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Suzuki-Type Fixed Point Results in b_2-Metric Spaces

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Abstract

A common fixed point theorem for Suzuki-type contractions in the setting of b_2-metric space is established in this paper. Our result extends some known results from metric spaces to b_2-metric space. The research is meaningful and I recommend it to be published in the journal.

Subject Areas

Mathematical Analysis

Keywords

Common Fixed Point, Complete b_2-Metric Space, Suzuki Contraction

1. Introduction

Banach fixed point principle [1] is simple but forceful, which is a classical tool for many aspects. There are many generalizations of this principle, see [2] [3] [4] [5], from which, an interesting generalization is introduced by Suzuki [6] in 2008.

Many generalized spaces of Metric space have been established. Among them, *b*-metric [7] and 2-metric [8] have been extensively researched. Both of these metrics of those spaces are not continuous functions of its variables. In order to solve this problem, the author of [9] established the notion of b_2-metric space generalizing from both spaces above. And in this paper, we proved a common fixed point result for two maps in b_2-metric space [9]. Our purpose is to present a fixed point result of two maps under a newly Suzuki-type contractive condition in this space, and the fixed point theory in b_2-metric space is perfected.

2. Preliminaries

The following definitions will be presented before giving our results.

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Definition 2.1. [9] Let X be a nonempty set, $s \geq 1$ be a real number and let $d: X \times X \times X \to R$ be a map satisfying the following conditions:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

2) If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$.

3) The symmetry:

$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,x,y)$$

for all $x, y, z \in X$.

4) The rectangle inequality:

$$d(x,y,z) \leq s\left[d(x,y,a) + d(y,z,a) + d(z,x,a)\right]$$

for all $x, y, z, a \in X$.

Then d is called a b_2 metric on X and (X,d) is called a b_2 metric space with parameter s. Obviously, for $s = 1$, b_2 metric reduces to 2-metric.

Definition 2.2. [9] Let $\{x_n\}$ be a sequence in a b_2 metric space (X,d).

1) A sequence $\{x_n\}$ is said to be b_2-convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if all $a \in X$ $\lim_{n\to\infty} d(x_n, x, a) = 0$.

2) $\{x_n\}$ is Cauchy sequence if and only if $d(x_n, x_m, a) \to 0$, when $n, m \to \infty$. for all $a \in X$.

3) (X,d) is said to be complete if every b_2-Cauchy sequence is a b_2-convergent sequence.

Definition 2.3. [9] Let (X,d) and (X',d') be two b_2-metric spaces and let $f: X \to X'$ be a mapping. Then f is said to be b_2-continuous, at a point $z \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $d(z,x,a) < \delta$ for all $a \in X$ imply that $d'(fz, fx, a) < \varepsilon$. The mapping f is b_2-continuous on X if it is b_2-continuous at all $z \in X$.

Definition 2.4. [9] Let (X,d) and (X',d') be two b_2-metric spaces. Then a mapping $f: X \to X'$ is b_2-continuous at a point $x \in X'$ if and only if it is b_2-sequentially continuous at x; that is, whenever $\{x_n\}$ is b_2-convergent to x, $\{fx_n\}$ is b_2-convergent to $f(x)$.

Lemma 2.5. [10] Let (X,d) be a b_2 metric space with $s \geq 1$ and let $\{x_n\}_{n=0}^{\infty}$ be a sequence in X such that

$$d(x_n, x_{n+1}, a) \leq \lambda d(x_{n-1}, x_n, a) \tag{2.1}$$

for all $n \in N$ and all $a \in X$, where $\lambda \in [0, 1/s)$. Then $\{x_n\}$ is a b_2-Cauchy sequence in (X,d).

3. Main Results

Theorem 3.1. Let (X,d) be a complete b_2 metric space and in each variable d is continuous. Let $f: X \to X$ be a selfmap and $\phi = \phi_s$: $[0,1) \to (1/(s+1), 1]$ be defined by:

$$\phi(\rho)=\begin{cases}1, 0\le\rho\le\frac{\sqrt{5}-1}{2},\\ \frac{1-\rho}{\rho^2}, \frac{\sqrt{5}-1}{2}\le\rho\le b_s,\\ \frac{1}{s+\rho}, b_s\le\rho<1,\end{cases} \tag{3.1}$$

where $b_s=\frac{1-s+\sqrt{1+6s+s^2}}{4}$ is the positive solution of $\frac{1-\rho}{\rho^2}=\frac{1}{s+\rho}$. If there exists $\rho\in[0,1)$ such that for each $x,y\in X$,

$$\phi(\rho)d(x,fx,a)\le d(x,y,a)\Rightarrow d(fx,fy,a)\le\frac{\rho}{s}N(x,y,a), \tag{3.2}$$

where

$$N(x,y,a)=\max\{d(x,y,a),d(x,fx,a),d(y,fy,a)\}$$

then *f* has a unique fixed point *z* in *X* and the sequence $\{T^n x\}$ converges to *z*.

Proof From (3.1) and take $y=fx$, we get the inequality as follows:

$$\begin{aligned}d(fx,f^2x,a)&\le\frac{\rho}{s}\max\{d(x,fx,a),d(x,fx,a),d(fx,f^2x,a)\}\\&=\frac{\rho}{s}\max\{d(x,fx,a),d(fx,f^2x,a)\}\end{aligned} \tag{3.2.1}$$

from the above relation, we get

$$d(fx,f^2x,a)\le\frac{\rho}{s}d(x,fx,a),\ \text{for each}\ x\in X \tag{3.3}$$

Given $v_0\in X$ and construct a sequence $\{v_n\}$ letting $v_{n+1}=fv_n=f^{n+1}v_0$, for all $n\in N$. Then by taking $x=v_{n-1}$ in (3.3) we get

$$d(v_n,v_{n+1},a)\le\frac{\rho}{s}d(v_{n-1},v_n,a) \tag{3.4}$$

since $\rho\in[0,1)$, we have $\frac{\rho}{s}<\frac{1}{s}$, by Lemma 2.6, we get the conclusion that $\{v_n\}$ is a Cauchy sequence, so there exists *z* in *X*, such that $fv_n=v_{n+1}\to z, n\to\infty$.

Since $v_n\to z$ and $fv_n\to z$, that is $d(v_n,fv_n,a)\to 0$ and by the continuity of *d*, we have $d(v_n,x,a)\to d(x,z,a)\ne 0, n\to\infty$, for every $x\ne z$, so there exists $n_0\in N$ such that $\phi(\rho)d(v_n,fv_n,a)<d(v_n,x,a)$, for each $n\ge n_0$, now for such above *n* and from the assumption (3.2) we get

$$d(fv_n,fx,a)\le\frac{\rho}{s}\max\{d(v_n,x,a),d(v_n,v_{n+1},a),d(x,fx,a)\},\ \text{for}\ x\ne z \tag{3.5}$$

taking $n\to\infty$ we have

$$d(fx,z,a)\le\frac{\rho}{s}\max\{d(x,z,a),d(x,fx,a)\} \tag{3.6}$$

In (3.3), take $x=f^{n-1}z$, we have

$$d(f^n z,f^{n+1}z,a)\le\frac{\rho}{s}d(f^{n-1}z,f^n z,a),\ \text{for}\ n\in N \tag{3.7}$$

by induction, we have

$$d\left(f^n z, f^{n+1} z, a\right) \le \frac{\rho^n}{s^n} d\left(z, fz, a\right) \tag{3.8}$$

Now we claim that

$$d\left(f^n z, z, a\right) \le d\left(fz, z, a\right), \text{ for every } n \in N \tag{3.9}$$

this inequality is true for $n=1$, assume (3.9) holds for some $n \in N$, if $f^n z = z$, then we have $f^{n+1} z = fz$ and

$$d\left(f^{n+1} z, z, a\right) = d\left(fz, z, a\right) \le d\left(fz, z, a\right) \tag{3.9.1}$$

if $f^n z \ne z$, then we can obtain the following inequality from (3.6), and that is:

$$d\left(f^{n+1} z, z, a\right) \le \frac{\rho}{s} \max\left\{d\left(f^n x, z, a\right), d\left(f^n x, f^{n+1} x, a\right)\right\} \tag{3.9.2}$$

By the induction hypothesis (3.9) for some $n \in N$ and (3.8), we have

$$\begin{aligned} d\left(f^{n+1} z, z, a\right) &\le \frac{\rho}{s} \max\left\{d\left(fx, z, a\right), \frac{\rho}{s} d\left(fx, z, a\right)\right\} \\ &= \frac{\rho}{s} d\left(fz, z, a\right) \le d\left(fz, z, a\right) \end{aligned} \tag{3.9.3}$$

Therefore, (3.9) is true for every $n \in N$.

Now we assume that $fz \ne z$ and consider the two following possible cases to prove that $fz = z$.

Case 1. Take $0 \le \rho < b_s$, therefore $\phi(\rho) \le \frac{1-\rho}{\rho^2}$. Firstly we claim that

$$d\left(f^n z, fz, a\right) \le \frac{\rho}{s} d\left(fz, z, a\right), \text{ for all } n \in N \tag{3.10}$$

It is obvious for $n=1$ and this follows from (3.8) for $n=2$.

From (3.9) we have $d\left(z, f^n z, fz\right) \le d\left(fz, z, fz\right) = 0$, that is,

$$d\left(z, f^n z, fz\right) = 0 \tag{3.11}$$

Now assume that (3.10) holds for some $n \ge 2$, then from part 4 of Definition 2.1 and (3.11) we have

$$\begin{aligned} d\left(z, fz, a\right) &\le s\left(d\left(z, f^n z, a\right) + d\left(f^n z, fz, a\right) + d\left(z, f^n z, fz\right)\right) \\ &\le s\left(d\left(z, f^n z, a\right) + d\left(f^n z, fz, a\right)\right) \\ &\le s\left(d\left(z, f^n z, a\right) + \frac{\rho}{s} d\left(fz, z, a\right)\right) \end{aligned} \tag{3.10.1}$$

and that is $d\left(z, fz, a\right) \le \frac{s}{1-\rho} d\left(z, f^n z, a\right)$, using (3.8), it follows that

$$\begin{aligned} &\phi(\rho) d\left(f^n z, f^{n+1} z, a\right) \\ &\le \frac{1-\rho}{\rho^2} d\left(f^n z, f^{n+1} z, a\right) \le \frac{1-\rho}{\rho^2} d\left(f^n z, f^{n+1} z, a\right) \\ &\le \frac{1-\rho}{\rho^n} \frac{\rho^n}{s^n} d\left(z, fz, a\right) \le \frac{1-\rho}{s^n} d\left(z, fz, a\right) \\ &\le \frac{1}{s^{n-1}} d\left(z, f^n z, a\right) \le d\left(f^n z, z, a\right) \end{aligned} \tag{3.10.2}$$

from (3.2)

$$\begin{aligned} d\left(f^{n+1}z, fz, a\right) &\le \frac{\rho}{s}\max\left\{d\left(f^{n}z, z, a\right), d\left(f^{n}z, f^{n+1}z, a\right), d\left(z, fz, a\right)\right\} \\ &\le \frac{\rho}{s} d\left(z, fz, a\right) \end{aligned} \tag{3.10.3}$$

By induction with using (3.8) and (3.9), it is easy for us to get the relation (3.10).

Now from $fz \ne z$ and (3.10), we get for each $n \in N$ $f^{n}z \ne z$, therefore, (3.6) and (3.8) show that

$$\begin{aligned} d\left(f^{n+1}z, fz, a\right) &\le \frac{\rho}{s}\max\left\{d\left(f^{n}z, z, a\right), d\left(f^{n}z, f^{n+1}z, a\right)\right\} \\ &\le \frac{\rho}{s}\max\left\{d\left(f^{n}z, z, a\right), \frac{\rho^{n}}{s^{n}} d\left(z, fz, a\right)\right\} \end{aligned} \tag{3.12}$$

From part 4 of Definition 2.1 and (3.11), we get

$$\begin{aligned} d\left(f^{n}x, z, a\right) &\le s\left(d\left(fz, f^{n}z, a\right) + d\left(f^{n}z, z, a\right) + d\left(fz, z, f^{n}z\right)\right) \\ &\le s\left(d\left(fz, f^{n}z, a\right) + d\left(f^{n}z, z, a\right)\right) \end{aligned} \tag{3.12.1}$$

It follows from (3.10) that

$$\begin{aligned} d\left(f^{n}z, z, a\right) &\ge \frac{1}{s} d\left(fz, z, a\right) - d\left(fz, f^{n}z, a\right) \\ &\ge \frac{1}{s} d\left(fz, z, a\right) - \frac{\rho}{s} d\left(fz, z, a\right) \ge \frac{1-\rho}{s} d\left(fz, z, a\right) \end{aligned} \tag{3.12.2}$$

There exists $n_1 \in N$, for $n \ge n_1$ and $0 \le \rho < b_s$ such that $1-\rho \ge \rho^{n}$, for such n, we get

$$d\left(f^{n}z, z, a\right) \ge \frac{\rho^{n}}{s} d\left(fz, z, a\right) \ge \frac{\rho^{n}}{s^{n}} d\left(fz, z, a\right) \tag{3.12.3}$$

Then taking $n \to \infty$ from (3.12) we have

$$d\left(f^{n+1}z, z, a\right) \le \frac{\rho}{s} d\left(f^{n}z, z, a\right) \le \cdots \le \left(\frac{\rho}{s}\right)^{n-n_1+1} d\left(f^{n_1 z}, z, a\right) \to 0 \tag{3.12.4}$$

That is, $f^{n}z \to z$, and from (3.10), we get

$$\lim_{n\to\infty} d\left(fz, z, a\right) \le \frac{\rho}{s} \lim_{n\to\infty} d\left(fz, z, a\right) \tag{3.12.5}$$

which is impossible except $fz = z$.

Case 2. Take $b_s \le \rho < 1$ and that is when $\phi(\rho) = \frac{1}{s+\rho}$, we will prove that we can find a subsequence $\left\{v_{n_i}\right\}$ of $\left\{v_n\right\}$ such that for each $i \in N$,

$$\phi(\rho) d\left(v_{n_i}, fv_{n_i}, a\right) = \phi(\rho) d\left(v_{n_i}, v_{n_i+1}, a\right) \le d\left(v_{n_i}, z, a\right), \tag{3.13}$$

we know for each $n \in N$ $d\left(v_n, v_{n+1}, a\right) \le \frac{\rho}{s} d\left(v_{n-1}, v_n, a\right)$ from (3.4), assume that for some $n \in N$

$$\frac{1}{s+\rho}d(v_n,v_{n-1},a)>d(v_{n-1},z,a), \tag{3.13.1}$$

and

$$\frac{1}{s+\rho}d(v_n,v_{n+1},a)>d(v_n,z,a) \tag{3.13.2}$$

then

$$\begin{aligned} d(v_{n-1},v_n,a) &\le s\left(d(v_{n-1},z,a)+d(v_n,z,a)+d(v_{n-1},v_n,z)\right) \\ &< \frac{s}{s+\rho}\left(d(v_{n-1},v_n,a)+d(v_n,v_{n+1},a)\right)+sd(v_n,v_{n-1},z) \end{aligned} \tag{3.13.3}$$

taking $n\to\infty$, we get a relation which is impossible. Therefore we have

$$\phi(\rho)d(v_n,v_{n-1},a)\le d(v_{n-1},z,a) \quad \text{or} \quad \phi(\rho)d(v_n,v_{n-1},a)\le d(v_{n-1},z,a)$$

for each $n\in N$. (3.13.4)

In other words, there is a subsequence $\{v_{n_i}\}$ for $\{v_n\}$ such that (3.13) is true for every $i\in N$, but from (3.2) we have

$$d(fv_{n_i},fz,a)\le\frac{\rho}{s}\max\left\{d(v_{n_i},z,a),d(v_{n_i},fv_{n_i},a),d(z,fz,a)\right\} \tag{3.13.5}$$

Taking $i\to\infty$, we have

$$d(z,fz,a)\le\frac{\rho}{s}d(z,fz,a) \tag{3.13.6}$$

which is possible only if $fz=z$.

Therefore, z is a fixed point of f. Let w be another fixed point of f, from (3.6), we have

$$d(w,z,a)=d(fw,z,a)\le\frac{\rho}{s}\max\left\{d(w,z,a),d(w,fw,z)\right\}=\frac{\rho}{s}d(w,z,a) \tag{3.14}$$

which is a contraction unless $d(w,z,a)=0$, and that is $w=z$, f has a unique common fixed point $z\in X$.

Corollary Let (X,d) be a complete b_2-metric space and d is continuous in every variable. Let $f:X\to X$ be a selfmap and $\phi:[0,1)\to(1/(s+1),1]$ be defined by (3.1). If there exists $\rho\in[0,1)$ such that for each x, y of X,

$$\phi(\rho)d(x,fx,a)\le d(x,y,a)\Rightarrow d(fx,fy,a)\le\frac{\rho}{s}d(x,y,a) \tag{3.15}$$

then f has a unique fixed point z in X and the sequence $\{f^n x\}$ converges to z, for each $x\in X$.

4. Conclusion

A known existence theorems of common fixed points for two maps was proved for the generalized Suzuki-type contractions in b_2-metric space. The results generalized and improved the field of fixed point theory for metric spaces and perfected the realization of the fixed point theory in this generalized space.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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