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Unique Common Fixed Point in b_2 Metric Spaces

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Abstract

We establish some common fixed and common coincidence point theorems for expansive type mappings in the setting of b_2 metric space. Our results extend some known results in metric spaces to b_2 metric space. The research is meaningful and I recommend it to be published when the followings have been improved.

Subject Areas

Mathematical Analysis

Keywords

b_2 Metric Space, Common Fixed Point, Coincidence Point

1. Introduction

The author in (see [1] [2] [3]) discuss coincidence and fixed point existence problems relating to expansive mappings in cone metric spaces (see [4] [5]), and also gives fixed point theories for expanding mappings. The author in (see [6]) gets the coincidence and common fixed point theories in 2 metric spaces (see [7] [8] [9]), using the method in (see [1] [2] [3]). In this paper, a known existence theorems of common fixed points for two mappings satisfying expansive conditions in b_2 metric space (see [10]), which is the generalization of both 2 metric space and b metric space (see [11] [12]).

2. Preliminaries

Before stating our main results, some necessary definitions might be introduced as follows.

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Definition 2.1. [11] [12] *Let* X *be a nonempty set and* $s \geq 1$ *be a given real number. A function* $d: X \times X \to \mathbb{R}^+$ *is a* b *metric on* X *if for all* $x, y, z \in X$, *the following conditions hold*:

1) $d(x, y) = 0$ if and only if $x = y$.

2) $d(x, y) = d(y, x)$.

3) $d(x, y) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b metric space.

Definition 2.2. [7] [8] [9] *Let* X *be an nonempty set and let* $d: X \times X \times X \to \mathbb{R}$ *be a map satisfying the following conditions*:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

2)If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$.

3) The symmetry:

$$d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$$

for all $x, y, z \in X$.

4) The rectangle inequality: $d(x, y, z) \leq d(x, y, a) + d(y, z, a) + d(z, x, a)$ for all $x, y, z, a \in X$.

Then d is called a 2 metric on X and (X, d) is called a 2 metric space.

Definition 2.3. [10] *Let* X *be a nonempty set,* $s \geq 1$ *be a real number and let* $d: X \times X \times X \to \mathbb{R}$ *be a map satisfying the following conditions*:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

2) If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$.

3) The symmetry:

$$d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$$

for all $x, y, z \in X$.

4) The rectangle inequality: $d(x, y, z) \leq s[d(x, y, a) + d(y, z, a) + d(z, x, a)]$ for all $x, y, z, a \in X$.

Then d is called a b_2 metric on X and (X, d) is called a b_2 metric space with parameter *s*. Obviously, for $s = 1$, b_2 metric reduces to 2 metric.

Definition 2.4. [10] *Let* $\{x_n\}$ *be a sequence in a* b_2 *metric space* (X, d).

1) A sequence $\{x_n\}$ is said to be b_2-convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if all $a \in X$, $\lim_{n\to\infty} d(x_n, x, a) = 0$.

2) $\{x_n\}$ is Cauchy sequence if and only if $d(x_n, x_m, a) \to 0$, when $n, m \to \infty$. for all $a \in X$.

3) (X, d) is said to be b_2-complete if every b_2-Cauchy sequence is a b_2-convergent sequence.

Definition 2.5. [10] *Let* (X, d) *and* (X', d') *be two* b_2 *metric spaces and let* $f: X \to X'$ *be a mapping. Then* f *is said to be* b_2*-continuous at a point* $z \in X$ *if for a given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $x \in X$ *and* $d(z, x, a) < \delta$ *for all* $a \in X$ *imply that* $d'(fz, fx, a) < \varepsilon$. *The mapping* f *is* b_2*-continuous on* X *if it is* b_2*-continuous at all* $z \in X$.

Definition 2.6. [10] *Let* (X,d) *and* (X',d') *be two* b_2 *metric spaces. Then a mapping* $f:X\to X'$ *is* b_2*-continuous at a point* $x\in X'$ *if and only if it is* b_2*-sequentially continuous at x; that is, whenever* $\{x_n\}$ *is* b_2*-convergent to* x, $\{fx_n\}$ *is* b_2*-convergent to* $f(x)$.

Definition 2.7. [13] *Let* f *and* g *be self maps of a set* X. *If* $w=fx=gx$ *for some* x *in* X, *then* x *is called a coincidence point of* f *and* g, *and* w *is called a point of coincidence of* f *and* g. f *and* g *be weakly compatible means if* $x\in X$ *and* $fx=gx$, *then* $fgx=gfx$.

Proposition 2.8. [13] *Let* f *and* g *be weakly compatible self maps of a set* X. *If* f *and* g *have a unique point of coincidence* $w=fx=gx$, *then* w *is the unique common fixed point of* f *and* g.

3. Main Results

Theorem 3.1. *Let* (X,d) *be a* b_2 *metric space. Suppose mappings* $f,g:X\to X$ *are onto and satisfy*

$$d(fx,fy,a)\ge \alpha d(gx,fx,a)+\beta d(gy,fy,a)+\gamma d(gx,gy,a) \tag{1}$$

for all $x,y,a\in X$ and $x\ne y$, where $\alpha,\beta\in\mathbb{R},\gamma>0$. Suppose the following hypotheses:

1) fX or gX is complete,

2) $\alpha+\beta+\gamma>2$,

3) $gX\subset fX$.

Then f and g have a coincidence point.

Proof. From 2), we get $\alpha+\gamma>0$ or $\beta+\gamma>0$. Indeed, if we suppose $\alpha+\gamma\le 0$ and $\beta+\gamma\le 0$, we have $\alpha+\beta+2\gamma\le 0$. Since $\gamma\ge 0$, we have $\alpha+\beta+\gamma\le 0$. That is a contradiction.

Let $x_0\in X$, since $gX\subset fX$, we take $x_1\in X$ such that $fx_1=gx_0$. Again, we can take $x_2\in X$ such that $fx_2=gx_1$. Continuing in the same way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n=fx_{n+1}=gx_n$ for all $n\in\mathbb{N}$.

If $gx_{m-1}=gx_m$ for some $m\in\mathbb{N}$, then $fx_m=gx_m$. Thus x_m is a coincidence point of f and g.

Now, assume that $x_{n-1}\ne x_n$ for all $n\in\mathbb{N}$.

Step 1: It is shown that $\lim_{n\to\infty} d(y_{n+1},y_{n+2},a)=0$.

Suppose $\beta+\gamma>0$, take $x=x_{n+1}$, $y=x_{n+2}$ into (1). we have

$$d(y_n,y_{n+1},a)\ge \alpha d(y_n,y_{n+1},a)+(\beta+\gamma)d(y_{n+1},y_{n+2},a) \tag{2}$$

Then

$$(1+\alpha)d(y_n,y_{n+1},a)\ge(\beta+\gamma)d(y_{n+1},y_{n+2},a),\ \text{for all } n\in\mathbb{N} \tag{3}$$

Since $\beta+\gamma>0$, $(1-\alpha)d(y_n,y_{n+1},a)\ge 0$. If $1-\alpha<0$, then $d(y_n,y_{n+1},a)=0$. If $1-\alpha=0$, then $d(y_{n+1},y_{n+2},a)=0$. Therefore $\{y_n\}$ is constant sequence when $1\le\alpha$. Suppose $1-\alpha>0$, then $0<\dfrac{1-\alpha}{\beta+\gamma}<1$ and

$$d\left(y_{n+1}, y_{n+2}, a\right) \leq \frac{1-\alpha}{\beta+\gamma} d\left(y_n, y_{n+1}, a\right) \tag{4}$$

Suppose $\alpha+\gamma>0$, take $x=x_{n+2}, y=x_{n+1}$ into (1). We have

$$d\left(y_n, y_{n+1}, a\right) \geq (\alpha+\gamma) d\left(y_{n+1}, y_{n+2}, a\right)+\beta d\left(y_n, y_{n+1}, a\right) \tag{5}$$

Then

$$(1-\beta) d\left(y_n, y_{n+1}, a\right) \geq (\alpha+\gamma) d\left(y_{n+1}, y_{n+2}, a\right), \text{ for all } n \in \mathbb{N} \tag{6}$$

Similarly, since $\alpha+\gamma>0$, suppose $1-\beta>0$, then $0<\frac{1-\beta}{\alpha+\gamma}<1$ and

$$d\left(y_{n+1}, y_{n+2}, a\right) \leq \frac{1-\beta}{\alpha+\gamma} d\left(y_n, y_{n+1}, a\right) \tag{7}$$

Let $h=\max \left\{\frac{1-\alpha}{\beta+\gamma}, \frac{1-\beta}{\alpha+\gamma}\right\}$, we know $0<h<1$, applying (4) and (7), we get

$$d\left(y_{n+1}, y_{n+2}, a\right) \leq h d\left(y_n, y_{n+1}, a\right) \leq \cdots \leq h^{n+1} d\left(y_0, y_1, a\right), n=0,1,2, \cdots \tag{8}$$

then $\lim_{n \to \infty} d\left(y_{n+1}, y_{n+2}, a\right)=0$.

Step 2: As $\left\{d\left(y_n, y_{n+1}, a\right)\right\}$ is decreasing, if $d\left(y_{n-1}, y_n, a\right)=0$, then $d\left(y_n, y_{n+1}, a\right)=0$. Since part 2 of Definition 2.3, $d\left(y_0, y_1, y_0\right)=0$, we have $d\left(y_n, y_{n+1}, y_0\right)=0$ for all $n \in \mathbb{N}$.

Since $d\left(y_{m-1}, y_m, y_m\right)=0$, we have

$$d\left(y_n, y_{n+1}, y_m\right)=0 \tag{9}$$

for all $n \geq m-1$. For $0 \leq n<m-1$, we have $m-1 \geq n+1$, and from(9) we have

$$d\left(y_{m-1}, y_m, y_{n+1}\right)=d\left(y_{m-1}, y_m, y_n\right)=0 \tag{10}$$

It implies that

$$\begin{aligned} & d\left(y_n, y_{n+1}, y_m\right) \\ & \leq s d\left(y_n, y_{n+1}, y_{m-1}\right)+s d\left(y_{n+1}, y_m, y_{m-1}\right)+s d\left(y_m, y_n, y_{m-1}\right) \\ & =s d\left(y_n, y_{n+1}, y_{m-1}\right) \end{aligned} \tag{11}$$

Since $d\left(y_n, y_{n+1}, y_{n+1}\right)=0$, from the above inequality, we have

$$d\left(y_n, y_{n+1}, y_m\right) \leq s^{m-n-1} d\left(y_n, y_{n+1}, y_{n+1}\right)=0 \tag{12}$$

for all $0 \leq n \leq m-1$. From (9) and (12), we have

$$d\left(y_n, y_{n+1}, y_m\right)=0 \tag{13}$$

for all $n, m \in \mathbb{N}$. Now, for all $i, j, k \in \mathbb{N}$ with $i<j$, we have

$$d\left(y_{j-1}, y_j, y_i\right)=d\left(y_{j-1}, y_j, y_k\right)=0 \tag{14}$$

From (14) and triangular inequality, Therefore

$$\begin{aligned} d\left(y_i, y_j, y_k\right) & \leq s\left[d\left(y_i, y_j, y_{j-1}\right)+d\left(y_j, y_k, y_{j-1}\right)+d\left(y_k, y_i, y_{j-1}\right)\right] \\ & =s d\left(y_i, y_{j-1}, y_k\right) \leq \cdots \leq s^{j-i} d\left(y_i, y_i, y_k\right)=0 \end{aligned} \tag{15}$$

This proves that for all $i, j, k \in \mathbb{N}$,

$$d\left(y_i, y_j, y_k\right)=0 \tag{16}$$

Step 3: It is proved that the sequence $\{y_n\}$ is a b_2-Cauchy sequence. Let $m, n \in \mathbb{N}$ with $m > n$. We claim that, there exists $n_0 \in \mathbb{N}$, such that

$$d\left(y_n, y_m, a\right)<\varepsilon \tag{17}$$

for all $m > n \geq n_0$, $a \in X$. This is done by induction on m.

Let $n \geq n_0$ and $m = n+1$. Then we get

$$d\left(y_n, y_m, a\right)=d\left(y_n, y_{n+1}, a\right)<d\left(y_{n-1}, y_n, a\right)<\varepsilon \tag{18}$$

Then (17) holds for $m = n+1$.

Assume now that (17) holds for some $m \geq n+1$. We will show that (17) holds for $m+1$. Take $x = x_n$, $y = x_{m+1}$

$$\begin{aligned} d\left(y_{n-1}, y_m, a\right) &= d\left(fx_n, fx_{m+1}, a\right) \\ &\geq \alpha d\left(gx_n, fx_n, a\right)+\beta d\left(gx_{m+1}, fx_{m+1}, a\right)+\gamma d\left(gx_n, gx_{m+1}, a\right) \\ &\geq \alpha d\left(y_n, y_{n-1}, a\right)+\beta d\left(y_{m+1}, y_m, a\right)+\gamma d\left(y_n, y_{m+1}, a\right) \end{aligned} \tag{19}$$

Then

$$\begin{aligned} &s\left[d\left(y_{n-1}, y_n, y_m\right)+d\left(y_{n-1}, y_n, a\right)+d\left(y_n, y_m, a\right)\right] \\ &\geq \alpha d\left(y_n, y_{n-1}, a\right)+\beta d\left(y_{m+1}, y_m, a\right)+\gamma d\left(y_n, y_{m+1}, a\right) \end{aligned} \tag{20}$$

We get

$$\begin{aligned} &\gamma d\left(y_n, y_{m+1}, a\right) \\ &\leq s\left[d\left(y_{n-1}, y_n, y_m\right)+d\left(y_{n-1}, y_n, a\right)+d\left(y_n, y_m, a\right)\right] \\ &\quad -\alpha d\left(y_n, y_{n-1}, a\right)-\beta d\left(y_{m+1}, y_m, a\right) \\ &\leq 2s\varepsilon-(\alpha+\beta)\varepsilon \end{aligned} \tag{21}$$

Then

$$d\left(y_n, y_{m+1}, a\right) \leq \frac{2s\varepsilon-(\alpha+\beta)\varepsilon}{\gamma}=\frac{2s-\alpha-\beta}{\gamma}\varepsilon<\varepsilon \tag{22}$$

Thus we have proved that (17) holds for $m+1$. From (17), we know $\{y_n\}$ is a Cauchy sequence in (X, d).

If fX is complete, there exists $u \in fX$ and $p \in X$ such that $y_n = gx_n = fx_{n+1} \to u = fp$.

If $\beta+\gamma > 0$, let $x = x_{n+1}$, $y = p$ into (1), We have

$$\begin{aligned} d\left(y_n, u, a\right) &\geq \alpha d\left(y_n, y_{n+1}, a\right)+\beta d\left(gp, u, a\right)+\gamma d\left(y_{n+1}, gp, a\right) \\ &\geq \alpha d\left(y_n, y_{n+1}, a\right)+\beta d\left(gp, u, a\right) \\ &\quad +\gamma\left[\frac{d\left(u, gp, a\right)}{s}-d\left(y_{n+1}, u, a\right)-d\left(y_{n+1}, u, gp\right)\right] \end{aligned} \tag{23}$$

Therefore

$$\begin{aligned} \left(\beta+\frac{\gamma}{s}\right) d\left(u, gp, a\right) &\leq d\left(y_n, u, a\right)+|\alpha| d\left(y_n, y_{n+1}, a\right) \\ &\quad +\gamma d\left(y_{n+1}, u, a\right)+\gamma d\left(y_{n+1}, u, gp\right) \end{aligned} \tag{24}$$

We take a natural number n_1 such that

$$d(y_n, u, a) \le \frac{\beta + \frac{\gamma}{s}}{4}\varepsilon, \quad |\alpha| d(y_n, y_{n+1}, a) \le \frac{\beta + \frac{\gamma}{s}}{4}\varepsilon,$$

$$d(y_{n+1}, u, a) \le \frac{\beta + \frac{\gamma}{s}}{4}\varepsilon, \quad d(y_{n+1}, u, gp) \le \frac{\beta + \frac{\gamma}{s}}{4}\varepsilon$$

for $n \ge n_1$. Thus, we obtain $d(u, gp, a) \le \varepsilon$. Therefore $fp = u = gp$.

If $\alpha + \gamma > 0$, let $x = p$, $y = x_{n+1}$ into (1), We get

$$d(y_n, u, a) \ge \alpha d(gp, u, a) + \beta d(y_n, y_{n+1}, a) + \gamma d(y_{n+1}, gp, a) \tag{25}$$

Therefore

$$\left(\alpha + \frac{\gamma}{s}\right) d(u, gp, a) \le d(y_n, u, a) + |\beta| d(y_n, y_{n+1}, a) + \gamma d(y_{n+1}, u, a) + \gamma d(y_{n+1}, u, gp) \tag{26}$$

We take a natural number n_2 such that

$$d(y_n, u, a) \le \frac{\alpha + \frac{\gamma}{s}}{4}\varepsilon, \quad |\beta| d(y_n, y_{n+1}, a) \le \frac{\alpha + \frac{\gamma}{s}}{4}\varepsilon,$$

$$d(y_{n+1}, u, a) \le \frac{\alpha + \frac{\gamma}{s}}{4}\varepsilon, \quad d(y_{n+1}, u, gp) \le \frac{\alpha + \frac{\gamma}{s}}{4}\varepsilon$$

for $n \ge n_2$. Thus, we obtain $d(u, gp, a) \le \varepsilon$. Therefore $fp = u = gp$.

In short, no matter what the situation is, u is always the point of coincidence of f and g, p is the coincidence point of f and g.

If gX is complete, there exists $u \in gX \subset fX$ and $p, q \in X$, such that $y_n = gx_n \to u = gq = fp$. The rest proof is the same as that fX is complete.

Theorem 3.2. *Let* (X, d) *be a* b_2 *metric space. Let* f, g *be mappings satisfying* $fX \supset gX$ *and* (1), *for all* $\alpha, \beta \in \mathbb{R}, \gamma > 1$. *If* 1). fX *or* gX *is complete*, 2). $\alpha + \beta + \gamma > 2$, 3). f *and* g *is weakly compatible. Then* f *and* g *have a common fixed point.*

Proof. According to Theorem 3.1, there exists $u, p \in X$ such that $u = fp = gp$. Suppose there also exists $v, z \in X$ such that $v = fz = gz$, choose $x = p$, $y = z$ into (1), we get

$$d(u, v, a) = d(fp, fz, a) \ge \gamma d(gp, gz, a) = \gamma d(u, v, a) \tag{27}$$

Therefore, there exists $d(u, v, a) = 0$, then $u = v$. f and g have the point of coincidence u. According to Proposition 2.8, u is the unique common fixed point of f and g.

Corollary 3.3. *Let* (X, d) *be a complete* b_2 *metric space. Let* f *be surjective mapping satisfying* $d(fx, fy, a) \ge \alpha d(x, fx, a) + \beta d(y, fy, a) + \gamma d(x, y, a)$, *for all* $x, y, a \in X$, $x \ne y$, *where with* $\alpha, \beta \in \mathbb{R}$, $\gamma > 0$ *and* $\alpha + \beta + \gamma > 2$, *then* f *has a fixed point, if* $\gamma > 0$, *then f has a unique fixed point.*

Proof. Follows from Theorem 3.1, by taking $g = 1_x$, identify map, then we get the result.

4. Conclusion

In this paper, a known existence theorems of common fixed points for two mappings satisfying expansive conditions in b_2 metric space were generalized and improved. Based on the research, a new method to discuss the existence problems of common fixed points for mappings with this type expansive condition was taken out. And the results show that the proposed method is better than the former ones.

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