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Modified Legendre Collocation Block Method for Solving Initial Value Problems of First Order Ordinary Differential Equations

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Abstract

In this paper, block procedure for some k-step linear multi-step methods, using the Legendre polynomials as the basis functions, is proposed. Discrete methods were given which were used in block and implemented for solving the initial value problems, being continuous interpolant derived and collocated at grid points. Some numerical examples of ordinary differential equations were solved using the derived methods to show their validity and the accuracy. The numerical results obtained show that the proposed method can also be efficient in solving such problems.

Subject Areas

Mathematical Analysis

Keywords

Block, Legendre Polynomials, Zero-Stable, Convergent

1. Introduction

Many problems in celestial and quantum mechanics, nuclear, theoretical physics, astrophysics, quantum chemistry and molecular dynamics, are of great interest to scientists and engineers. These problems are mathematically modelled by using ordinary differential equation of the form:

$$f\left(x, y, y', y'', \cdots, y^{(n)}\right), y(a) = y_0, y'(a) = y_1, \cdots, y^{(n-1)}(a) = y_n \qquad (1)$$

where on the interval $[a,b]$ has given rise to two major discrete variable methods namely, one step and multistep methods commonly known as linear multi-step methods. Many authors have worked on the direct solution of (1), among which are Lambert [1], Fatunla [2], Sarafyan [3], Awoyemi [4] and Kayode [5]. Each of them worked on the development of several methods for solving Equation (1) directly without having to reduce to system of first order differential equations. For instance, in Awoyemi [4], methods were developed to solve second order initial value problems which are the mathematical formulation for systems without dissipation. Fatunla [2] considered a step-by-step method based on the classical Runge-Kutta method; Hairer and Wanner [6] developed Nystrom type method for initial value problem for first order differential equations in which the conditions for the determination of the parameters of the methods were mentioned. Also, Henrici [7] and Lambert [1] improved the derivation of linear multi-step methods with constant coefficients for solving first order equation with initial conditions.

In Awoyemi [8], linear multi-step methods with continuous coefficient for initial value problem of the first order differential equations in the predictor-corrector mode were proposed, based on collocation method with power series polynomial as basis function, and Taylor series algorithm to supply starting values. Continuous linear multi-step method is useful in reducing the step number of a method and still remains zero-stable; it has greater advantage in the sense that better error estimates guaranteed easy approximation of solution to all points of integration interval. Moreover, Awoyemi (1995) adopted the hybrid methods and proposed a two-step hybrid multi-step method with continuous coefficients for the solution of a first order initial value problem based on the collocation at selected grid points, using off-grid points to improve the order of the method implemented on the predictor-corrector mode. Other researchers who have studied hybrid method include Adee and Onumanyi [9], and Yahaya and Badmus [10].

Furthermore, many researchers had developed interest on improving the numerical solution of initial value problems of ordinary differential equation. Consequently, the development of a class of methods called block method is one of the outcomes. This was proposed by Milne [11], and it was found that it generates approximations continuously at different grid points in the interval of integration; it is less expensive in terms of the number of function evaluations compared to the linear multi-step methods. Chu and Hamilton [12] also proposed a generalization of the linear multi-step method to a class of multi-block methods where step values are obtained all together in a single block. Jator (2007) and Jator *et al.* (2005) proposed five-step and four-step self starting methods which adopt continuous linear multi-step method to obtain finite difference method applied respectively as a block for the direct solution of the first order initial value problem. Also, in Yahaya and Mohammed (2010), Chebyshev polynomial was considered as trial function. Ajileye *et al.* adopted Laguerre collocation approach for continuous hybrid block method. Other scholars that adopted block methods include Omar and Suleiman [13] [14] [15] and Areo and

Adeniyi [16]. Abualnaja (2015) developed a block procedure with linear multi-steps using Legendre polynomials but did not include the block schemes. Thus, in this paper, Legendre polynomial is used as a basis function to derive some block methods for the solution of first order ordinary differential equation, which extends the work of Abualnaja (2005).

2. Derivation of the Method

In this section, we consider the approximate solution of the form

$$y_k(x) = \sum_{i=0}^{k} c_i \psi_i(x).$$

Perturbing the equation above, we have

$$\sum_{i=0}^{k} c_i \psi_i(x) = f(x, y) + \lambda L_k(x) \tag{2}$$

where, $\psi_i(x) = x^i, i = 0, 1, \cdots, k$ and $L_k(x)$ is the Legendre polynomial of degree k, which is defined on the interval $[-1, 1]$, and can be determined with the aid of the recurrence formula:

$$L_{i+1}(x) = \frac{2i+1}{i+1} x L_i(x) - \frac{i}{i+1} L_{i-1}(x), i = 1, 2, \cdots \tag{3}$$

So that

$$L_0(x) = 1, L_1(x) = x, L_2(x) = \frac{3x^2 - 1}{2}, L_3(x) = \frac{5x^3 - 3x}{2}, L_4(x) = \frac{35x^4 - 30x^2 + 3}{2}$$

We define a shifted Legendre polynomials by introducing the change of variable

$$x = \frac{2\bar{x} - (x_{n+k} + x_n)}{x_{n+k} - x_n}, k = 1, 2, 3, 4 \tag{4}$$

For $k = 1$

using Equation (4), taking $L_1(x) = x$, and collocating at x_n and x_{n+1}, we obtain

$$L_1(x_n) = -1, L_1(x_{n+1}) = 1 \tag{5}$$

hence,

$$L_1(x_n) = \frac{2x_n - x_{n+1} - x_n}{x_{n+1} - x_n} = -1 \tag{6}$$

and,

$$L_1(x_{n+1}) = \frac{2x_{n+1} - x_{n+1} - x_n}{x_{n+1} - x_n} = 1 \tag{7}$$

Deducing $\psi_0(x) = 0, \psi_1(x) = 1$ from Equation (1), it follows that Equation (2) becomes

$$f(x, y) = c_1 - \lambda L_1(x) \tag{8}$$

Solving the above systems we obtain

$$\lambda = \frac{1}{2}(f_n - f_{n+1}), c_1 = f_n - \lambda, c_0 = y_n - x_n(f_n - \lambda)$$

The required numerical scheme of the method will be obtained if we collocate $y(x)=c_0+c_1x$ at $x=x_{n+1}$ and substitute c_0,c_1,λ as follows

$$y_{n+1}=y_n+\frac{h}{2}\left(f_{n+2}+f_n\right) \tag{9}$$

***k* = 2**

taking $L_2(x)=\frac{1}{2}\left(3x^2-1\right)$, and collocating at x_n, x_{n+1}, and x_{n+2}

we get

$$L_2(x_n)=1, L_2(x_{n+1})=\frac{-1}{2}, L_2(x_{n+2})=1$$

From Equation (1), it can be deduced that $\psi_0(x)=0,\psi_1(x)=1,\psi_2(x)=2x$, then Equation (2) becomes

$$f(x,y)=c_1+2xc_2-\lambda L_2(x) \tag{10}$$

Collocating Equation (10) at $x_{n+i}, i=0,1,2$ and interpolate

$$y_k(x)=\sum_{i=0}^{k}c_i\psi_i(x), x_n\le x\le x_{n+k} \tag{11}$$

at $x=x_n$, we get a system of four equations with $c_i\left(i=0,1,2\right)$ and parameter λ

$$\begin{aligned} y_n &= c_0+c_1x_n+c_2x_n^2 \\ f_n &= c_1+2c_2x_n-\lambda \\ f_{n+1} &= c_1+2c_2x_{n+1}-\frac{1}{2}\lambda \\ f_n &= c_1+2c_2x_{n+2}-\lambda \end{aligned} \tag{12}$$

Hence, solving Equation (12), we get

$$\begin{aligned} \lambda &= \frac{1}{3}\left(-f_n+2f_{n+1}-f_{n+2}\right) \\ c_0 &= \frac{-1}{12h}\left(-12hy_n+8t_nhf_{n+1}-4t_nhf_{n+2}+8t_nhf_n-3t_n^2f_{n+2}+3t_n^2f_n\right) \\ c_1 &= \frac{1}{6h}\left(4hf_{n+1}-2hf_{n+2}+4hf_n-3t_nf_{n+2}+3t_nf_n\right) \\ c_2 &= \frac{-1}{4h}\left(f_n-f_{n+2}\right) \end{aligned} \tag{13}$$

From $y_k(x)=\sum_{i=0}^{k}c_i\psi_i(x)$, we have

$$\bar{y}=c_0+c_1x+c_2x^2 \tag{14}$$

Hence, the required numerical scheme is obtained by collocating Equation (14) above at $x=x_{n+1}$ and substituting c_0,c_1,c_2,λ as follows

$$y_{n+1}=y_n+\frac{h}{12}\left(5f_n+8f_{n+1}-f_{n+2}\right) \tag{15}$$

***k* = 3**

Taking the polynomial $L_3=\frac{1}{2}\left(5x^3-3x\right)$ and use the Equation (4), then collocating this at x_n,x_{n+1},x_{n+2} and x_{n+3}, we obtain

$L_3(x_n) = -1$, $L_3(x_{n+1}) = \frac{11}{27}$, $L_3(x_{n+2}) = \frac{-11}{27}$, $L_3(x_{n+3}) = 1$. From Equation (1), it can be deduced that $\psi_0(x) = 0, \psi_1(x) = 1, \psi_2(x) = 2x, \psi_3(x) = 3x^2$, then Equation (2) is reduced to the form

$$f(x, y) = c_1 + 2xc_2 + 3c_3x^2 - \lambda L_3(x) \tag{16}$$

Hence, collocating Equation (16) at $x_{n+i}, i = 0,1,2,3$ and interpolate Equation (11) at $x = x_n$, we get the system of equations with $c_i, (i = 0,1,2,3)$ and parameter λ

$$\begin{aligned} y_n &= c_0 + c_1x_n + c_2x_n^2 + c_3x_n^3 \\ f_n &= c_1 + 2c_2x_n + 3c_3x_n^2 + \lambda \\ f_{n+1} &= c_1 + 2c_2x_{n+2} + 3c_3x_{n+1}^2 - \frac{11}{27}\lambda \\ f_{n+2} &= c_1 + 2c_2x_{n+2} + 3c_3x_{n+2}^2 + \frac{11}{27}\lambda \\ f_{n+3} &= c_1 + 2c_2x_{n+3} + 3c_3x_{n+3}^2 - \lambda \end{aligned} \tag{17}$$

Solving the above system of equations, we obtain

$$\lambda = \frac{9}{40}(f_n - 3f_{n+1} + 3f_{n+2} - f_{n+3})$$

$$\begin{aligned} c_0 = -\frac{1}{120}\big(&-81t_nh^2f_{n+2} + 27t_nf_{n+3}h^2 + 56t_n^2h^2f_n + 81t_nh^2f_{n+1} + 93t_nh^2f_n - 18t_n^2hf_{n+1} \\ &+ 34t_n^2hf_{n+3} - 72t_n^2hf_{n+2} - 120t_n^2hf_{n+2} - 10t_n^3f_{n+2} + 10t_n^3hf_{n+3} - 10t_n^3f_{n+1}\big) \end{aligned}$$

$$\begin{aligned} c_1 = \frac{1}{120}\big(&68t_nhf_{n+3} + 112t_nf_nh - 36t_n^2hf_{n+1} - 144t_nhf_{n+2} + 81h^2f_{n+1} + 27h^2f_{n+3} \\ &+ 93h^2hf_n - 81h^2f_{n+2} - 30t_n^2f_{n+2} + 30t_n^2f_{n+3} + 30t_n^2f_n - 30t_n^2f_{n+1}\big) \end{aligned}$$

$$c_2 = -\frac{1}{60h^2}\left(-15t_nf_{n+2} + 15t_nf_n - 15t_nf_{n+1} - 36hf_{n+2} + 28hf_n + 17hf_{n+3} - 9hf_{n+1}\right)$$

$$c_3 = -\frac{1}{12h^2}\left(f_n - f_{n+1} - f_{n+2} + f_{n+3}\right)$$

From $y_k(x) = \sum_{i=0}^{k} c_i\psi_i(x)$, we have

$$\bar{y} = c_0 + c_1x + c_2x^2 + c_3x^3 \tag{18}$$

Hence, the required numerical scheme is obtained by collocating Equation (18) above at $x = x_{n+1}$ and substituting $c_0, c_1, c_2, c_3, \lambda$ as follows

$$y_{n+1} = y_n + \frac{h}{120}\left(47f_n + 89f_{n+1} - 19f_{n+2} + 3f_{n+3}\right) \tag{19}$$

***k* = 4**

In this case, we take the polynomial $L_4 = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$ and use the Equation (4), then collocating this at $x_n, x_{n+1}, x_{n+2}, x_{n+3}$ and x_{n+4}, we obtain $L_4(x_n) = 1$, $L_4(x_{n+1}) = \frac{-37}{128}$, $L_4(x_{n+3}) = \frac{3}{8}$, $L_4(x_{n+4}) = 1$. From Equation (1), we can deduce that $\psi_0(x)=0, \psi_1(x)=1, \psi_2(x)=2x, \psi_3(x)=3x^2, \psi_4(x)=4x^3$, then Equation (2) is reduced to the form

$$f(x,y)=c_1+2xc_2+3c_3x^2+4c_4x^3-\lambda L_4(x) \tag{20}$$

Hence, collocating Equation (20) at $x_{n+i}, i=0,1,2,3,4$ and interpolate Equation (11) at $x=x_n$, we get the system of equations with $c_i,(i=0,1,2,3,4)$ and parameter λ

$$\begin{aligned}
&y_n=c_0+c_1x_n+c_2x_n^2+c_3x_n^3+c_4x_n^4\\
&f_n=c_1+2c_2x_n+3c_3x_n^2+4c_4x_n^3-\lambda\\
&f_{n+1}=c_1+2c_2x_{n+1}+3c_3x_{n+1}^2+4c_4x_{n+1}^3+\frac{37}{128}\lambda\\
&f_{n+2}=c_1+2c_2x_{n+2}+3c_3x_{n+2}^2+4c_4x_{n+2}^3-\frac{3}{8}\lambda\\
&f_{n+3}=c_1+2c_2x_{n+3}+3c_3x_{n+3}^2+4c_4x_{n+3}^3+\frac{37}{128}\lambda\\
&f_{n+4}=c_1+2c_2x_{n+4}+3c_3x_{n+4}^2+4c_4x_{n+3}^3-\lambda
\end{aligned} \tag{21}$$

Solving the above system of equations with a suitable method, we obtain

$$\lambda=\frac{16}{105}\left(-f_n+4f_{n+1}-6f_{n+2}+4f_{n+3}-f_{n+4}\right)$$

$$\begin{aligned}
c_0=-\frac{1}{5040h^3}\big(&768x_nh^3f_{n+4}-4272x_nf_nh^3-3072x_nh^3f_{n+3}\\
&-3072x_nh^3f_{n+1}+4608x_nh^3f_{n+2}+2400x_n{}^2h^2f_{n+1}-3330x_n^2h^2f_n\\
&+1290x_n^2h^2f_{n+4}-4320x_n^2h^2f_{n+3}+1520x_n^3hf_{n+1}-1840x_n^3hf_{n+3}\\
&+670x_n^3hf_{n+4}-1010x_n^3hf_n+660x_n^3hf_{n+2}+3960x_n^2h^2f_{n+2}\\
&+5040y_nh^3+210x_n^4f_{n+1}-210x_n^4f_{n+3}+105x_n^4f_{n+4}-105x_n^4f_n\big)
\end{aligned}$$

$$\begin{aligned}
c_1=\frac{1}{840h^3}\big(&-1320x_nh^2f_{n+2}-330x_n^2f_{n+2}h+505x_n^2f_nh-335x_n^2f_{n+4}h\\
&+920x_n^2f_{n+3}h+760x_n^2hf_{n+1}+1440x_nh^2f_{n+3}-430x_nh^2f_{n+4}+1110x_nh^2f_n\\
&+800h^2x_nf_{n+1}-128h^3hf_{n+4}-768h^3hf_{n+2}+512h^3f_{n+1}+512h^3f_{n+3}\\
&+712h^3f_n-140x_n^3f_{n+1}+140x_n^3f_{n+3}-70x_n^3f_{n+4}+70x_n^3f_n\big)
\end{aligned}$$

$$\begin{aligned}
c_2=-\frac{1}{168h^3}\big(&-66x_nhf_{n+2}+101x_nhf_n-67x_nhf_{n+4}+184x_nhf_{n+3}-152x_nhf_{n+1}\\
&+43h^2f_{n+4}+111f_nh^2-42x_n^2f_{n+1}+42x_n^2f_{n+1}+42x_n^2f_{n+3}\\
&-21x_n^2f_{n+4}+21x_n^2f_n-80h^2f_{n+1}+144h^2f_{n+3}-132h^2f_{n+3}\big)
\end{aligned}$$

$$\begin{aligned}
c_3=-\frac{1}{504h^3}\big(&-84x_nf_{n+1}+84x_nf_{n+3}-42x_nf_{n+4}+42x_nf_n\\
&-152hf_{n+1}+184hf_{n+3}-67hf_{n+4}+101hf_n-66hf_{n+2}\big)
\end{aligned}$$

$$c_4=-\frac{1}{48h^3}\left(f_n-2f_{n+1}+2f_{n+3}-f_{n+4}\right)$$

From $y_k(x)=\sum_{i=0}^{k}c_i\psi_i(x)$, we have

$$\bar{y} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \quad (22)$$

Hence, the required numerical scheme is obtained by collocating Equation (22) above at $x = x_{n+1}$ and substituting $c_0, c_1, c_2, c_3, c_4, \lambda$ as follows

$$y_{n+1} = y_n + \frac{h}{5040}\left(1847 f_n + 4162 f_{n+1} - 1308 f_{n+2} + 382 f_{n+3} - 43 f_{n+4}\right) \quad (23)$$

Formulating the Block Scheme of Cases *k* = 2, 3 and 4

If *k* = 2

We collocate Equation (14) at $x = x_{n+1}, x_{n+2}, x_{n+3}$ to give us

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{12}\left(5 f_n + 8 f_{n+1} - f_{n+2}\right) \\ y_{n+2} &= y_n + \frac{h}{3}\left(f_n + 4 f_{n+1} + f_{n+2}\right) \\ y_{n+3} &= y_n + \frac{h}{4}\left(8 f_{n+1} - f_n + f_{n+2}\right) \end{aligned} \quad (24)$$

If *k* = 3

We collocate Equation (18) at $x = x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$ to give us

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{120}\left(47 f_n + 89 f_{n+1} - 19 f_{n+2} + 3 f_{n+3}\right) \\ y_{n+2} &= y_n + \frac{h}{60}\left(21 f_n + 77 f_{n+1} + 23 f_{n+2} - f_{n+3}\right) \\ y_{n+3} &= y_n + \frac{h}{8}\left(3 f_n + 9 f_{n+1} + 9 f_{n+2} + 3 f_{n+3}\right) \\ y_{n+4} &= y_n + \frac{h}{30}\left(29 f_n - 7 f_{n+1} + 47 f_{n+2} + 51 f_{n+3}\right) \end{aligned} \quad (25)$$

If *k* = 4

We collocate Equation (22) at $x = x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$ to give us

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{5040}\left(1847 f_n + 4162 f_{n+1} - 1308 f_{n+2} + 382 f_{n+3} - 43 f_{n+4}\right) \\ y_{n+2} &= y_n + \frac{h}{90}\left(29 f_n + 124 f_{n+1} + 24 f_{n+2} + 4 f_{n+3} - f_{n+4}\right) \\ y_{n+3} &= y_n + \frac{h}{560}\left(179 f_n + 377 f_{n+1} + 444 f_{n+2} + 334 f_{n+3} - 31 f_{n+4}\right) \\ y_{n+4} &= y_n + \frac{h}{45}\left(14 f_n + 64 f_{n+1} + 24 f_{n+2} + 64 f_{n+3} + 14 f_{n+4}\right) \\ y_{n+5} &= y_n + \frac{h}{1008}\left(-253 f_n + 3322 f_{n+1} - 1308 f_{n+2} + 1222 f_{n+3} + 2057 f_{n+4}\right) \end{aligned} \quad (26)$$

3. Basic Properties of the Method

3.1. Order, Error Constant and Consistency of the Methods

The schemes developed belong to the class of Linear Multi-step Method (LMM) of the form

$$\sum_{j=0}^{k} \alpha_j(x) y(x_{n+j}) = h \sum_{j=0}^{k} \beta_j(x) f(x_{n+j}) \quad (27)$$

Equation (27) is a method associated with a linear difference operator

$$L\left[y(x);h\right]=\sum_{j=0}^{k}\left(\alpha_j y(x+jh)=h\beta_j y'(x+jh)\right) \tag{28}$$

where $y(x)$ is continuously differentiable on the interval $[a,b]$, and the Taylor series expansion about the point x is expressed as

$$L\left[y(x);h\right]=c_0 y(x)+c_1 h y'(x)+c_2 h^2 y''(x)+\cdots+c_q h^q y^q(x) \tag{29}$$

In line with [12], schemes (15, 19, 23) are said to be of order P if $C_0=C_1=C_2=\cdots=C_p=0$ and the error constant is $C_{p+1}\neq 0$. Hence, we establish that (15),(19), and (23) is of the following orders respectively

when $k=2$, $P=3$ and $C_{p+1}=0.041667$

when $k=3$, $P=3$ and $C_{p+1}=0.016$

when $k=4$, $P=2$ and $C_{p+1}=0.041667$

3.2. Stability Analysis

The scheme can be expressed as:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n-2}\\y_{n-1}\\y_n\end{bmatrix}+h\begin{bmatrix}\frac{8}{12}&-\frac{1}{12}&0\\\frac{4}{3}&\frac{1}{3}&0\\2&\frac{5}{4}&0\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix}$$

$$+\begin{bmatrix}0&0&\frac{5}{12}\\0&0&\frac{1}{3}\\0&0&-\frac{1}{4}\end{bmatrix}\begin{bmatrix}f_{n-2}\\f_{n-1}\\f_n\end{bmatrix}$$

where,

$$A^{(0)}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},A^{(1)}=\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix},B^{(0)}=\begin{bmatrix}\frac{8}{12}&-\frac{1}{12}&0\\\frac{4}{3}&\frac{1}{3}&0\\2&\frac{5}{4}&0\end{bmatrix}$$

and

$$B^{(1)}=\begin{bmatrix}0&0&\frac{5}{12}\\0&0&\frac{1}{3}\\0&0&-\frac{1}{4}\end{bmatrix}$$

The first characteristics polynomial of the scheme is

$$\rho(\lambda)=det\left[\lambda A^0-A^1\right]$$

$$\rho(\lambda)=det\left[\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}-\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}\right]$$

$$=det\begin{bmatrix}\lambda & 0 & -1\\ 0 & \lambda & -1\\ 0 & 0 & \lambda-1\end{bmatrix}$$

$$\begin{vmatrix}\lambda & 0 & -1\\ 0 & \lambda & -1\\ 0 & 0 & \lambda-1\end{vmatrix}=0$$

$$\lambda^2(\lambda-1)=0$$

$$\lambda_1=\lambda_2 \quad \text{or} \quad \lambda_3=1$$

3.3. Zero-Stability for *k* = 3

A block method is said to be stable as $h \to 0$ if the roots of the first characteristics polynomial defined by

$$\rho\lambda=det\left[\lambda A^0-A^1\right]$$

satisfies $|r_s|=1$

The scheme can be expressed as

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\end{bmatrix}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-3}\\y_{n-2}\\y_{n-1}\\y_n\end{bmatrix}$$

$$+h\begin{bmatrix}\frac{89}{120} & -\frac{19}{120} & \frac{1}{40} & 0\\ \frac{124}{90} & \frac{4}{15} & \frac{2}{45} & -\frac{1}{90}\\ \frac{77}{60} & \frac{23}{60} & \frac{-1}{60} & 0\\ \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0\\ -\frac{7}{30} & \frac{47}{30} & \frac{17}{10} & 0\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\end{bmatrix}+\begin{bmatrix}0&0&0&\frac{47}{120}\\0&0&0&\frac{7}{20}\\0&0&0&\frac{3}{8}\\0&0&0&\frac{29}{30}\end{bmatrix}\begin{bmatrix}f_{n-3}\\f_{n-2}\\f_{n-1}\\f_n\end{bmatrix}$$

where,

$$A^{(0)}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},A^{(1)}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix},B^{(0)}=\begin{bmatrix}\frac{89}{120} & -\frac{19}{120} & \frac{1}{40} & 0\\ \frac{77}{60} & \frac{23}{60} & \frac{-1}{60} & 0\\ \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0\\ -\frac{7}{30} & \frac{47}{30} & \frac{17}{10} & 0\end{bmatrix}$$

and

$$B^{(1)} = \begin{bmatrix} 0 & 0 & 0 & \frac{47}{120} \\ 0 & 0 & 0 & \frac{7}{20} \\ 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{29}{30} \end{bmatrix}$$

The first characteristics polynomial of the scheme is

$$\rho(\lambda) = det\left[\lambda A^0 - A^1\right]$$

$$\rho(\lambda) = det\left[\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] = det\begin{bmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda-1 & 0 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix}$$

$$\begin{vmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda-1 \end{vmatrix} = 0$$

$$\lambda^3(\lambda-1) = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 0 \quad \text{or} \quad \lambda_4 = 1$$

3.4. Zero-Stability for *k* = 4

A block method is said to be stable as $h \to 0$ if the roots of the first characteristics polynomial defined by

$$\rho\lambda = det\left[\lambda A^0 - A^1\right]$$

satisfies $|r_s| = 1$

The scheme can be expressed as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}$$

$$+h\begin{bmatrix} \frac{56150}{144} & -\frac{84084}{144} & \frac{56042}{144} & -\frac{14009}{144} & 0 \\ \frac{124}{90} & \frac{4}{15} & \frac{2}{45} & -\frac{1}{90} & 0 \\ \frac{754}{560} & \frac{444}{560} & \frac{334}{560} & -\frac{31}{560} & 0 \\ \frac{64}{45} & \frac{8}{9} & \frac{64}{45} & \frac{14}{45} & 0 \\ \frac{3322}{1008} & -\frac{1308}{1008} & \frac{1222}{1008} & \frac{2057}{1008} & 0 \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{13955}{144} \\ 0 & 0 & 0 & 0 & \frac{29}{90} \\ 0 & 0 & 0 & 0 & \frac{179}{560} \\ 0 & 0 & 0 & 0 & \frac{14}{45} \\ 0 & 0 & 0 & 0 & -\frac{253}{1008} \end{bmatrix} \begin{bmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

where,

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B^{(0)} = \begin{bmatrix} \frac{56150}{144} & -\frac{84084}{144} & \frac{56042}{144} & -\frac{14009}{144} & 0 \\ \frac{124}{90} & \frac{4}{15} & \frac{2}{45} & -\frac{1}{90} & 0 \\ \frac{754}{560} & \frac{444}{560} & \frac{334}{560} & -\frac{31}{560} & 0 \\ \frac{64}{45} & \frac{8}{9} & \frac{64}{45} & \frac{14}{45} & 0 \\ \frac{3322}{1008} & -\frac{1308}{1008} & \frac{1222}{1008} & \frac{2057}{1008} & 0 \end{bmatrix}$$

and

$$B^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{13955}{144} \\ 0 & 0 & 0 & 0 & \frac{29}{90} \\ 0 & 0 & 0 & 0 & \frac{179}{560} \\ 0 & 0 & 0 & 0 & \frac{14}{45} \\ 0 & 0 & 0 & 0 & -\frac{253}{1008} \end{bmatrix}$$

The first characteristics polynomial of the scheme is

$$\rho(\lambda) = det\left[\lambda A^0 - A^1\right]$$

$$\rho(\lambda) = det\left[\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] = det\begin{bmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda-1 & 0 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix}$$

$$\begin{vmatrix} \lambda & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & \lambda-1 \end{vmatrix} = 0$$

$$\lambda^4(\lambda-1)=0$$

$$\lambda_1=\lambda_2=\lambda_3=\lambda_4=0 \quad \text{or} \quad \lambda_5=1$$

4. Numerical Experiments

In order to confirm the accuracy and efficiency of the scheme, we consider the following initial value problems: Tables 1-4.

Table 1. Results and errors of problem 1 for $K = 3$.

N	x	YC	YEX	|YEX − YC|
0	0.000000	1.000000	1.000000	0.000000
1	0.100000	0.904874	0.904837	3.7×10^{-5}
2	0.200000	0.818800	0.818730	7.0×10^{-5}
3	0.300000	0.740889	0.740818	7.1×10^{-5}
4	0.400000	0.670418	0.670320	9.8×10^{-5}
5	0.500000	0.606653	0.606531	1.22×10^{-4}
6	0.600000	0.548956	0.548811	1.45×10^{-4}
7	0.700000	0.496749	0.496585	1.64×10^{-4}
8	0.800000	0.449511	0.449328	1.83×10^{-4}
9	0.900000	0.406768	0.406569	1.99×10^{-4}
10	1.000000	0.368093	0.367879	2.14×10^{-4}

Table 2. Results and errors of problem 1 for $K = 4$.

N	x	YC	YEX	|YEX − YC|
0	0.000000	1.000000	1.000000	0.000000
1	0.100000	0.905695	0.904837	8.58×10^{-4}
2	0.200000	0.820365	0.818730	1.635×10^{-3}
3	0.300000	0.743155	0.740818	2.337×10^{-3}
4	0.400000	0.673292	0.670320	2.972×10^{-3}
5	0.500000	0.610078	0.606531	3.547×10^{-3}
6	0.600000	0.552879	0.548811	4.068×10^{-3}
7	0.700000	0.501124	0.496585	4.539×10^{-3}
8	0.800000	0.463649	0.449328	1.4321×10^{-2}
9	0.900000	0.429739	0.406569	2.3170×10^{-2}
10	1.000000	0.399057	0.367879	3.1178×10^{-2}

Table 3. Results and errors of problem 2 for $K = 3$.

N	x	YC	YEX	|YEX − YC|
0	0.000000	0.000000	0.000000	0.000000
1	0.100000	0.004993	0.004988	5.0×10^{-6}
2	0.200000	0.019806	0.019801	5.0×10^{-6}
3	0.300000	0.044003	0.044003	0.00000
4	0.400000	0.076890	0.076884	6.0×10^{-6}
5	0.500000	0.117509	0.117503	6.0×10^{-6}
6	0.600000	0.164732	0.164729	3.0×10^{-6}
7	0.700000	0.217303	0.217295	8.0×10^{-6}
8	0.800000	0.273858	0.273851	7.0×10^{-6}
9	0.900000	0.333026	0.333023	3.0×10^{-6}
10	1.000000	0.393477	0.393469	8.0×10^{-6}

Table 4. Results and errors of problem 2 for $K = 4$.

N	x	YC	YEX	|YEX − YC|
0	0.000000	0.000000	0.000000	0.000000
1	0.100000	0.004434	0.004988	5.5×10^{-4}
2	0.200000	0.019809	0.019801	8.0×10^{-6}
3	0.300000	0.044009	0.044003	6.0×10^{-6}
4	0.400000	0.076891	0.076884	7.0×10^{-6}
5	0.500000	0.129887	0.117503	1.2×10^{-2}
6	0.600000	0.168705	0.164729	3.9×10^{-3}
7	0.700000	0.216646	0.217295	6.5×10^{-4}
8	0.800000	0.272938	0.273851	9.1×10^{-4}
9	0.900000	0.344677	0.333023	1.1×10^{-2}
10	1.000000	0.395906	0.393469	2.4×10^{-3}

Problem 1:

$$y' = -y(x), h = 0.1, y(0) = 1 \tag{30}$$

Exact solution: $y(x) = \mathrm{e}^{-x}$ (see K.M. Abualnaja, 2015).

YC: approximate solution

YEX: exact solution

Problem 2:

$$y'(x) = -x(1-y), h = 0.1, y(0) = 0 \tag{31}$$

Exact solution: $y(x) = 1 - \mathrm{e}^{-\frac{x^2}{2}}$ (see K.M. Abualnaja, 2015).

YC: approximate solution

YEX: exact solution

5. Conclusion

In this research work, a class of implicit block collocation methods for the direct solution of initial value problems of general first order ordinary differential equations was developed using Legendre collocation approach. The collocation technique yielded a consistent and zero stable implicit block multi-step method with continuous coefficients. The method is implemented without the need for the development of correctors.

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