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Radiation Reaction Formula in Electrodynamics from the Momentum Carried Out by the Electromagnetic Field Due to an Accelerating Charge

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Abstract

As a follow up of the author's earlier papers the new theory of radiation reaction is placed on a sound theoretical basis. This includes the nature of the equation of motion both under a relativistic Lorentz transformation as well as under a non-relativistic Galilean transformation. It is claimed that both momentum and energy carried out by the radiated fields need to be considered to reach the desired results. It is also shown that the radiation reaction force is non linearly dependent on the externally applied force.

Subject Areas

Theoretical Physics

Keywords

Radiation Reaction, Classical Electrodynamics

1. Introduction

The theory of radiation reaction force in classical electrodynamics as proposed by Lorentz, Abraham and Dirac (LAD—see for example Gron [1]) has the problem of pre-acceleration (see for example [2]). Thus it is possible to change the past history of the system however small it may be, by taking a sudden decision to switch on or off an electromagnetic field. Observational physics has not so far produced any evidence of such a phenomenon. Yaghjian [3] has tried to use certain transition forces to cancel pre-acceleration quite successfully. However these transition forces cannot be exactly modeled as seen on page 83 of ref. [3] in the transition interval Δt_α. This in my opinion can produce some ambiguity in the predicted dynamics of the particle.

Following a suggestion by Hammond [4] [5] [6], I tried to develop an alternative theory of radiation reaction [7] [8] which in the non relativistic limit does not yield a force law which is unchanged under a Galilean transformation connecting two inertial reference frames. This theory was however free of pre-acceleration and/or runaway solutions. In this present paper we try to take the electromagnetic momentum conservation into account. This is now taken along with the energy conservation which was already considered in my previous papers to evolve an equation of motion which is independent of the velocity and thus preserves the Newtonian concept of force in different inertial frames. The theory however may be a crude starting point and will necessitate refinements to replace the LAD equations.

2. The Electromagnetic Field of an Accelerating Point Charge and the Momentum Carried Out by the Radiation Fields

The electric field of a point charge [9] with some change in notations from what is given in this reference can be written as (see Equation (10.65) of this book)

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{R}{\left(Rc - \vec{R}.\vec{v}\right)^3}\left[\left(c^2 - v^2\right)\left(c\hat{R} - \vec{v}\right) + \vec{R}\times\left(c\hat{R} - \vec{v}\right)\times\vec{a}\right], \tag{1}$$

where $\vec{R}$ is a function of the time t and given at the retarded time t_r as

$$\vec{R}(t_r) = \vec{r} - \vec{w}(t_r) \approx \vec{r} - \vec{w}(t) + \vec{v}(t)\left|\vec{r} - \vec{w}(t)\right|/c .$$

Here $\vec{r}$ is the position vector of the field point, $\vec{w}(t_r)$ is the position vector of the particle as a function of time and $\vec{v}(t)$ is its velocity. The magnetic field is (see Equation (10.66) of reference [9])

$$\vec{B} = \frac{1}{c}\hat{R}\times\vec{E} \tag{2}$$

In Equation (1) and Equation (2) we are interested only in the radiation fields that is fields which depend on acceleration $\vec{a}$ and falls off at the rate $\frac{1}{R}$ from the position of the source. Also specifically for the purpose of the present study we consider terms up to first degree in velocity only, that is we consider fields due to non-relativistic slowly moving particles. Within these set of assumptions we can write the electric and magnetic fields of a particle accelerated along the z direction and in a reference frame in which it is at the origin at time t. It is also possible to select the reference frame in a way such that at some initial time the velocity is zero and hence by kinematical laws the velocity will be z directed for all times. The electric and magnetic field expressions are then

$$E_x = \frac{qa\cos\theta\sin\theta\cos\phi}{4\pi\varepsilon_0 c^2 r}\left[1 + \frac{v}{c\cos\theta}\right]$$

$$E_y = \frac{qa\cos\theta\sin\theta\sin\phi}{4\pi\varepsilon_0 c^2 r}\left[1+\frac{v}{c\cos\theta}\right]$$

$$E_z = -\frac{qa\sin^2\theta}{4\pi\varepsilon_0 c^2 r} \quad (3)$$

$$B_x = -\frac{qa\sin\theta\sin\phi}{4\pi\varepsilon_0 c^3 r}\left[1+\frac{v\cos\theta}{c}\right]$$

$$B_y = \frac{qa\sin\theta\cos\phi}{4\pi\varepsilon_0 c^3 r}\left[1+\frac{v\cos\theta}{c}\right]$$

expressed in terms of spherical coordinates. We have been asked by the referee to explain why the magnetic field does not have a z component for a particle whose motion is along the z axis at all times. The reason is that the $\vec{B}$ given by Equation (2) for the slow motion approximation and for the far zone field for this particular case is calculated by us to be

$$\vec{B} = \frac{qa}{4\pi\varepsilon_0 c^3 r}\left[-\hat{r}\times\hat{u}_z\left(1+\frac{v\cos\theta}{c}\right)\right]$$

where $\hat{r}$ and $\hat{u}_z$ are the unit vectors in the direction of the field point from the origin and along the z axis respectively. If the referee points out the mistake in our calculation we will be happy to revise our entire manuscript as the following results are all dependent on this formula. We also state that the angles θ and ϕ which were not clearly explained in our first manuscript are the spherical coordinates that are used in standard textbooks the co-latitude and the azimuth respectively.

The electromagnetic momentum conservation equation is again obtained from ref. [9] (see chapter 8)

$$\int_V \left(\rho\vec{E}+\vec{J}\times\vec{B}\right)d^3x = \int_S d\,\vec{s}\cdot\overleftrightarrow{T} - \frac{1}{c^2}\frac{d}{dt}\int_V \vec{E}\times\vec{H}d^3x \quad (4)$$

where $-\int_S d\,\vec{s}\cdot\overleftrightarrow{T}$ is to be considered as the rate of momentum radiated out through the enclosing surface S of volume V in which all charges are supposed to be confined. The Maxwell stress tensor $\overleftrightarrow{T}$ is given by Equation (8.19) of ref. [9]. In our calculation of momentum radiated out we make the following assumption that the system is approximately in steady state that is any change in $\int_V \vec{E}\times\vec{H}d^3x$ can be neglected. This assumption is nothing unusual and is made while calculating the total energy loss rate or radiated power given by Larmor formula. Thus we obtain an expression for the loss rate of the momentum which we denote by $\vec{P}_L$ to be

$$\vec{P}_L = \hat{u}_z\frac{q^2a^2v}{5\pi c^5\varepsilon_0} \quad (5)$$

for a charged particle with z-directed velocity and acceleration and can be considered as the momentum counterpart of Larmor formula.

3. The Radiation Reaction Force and the Equation of Motion Obtained from Momentum Balance

Let us first see the consequences of defining the radiation reaction force $\vec{F}_{rad}$ for motion in one dimension to be equal to the (negative of) momentum loss rate, that is

$$\vec{F}_{rad} = -\vec{P}_L = -\frac{q^2 a^2 v}{5\pi c^5 \varepsilon_0}\hat{u}_z \tag{6}$$

We find from Equation (1) of ref. [7] which is the present author's own published work that $\vec{F}_{rad}$ for one dimensional motion for similarly directed velocity and acceleration (for this case along the z axis) can be expressed as

$$\vec{F}_{rad} = -\frac{\mu_0 q^2 a^2}{6\pi c v}\hat{u}_z = -\frac{q^2 a^2}{6\pi c^3 \varepsilon_0 v}\hat{u}_z \tag{7}$$

where the right hand side is nothing but the energy radiation rate given by Larmor formula divided by the velocity. For a point particle like the electron which can accumulate no internal energy this formula will be valid for any kind of force acting on it that is the force multiplied by the velocity must be a contribution to the rate of change of its kinetic energy. And specifically for the radiation reaction force this contribution to the rate of change of kinetic energy is nothing but equal to the (negative of) radiated power. Also we should keep in mind as discussed in ref. [8] that Equation (7) above which is valid for the similarly directed velocity and acceleration case will hold in a physical situation where it is loosing potential energy in an externally applied force field. The referee has asked us to explain why the right hand sides of Equation (6) and Equation (7) are different even if they are supposed to represent the same quantity $\vec{F}_{rad}$. If from the above discussion it is not clear why it is so we again emphasize that these two expressions are derived from different conservation principles momentum and energy respectively. The laws of physics requires an unified description of both which we give in the next paragraph.

Now we are in a position to compare Equation (6) and Equation (7) to find that they are unequal except for a certain value of v or in other words in a particular reference frame. The problem of choosing a correct reference frame was discussed at length in references [7] and [8] for this present theory of radiation reaction to hold. We find this particular value of v to be $\sqrt{\frac{5}{6}}c$. Since we are using a non-relativistic description of motion this extremely high speed of a Galilean reference frame nearly approaching c should not come as a surprise. In fact it is the only possible starting point to obtain a non-relativistic equation of motion which is independent of the velocity of the inertial reference frame in which it is formulated. Once a non-relativistic description of particle motion respecting Newton's laws is obtained it will be possible as we show in Section 4. to make a relativistic generalization of the formulation. The equation of motion given by Equation (2) of ref. [8] will need to be modified by substituting this value of v that is $\sqrt{\frac{5}{6}}c$ into it to yield

$$\frac{dv}{dt}=\frac{-m+m\sqrt{1+\frac{2\sqrt{2}}{\sqrt{15}}\frac{\mu_0 q^2}{\pi c^2 m^2}\left|\vec{F}_{ext}\right|}}{\frac{\sqrt{2}}{\sqrt{15}}\frac{\mu_0 q^2}{\pi c^2}} \tag{8}$$

At last we have reached an equation of motion which approximately takes into account the radiation reaction of a charged particle and all physical requirements for non-relativistic mechanics. Its form is unchanged under Galilean transformation between different inertial frames ensuring the same value of $\frac{dv}{dt}$ in these frames. It is free of problems like causality violation of LAD theory in the sense that $\frac{dv}{dt}=0$ when $\left|\vec{F}_{ext}\right|=0$. For the other physical case of a particle gaining potential energy in an externally applied field one can substitute $v=-\sqrt{\frac{5}{6}}c$ into Equation (3.b) of ref. [8].

4. Three Dimensional and Relativistic Generalization of the Equation of Motion

The three dimensional generalization of Equation (8) is simple as it shows that $\frac{d\vec{v}}{dt}$ and $\vec{F}_{ext}$ are perfectly aligned (here in Equation (8) along the z-axis). Thus,

$$\frac{d\vec{v}}{dt}=\frac{-m+m\sqrt{1+\frac{2\sqrt{2}}{\sqrt{15}}\frac{\mu_0 q^2}{\pi c^2 m^2}\left|\vec{F}_{ext}\right|}}{\frac{\sqrt{2}}{\sqrt{15}}\frac{\mu_0 q^2}{\pi c^2}}\frac{\vec{F}_{ext}}{\left|\vec{F}_{ext}\right|} \tag{9}$$

For a relativistic generalization of Equation (9) a simple method cannot be devised because of the presence of the square root. If we are able to write all the components of the four vectors on both sides in Equation (9) then a transformation to a reference frame using four vector transformation law will yield the necessary equation of motion. First we note that Equation (9) holds in the rest frame of the particle. In the four vector notation for the spatial components of the velocity four vector the left hand side of this equation can be written as $\frac{dv^\alpha}{d\tau}$ where $\alpha=1,2,3$ and where τ is the proper time which is an invariant. Given that the zeroth or time component of the four velocity vector in the rest frame is 1 and its spatial components are equal to the components of its ordinary velocity (momentarily zero) we make a transformation to a frame which is moving with ordinary velocity $\vec{U}$ with respect to the rest frame according to the Lorentz transformation law [10]

$$A^{0\prime} = \frac{1}{\sqrt{1-\frac{U^2}{c^2}}}\left(A^0 - \frac{\vec{U}\cdot\vec{A}^{(1,2,3)}}{c}\right)$$

$$\vec{A}^{(1,2,3)\prime} = \left(\frac{1}{\sqrt{1-\frac{U^2}{c^2}}}-1\right)\frac{\vec{U}\cdot\vec{A}^{(1,2,3)}}{U^2}\vec{U} + \vec{A}^{(1,2,3)} - \frac{1}{\sqrt{1-\frac{U^2}{c^2}}}\frac{\vec{U}}{c}A^0$$

where $\vec{A}^{(1,2,3)}$ is a short hand notation for the three spatial components of the four vector A^i $(i = 0,1,2,3)$ and just like any ordinary three dimensional vector can be expressed as $\vec{A}^{(1,2,3)} == A_1\hat{u}_x + A_2\hat{u}_y + A_3\hat{u}_z$. Using these relationships, in terms of the primed frame quantities the L.H.S. of Equation (9) appears as (we are making a transformation to a frame from the rest frame of the particle where its velocity $v = 0$ such that in the new frame its velocity is v' and as per the request of the referee we again state LHS below stands for the left hand side of Equation (9) expressed in terms of v' and the transformed time t')

$$LHS = \vec{v}'\left(\frac{1}{\sqrt{1-\frac{v'^2}{c^2}}}-1\right)\frac{1}{\sqrt{1-\frac{v'^2}{c^2}}}\frac{\vec{v}'}{v'^2}\cdot\frac{d}{dt'}\frac{\vec{v}'}{\sqrt{1-\frac{v'^2}{c^2}}} + \frac{1}{\sqrt{1-\frac{v'^2}{c^2}}}\frac{d}{dt'}\frac{\vec{v}'}{\sqrt{1-\frac{v'^2}{c^2}}} + \frac{\vec{v}'}{\left(1-\frac{v'^2}{c^2}\right)c}\frac{d}{dt'}\frac{1}{\sqrt{1-\frac{v'^2}{c^2}}} \quad (10)$$

where $\vec{v}'$ is the ordinary velocity in the new frame. The R.H.S. can be similarly written noting that the four force (having three non zero spatial components only) in the rest frame is expressed in terms of the primed frame quantities as

$$\vec{F}_{ext} = \vec{v}'\left(\frac{1}{\sqrt{1-\frac{v'^2}{c^2}}}-1\right)\frac{\vec{v}'}{v'^2}\cdot\vec{F}_{ext}^{(1,2,3)\prime} + \vec{F}_{ext}^{(1,2,3)\prime} + \frac{1}{\sqrt{1-\frac{v'^2}{c^2}}}\frac{\vec{v}'}{c}F_{ext}^{0\prime} \quad (11)$$

Its subsequent substitution into the R.H.S. of Equation (9) gives an expression in terms of the primed quantities. This new R.H.S. expression in terms of primed quantities can be equated with the L.H.S. expression of Equation (10) to yield the relativistic equation of motion which includes radiation reaction. One can verify that in the limit $\vec{F}_{ext} \to 0$ this equation of motion reduces to the relativistic equation of motion without radiation reaction.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Gron, O. (2012) Electrodynamics of Radiating Charges. *Advances in Mathematical Physics*, **2012**, Article ID: 528631. https://doi.org/10.1155/2012/528631

[2] Roy, R. (2016) Some Studies on the Lorentz Theory of Radiation Reaction in Relation to a Charged Particle Acted on by a Constant Force in a Finite Time Interval. *Open Access Library Journal*, **3**, e2334.

[3] Yaghjian, A.D. (2006) Relativistic Dynamics of a Charged Sphere: Updating the Lorentz-Abraham Model. 2nd Edition, Springer-Verlag, New York. https://doi.org/10.1007/b98846

[4] Hammond, R.T. (2010) Relativistic Particle Motion and Radiation Reaction in Electrodynamics. *Electronic Journal of Theoretical Physics*, **7**, 221.

[5] Hammond, R.T. (2011) Charged Particle in a Constant, Uniform Electric Field with Radiation Reaction. *Advanced Studies in Theoretical Physics*, **5**, 275

[6] Hammond, R.T. (2013) Electrodynamics and Radiation Reaction. *Foundations of Physics*, **43**, 201-209. https://doi.org/10.1007/s10701-012-9687-z

[7] Roy, R. (2014) A New Approach to Radiation Reaction in Classical Electrodynamics. *Advanced Studies in Theoretical Physics*, **8**, 77-82. https://doi.org/10.12988/astp.2014.312141

[8] Roy, R. (2014) Further Studies on the New Approach to Radiation Reaction. *Advanced Studies in Theoretical Physics*, **8,** 955-960. https://doi.org/10.12988/astp.2014.48109

[9] Griffiths, D.J. (1999) Introduction to Electrodynamics. 3rd Edition, Pearson Education Inc., New Jersey, Indian Reprint by Prentice Hall India, 438.

[10] Mould, R.A. (1994) Basic Relativity. 2nd Indian Reprint, Springer-Verlag, New York. https://doi.org/10.1007/978-1-4612-4326-7