r B^r\right). \end{array}\right\} \quad (2.5)$$

We have the following lemmas

Lemma 2.1. [2] [5]. *If $a_{ij}y^i y^j$ contains $b_i(x)y^i$ as a factor i.e.*
$\alpha^2 \equiv 0\left(mod\,\beta\right)$, *then the dimension $n = 2$ and b^2 vanishes. In this case we have 1-form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

Lemma 2.2. [4]. *We consider the two dimension case.*

1) *If $b^2 \neq 0$, then $\exists$ a sign $\epsilon = \pm 1$ and $\delta = d_i(x)y^i$ $\square$ $\alpha^2 = \dfrac{\beta^2}{b^2} + \epsilon\delta^2$*

and $d_i b^i = 0$.

2) *If $b^2 = 0$, then $\exists$ $\delta = d_i(x)y^i$ $\square$ $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

If two functions $f(x)$ and $g(x)$ satisfies $f\alpha^2 + g\beta^2 = 0$, then it is cleared that $f = g = 0$ because $f \neq 0$ gives a contradiction $\alpha^2 = -\dfrac{g}{f}\beta^2$.

Throughout the chapter, for brevity we shall say "homogeneous polynomial (*s*) in y^i of degree *r*". Hence γ_{00}^i are $hp(2)$.

3. Berwald Space

In this section, Let us consider an *n*-dimensional Finslerian space
$F^n = \left(M^n, L(\alpha, \beta)\right)$ with the following special (α, β)-metric

$$L(\alpha, \beta) = \alpha + \frac{\alpha^2}{\beta}. \quad (3.1)$$

First we shall assume $b^2 \neq 0$.

Suppose if $b^2 = 0$, then from lemma (2.2), we have $\alpha^2 = \beta\delta$, then
$L(\alpha, \beta) = \alpha + \delta$, which is a Randers metric. So the assumption $b^2 \neq 0$ is reasonable.

Then from the above, we have

$$L_\alpha = 1 + 2\frac{\alpha}{\beta},$$

$$L_\beta = -\frac{\alpha^2}{\beta^2},$$

$$L_{\alpha\alpha} = \frac{2}{\beta},$$

$$L_{\beta\beta} = 2\frac{\alpha^2}{\beta^3}. \tag{3.2}$$

Substituting (3.2) into (3.3), we obtain

$$\beta^2 B_{ji}^k y^i y_k + \alpha \left\{ 2\beta B_{ji}^k y^i y_k + \alpha^2 \left(b_{j;i} - B_{ji}^k b_k \right) y^j \right\} = 0. \tag{3.3}$$

Assume that the Finsler space with metric (3.1) be a Berwald space, *i.e.*, $G_{jk}^i = G_{jk}^i(x)$.

Then we have $B_{ji}^k = B_{ji}^k(x)$, so LHS of (3.3) has a form

$$P(x,y) + \alpha Q(x,y) = 0,$$

where P and Q are polynomials in y^i while α is irrational in y^i. Hence the Equation (3.3) shows $P = Q = 0$.

By Lemma (2.1), we have

$$B_{ji}^k a_{kh} y^j y^h = 0, \left(b_{j;i} - B_{ji}^k b_k \right) y^i = 0.$$

The former yields $B_{ji}^k a_{kh} + B_{hi}^k a_{kj} = 0$, so we have $B_{ji}^k = 0$. Then the latter leads to $b_{j;i} = 0$ directly.

Conversely, if $b_{j;i} = 0$, by well known Okada's axioms $\left(\gamma_{jk}^i, \gamma_{0j}^i, 0 \right)$ becomes the Berwald connection of F^n. Thus F^n is a Berwald space.

Hence we have the following result

Theorem 3.1. *The Finsler space F^n with special (α, β)-metric (3.1) satisfying $b^2 \neq 0$ is a Berwald space if and only if $b_{j;i} = 0$, then Berwald connection is essentially Riemannian $\left(\gamma_{jk}^i, \gamma_{0j}^i, 0 \right)$.*

4. Two-Dimensional Landsberg Space

In this section, Let us consider an n-dimensional Finslerian space $F^n = \left(M^n, L(\alpha, \beta) \right)$ with the following special (α, β)-metric

$$L(\alpha, \beta) = \alpha + \frac{\alpha^2}{\beta}. \tag{4.1}$$

By means of (2.4) and (3.2), the difference vector B^i [6] of the Finsler space becomes

$$2B^i = \frac{A}{\beta(\beta + 2\alpha)L\Omega} \left\{ By^i + 2\alpha^3 Lb^i \right\} - \frac{2\alpha^3}{\beta(\beta + 2\alpha)} s_0^i, \tag{4.2}$$

where

$$A = \beta(\beta + 2\alpha)r_{00} + 2\alpha^3 s_0,$$

$$B = -3\alpha^2\beta - 4\alpha^3,$$

$$\Omega = \beta^3 + 2b^2\alpha^3.$$

It is trivial that $\beta \neq 0$, $\beta + 2\alpha \neq 0$ and $\Omega \neq 0$, because α is irrational in y^i. From (4.2) it follows that

$$r_{00} - 2b_r B^r = \frac{\alpha(\beta + 2\alpha)A}{L\Omega}.$$

In two-dimensional case, the main scalar of a general Finsler space $I(x, y)$ satisfies $I_{|i} y^i = 0$ if and only if general Finsler space is a Landsberg space [7]. If F^2 with (4.1), then the main scalar I is obtained as follows

$$\varepsilon I^2 = \frac{9\gamma^2 M^2}{4\alpha\beta L\Omega^3},\tag{4.3}$$

where

$$M = -\alpha^2\beta^3 - 2b^2\alpha^4\beta - 4b^2\alpha^5,$$

$$\Omega = \beta^3 + 2b^2\alpha^3.$$

The covariant differentiation of (4.3) leads to

$$4\alpha^2\beta^2 L\Omega^4 \varepsilon I_{|i}^2 = 9M\left(\alpha\beta\Omega M\gamma_{|i}^2 + 2\alpha\beta\Omega\gamma^2 M_{|i} - \beta\Omega\gamma^2 M\alpha_{|i} \right.$$
$$\left. - \alpha\Omega\gamma^2 M\beta_{|i} - 3\alpha\beta\gamma^2 M\Omega_{|i}\right).\tag{4.4}$$

Transvecting (4.4) by y^i, we have

$$4\alpha^2\beta^2 L\Omega^4 \varepsilon I_{|i}^2 = 9M\left(U\gamma_{|i}^2 y^i + QM_{|i} y^i - R\alpha_{|i} y^i - S\beta_{|i} y^i - T\Omega_{|i} y^i\right),\tag{4.5}$$

where

$$U = -\alpha^3\beta^7 - 2b^2\alpha^5\beta^5 - 6b^2\alpha^6\beta^4 - 4b^4\alpha^8\beta^2 - 8b^4\alpha^9\beta,$$

$$Q = -2\alpha\beta^6 + 2b^2\alpha^3\beta^4 - 4b^2\alpha^4\beta^3 + 4b^4\alpha^6\beta,$$

$$R = \alpha^2\beta^9 + b^2\alpha^4\beta^7 + 6b^2\alpha^5\beta^6 - 2b^4\alpha^6\beta^5 - 2b^4\alpha^7\beta^4$$
$$+ 8b^4\alpha^8\beta^3 - 4b^6\alpha^9\beta^2 - 8b^6\alpha^{10}\beta,$$

$$S = \alpha^3\beta^8 + b^2\alpha^5\beta^6 + 6b^2\alpha^6\beta^5 - 2b^4\alpha^7\beta^4 - 2b^4\alpha^8\beta^3$$
$$+ 8b^4\alpha^9\beta^2 - 4b^6\alpha^{10}\beta - 8b^6\alpha^{11},$$

$$T = 3\alpha^3\beta^6 + 3b^2\alpha^5\beta^4 + 12b^2\alpha^6\beta^3 - 6b^4\alpha^7\beta^2 - 12b^4\alpha^8\beta.$$

Hence (4.5) can be put in the form

$$4\alpha^2\beta^2 L\Omega^4 \varepsilon I_{|i}^2 y^i = 9M\left(U\gamma_{|i}^2 y^i + V\alpha_{|i} y^i + W\beta_{|i} y^i + Xb_{|i}^2 y^i\right),$$

where

$$V = 3\alpha^2\beta^9 + 11b^2\alpha^4\beta^7 + 24b^2\alpha^5\beta^6 - 14b^4\alpha^6\beta^5 - 32b^4\alpha^7\beta^4 + 8b^6\alpha^9\beta^2,$$

$$W = -4\alpha^3\beta^8 - 12b^2\alpha^5\beta^6 - 30b^2\alpha^6\beta^5 + 16b^4\alpha^7\beta^4 + 34b^4\alpha^8\beta^3$$
$$- 8b^4\alpha^9\beta^2 - 4b^6\alpha^{10}\beta + 8b^6\alpha^{11},$$

$$X = 4\alpha^5\beta^7 + 5\alpha^6\beta^6 - 4b^2\alpha^7\beta^5 - 3b^2\alpha^8\beta^4 + 4b^2\alpha^9\beta^3 - 2b^4\alpha^{10}\beta^2 - 4b^4\alpha^{11}\beta.$$

Consequently, the Finslerian space F^2 with special (α, β)-metric (4.1) is Landsberg space if and only if

$$U\gamma_{|i}^2 y^i + V\alpha_{|i} y^i + W\beta_{|i} y^i + Xb_{|i}^2 y^i = 0,$$

since $M \neq 0$.

If $M = 0$, then $b^2 = 0$ which is a contradiction.

In view of (2.5), the above equation written as

$$2\left(\beta^2 + 2\alpha\beta\right)\left(\alpha^2 U + X\right)(r_0 + s_0)$$
$$+ \left[V\alpha^2 + W\left(\beta^2 + 2\alpha\beta\right) - 2\left\{\beta^3 + 2\alpha\beta^2 - b^2\alpha^3\right\}U\right]\left(r_{00} - 2b_r B^r\right) = 0.\tag{4.6}$$

Substituting the values of U, V, W, X and $\left(r_{00} - 2b_r B^r\right)$ in (4.6), we obtain

$$\alpha^6 \beta^3 \Big[6\beta^{10} + \left(42 - 12b^2\right)\alpha^2\beta^8 - 14\alpha^4\beta^6 + \left(24b^2 - 168b^4\right)\alpha^6\beta^4$$
$$+ \left(-136b^4 - 24b^6\right)\alpha^8\beta^2 - 192b^6\alpha^{10} \Big](r_0 + s_0) + \alpha^7\beta^2 \Big[28\beta^{10} + \left(20 - 42b^2\right)\alpha^2\beta^8$$
$$+ \left(40b^2 - 36b^4\right)\alpha^4\beta^6 - 236b^4\alpha^6\beta^4 + \left(-120b^6 - 32b^4\right)\alpha^8\beta^2 - 96b^6\alpha^{10} \Big](r_0 + s_0)$$
$$+ \alpha^4\beta^6 \Big[256b^4\alpha^8 - 16b^6\alpha^8 + 336b^4\alpha^6\beta^2 - 144b^2\alpha^6\beta^2 - 168b^2\alpha^4\beta^4$$
$$+ 16b^4\alpha^4\beta^4 - 12\alpha^2\beta^6 - 2\beta^8 \Big]r_{00} + \alpha^5\beta^5 \Big[-9\beta^8 - 57b^2\alpha^2\beta^6 - 4\alpha^2\beta^6$$
$$+ 110b^4\alpha^4\beta^4 - 244b^2\alpha^4\beta^4 + 432b^4\alpha^6\beta^2 - 4b^6\alpha^6\beta^2 + 64b^4\alpha^8 - 16b^6\alpha^8 \Big]r_{00}$$
$$+ \alpha^8\beta^5 \Big[-10\beta^6 - 178b^2\alpha^2\beta^4 - 144b^2\alpha^4\beta^2 + 176b^4\alpha^4\beta^2 + 224b^4\alpha^6 - 8b^6\alpha^6 \Big]s_0$$
$$+ \alpha^7\beta^4 \Big[-4\beta^8 - 16b^2\alpha^2\beta^6 - 32b^4\alpha^4\beta^4 - 172b^2\alpha^4\beta^4 + 32b^4\alpha^6\beta^2 - 16b^6\alpha^8 \tag{4.7}$$
$$+ 64b^4\alpha^8 \Big]s_0 = 0.$$

Separating (4.7) as rational and irrational terms with respect to (y^i), we obtain

$$\Big[\alpha^6\beta^3 D_1 (r_0 + s_0) + \alpha^4\beta^6 E_1 r_{00} + \alpha^8\beta^5 F_1 s_0 \Big]$$
$$+ \alpha \Big[\alpha^6\beta^2 D_2 (r_0 + s_0) + \alpha^4\beta^5 E_2 r_{00} + \alpha^6\beta^4 F_2 s_0 \Big] = 0, \tag{4.8}$$

where

$$D_1 = 6\beta^{10} + \left(42 - 12b^2\right)\alpha^2\beta^8 - 14\alpha^4\beta^6 + \left(24b^2 - 168b^4\right)\alpha^6\beta^4$$
$$+ \left(-136b^4 - 24b^6\right)\alpha^8\beta^2 - 192b^6\alpha^{10},$$

$$D_2 = 28\beta^{10} + \left(20 - 42b^2\right)\alpha^2\beta^8 + \left(40b^2 - 36b^4\right)\alpha^4\beta^6 - 236b^4\alpha^6\beta^4$$
$$+ \left(-120b^6 - 32b^4\right)\alpha^8\beta^2 - 96b^6\alpha^{10},$$

$$E_1 = 256b^4\alpha^8 - 16b^6\alpha^8 + 336b^4\alpha^6\beta^2 - 144b^2\alpha^6\beta^2 - 168b^2\alpha^4\beta^4$$
$$+ 16b^4\alpha^4\beta^4 - 12\alpha^2\beta^6 - 2\beta^8,$$

$$E_2 = -9\beta^8 - 57b^2\alpha^2\beta^6 - 4\alpha^2\beta^6 + 110b^4\alpha^4\beta^4 - 244b^2\alpha^4\beta^4$$
$$+ 432b^4\alpha^6\beta^2 - 4b^6\alpha^6\beta^2 + 64b^4\alpha^8 - 16b^6\alpha^8,$$

$$F_1 = -10\beta^6 - 178b^2\alpha^2\beta^4 - 144b^2\alpha^4\beta^2 + 176b^4\alpha^4\beta^2 + 224b^4\alpha^6 - 8b^6\alpha^6,$$

$$F_2 = -4\beta^8 - 16b^2\alpha^2\beta^6 - 32b^4\alpha^4\beta^4 - 172b^2\alpha^4\beta^4 + 32b^4\alpha^6\beta^2 - 16b^6\alpha^8 + 64b^4\alpha^8.$$

The Equation (4.8) yields two equations as follows

$$\alpha^2 D_1 (r_0 + s_0) + \beta^3 E_1 r_{00} + \alpha^4\beta^2 F_1 s_0 = 0, \tag{4.9}$$

$$\alpha^2 D_2 (r_0 + s_0) + \beta^3 E_2 r_{00} + \alpha^2\beta^2 F_2 s_0 = 0. \tag{4.10}$$

From (4.10), we obtain

$$-9\beta^{11} r_{00} \equiv 0 \left(\bmod \alpha^2\right).$$

Then $\exists$ a function $f(x)$ $\Box$

$$r_{00} = \alpha^2 f(x).$$

Thus, we have

$$r_{ij} = a_{ij} f(x). \tag{4.11}$$

Transvecting above by $b^i y^j$ leads to

$$r_0 = \beta f(x); r_j = b_j f(x).$$

Eliminating $(r_0 + s_0)$ from (4.9) and (4.10), from (4.11), we have

$$\alpha^2 \beta^3 f(x)(D_2 E_1 - D_1 E_2) + \alpha^2 \beta^3 g(x)(\alpha^2 D_2 F_1 - D_1 F_2) = 0. \tag{4.12}$$

From $\alpha^2 \neq 0 (\mathrm{mod}\,\beta)$ it follows that $\exists$ a function $g(x)$ $\square$ $s_0 = g(x)\beta$. Hence (4.12) is reduces to

$$\alpha^2 D_2 F_1 g(x) + \{ f(x)(D_2 E_1 - D_1 E_2) - g(x) D_1 F_2 \} = 0. \tag{4.13}$$

Since only the term $-2(f(x) - 12g(x))\beta^{18}$ of

$$f(x)(D_2 E_1 - D_1 E_2) - g(x) D_1 F_2$$

seemingly does not contain α^2, we must have $hp(16)V_{16}$ such that $\beta^{18} = \alpha^2 V_{16}$. Thus it is a contradiction because of $\alpha^2 \neq 0 (\mathrm{mod}\,\beta)$, that is,

$$f(x)(D_2 E_1 - D_1 E_2) - g(x) D_1 F_2$$

does not contain α^2 as a factor.

Thus, from (4.13) we have $g(x) = 0$, which leads to $s_0 = 0$ and $s_i = 0$. Hence

$$f(x)(D_2 E_1 - D_1 E_2) - g(x) D_1 F_2 = 0, \tag{4.14}$$

which implies $f(x) = g(x) = 0$, which leads to $s_0 = 0$ and $s_i = 0$. From (4.11), we get $r_{ij} = 0$.

Summarizing up, we obtain $r_{ij} = 0$ and $s_i = 0$, that is,

$$b_{i;j} + b_{j;i} = 0, b^r b_{r;i} = 0.$$

Therefore $b_i(x)$ is the so-called killing vector field with a constant length.

According to Hashiguchi, Hojo and Matsumoto [4], the condition (4.14) is equivalent to $b_{i;j} = 0$.

Hence, we have the following result

Theorem 4.2. *If a two dimensional Finsler space* F^2 *with a special* (α, β) *-metric* (4.1) *satisfying* $b^2 \neq 0$, *is a Landsberg space then* F^2 *is a Berwald space.*

5. Conclusion

In this paper, first we found a condition for a Finslerian space with special (α, β) -metric $L(\alpha, \beta) = \alpha + \dfrac{\alpha^2}{\beta}$ to be a Berwald space. Further we have proved that two-dimensional Finslerian space with a special (α, β) -metric $L(\alpha, \beta) = \alpha + \dfrac{\alpha^2}{\beta}$ is a Landsberg space, then it is a Berwald space.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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