Open Access Library Journal
2017, Volume 4, e3716
ISSN Online: 2333-9721
ISSN Print: 2333-9705

On Certain Subclass of Analytic Functions Based on Convolution of Ruscheweyh and Generalized Salagean Differential Operator

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How to cite this paper: Terwase, A.P. (2017) On Certain Subclass of Analytic Functions Based on Convolution of Ruscheweyh and Generalized Salagean Differential Operator. *Open Access Library Journal*, **4**: e3716.
https://doi.org/10.4236/oalib.1103716

Received: June 7, 2017
Accepted: September 12, 2017
Published: September 15, 2017

Abstract

In this paper, we obtain certain properties of an operator defined by the convolution between Ruscheweyh and Salagean differential operator; coefficient inequality, extreme point growth and distortion among other properties are investigated.

Subject Areas

Geometry, Mathematical Analysis

Keywords

Analytic Function, Surbodination, Hadamard Product, Linear Combination

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let S be the subclass of $\mathcal{A}$ consisting of analytic univalent function of the form (1.1).

The study of normalised analytic univalent functions is enhanced by the used of operators, mostly, differential and integral operators. In this study, we have implored the used of convulation of well known differential operators to defined our class. For more works on operators see[1] [2] [3].

Definition 1

Let T denotes the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \left(a_k \geq 0\right) \tag{2}$$

Further we define the class $T_n(\mu, \beta, \gamma\zeta)$ by

$$T_n(\mu, \beta, \gamma\zeta) = B_\lambda^n(\mu, \beta, \gamma, \zeta) \cap T \tag{3}$$

Definition 2 ([4])

For $f \in \mathcal{A}$ and f of the form (1.1) $\lambda \geq 0$ and $n \in \mathbb{N}$, the operator D_λ^n is defined by

$$D_\lambda^n : \mathcal{A} \to \mathcal{A}$$

$$D_\lambda^0 f(z) = f(z)$$

$$D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z)$$

$$= D_\lambda^n\left(D_\lambda^{n-1} f(z)\right), \quad \text{for } z \in U$$

i.e.

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} \left[1 + (k-1)\lambda\right]^n a_k z^k, \ z \in U \tag{4}$$

Definition 2 [3] Let $f \in \mathcal{A}$ $n \in N$, the operator R^n is defined by

$$R^n : \mathcal{A} \to \mathcal{A}$$

$$R^0 f(z) = f(z)$$

$$R' f(z) = z f'(z)$$

$$\cdots$$

$$(n+1) R^{n+1} f(z) = z\left(R^n f(z)\right) + n R^n f(z), \, z \in U$$

Thus it is obvious to see from above that

$$R^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k \tag{5}$$

where

$$\delta(n,k) = \binom{n+k-1}{n} = \frac{(n+1)_{k-1}}{(1)_{k-1}}.$$

Thus by convolution as earlier defined by [5] we have

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} \left[1 + (k-1)\lambda\right]^n \delta(n,k) a_k z^k \tag{6}$$

We now defined a class $B_\lambda^n(\mu, \beta, \gamma, \zeta)$, which consist of functions $f \in S$ such that the following inequality is satisfy

$$\left| \frac{\left(D_\lambda^n f(z)\right) - 1}{\beta\left(D_\lambda^n f(z)\right) + 1 - \lambda\mu} \right| < \zeta, 0 \leq \gamma \leq 1, 0 < \zeta < 1, 0 \leq \mu < 1$$

Motivated here by the works of [1] [6], we characterize our class using well know existing geometric properties.

2. Properties of the Class $B_\lambda^n(\mu,\beta,\zeta,\gamma)$

2.1. Coefficient Inequality

Theorem 2.1.

let $f \in S$. Then $f \in B_\lambda^n(\mu,\beta,\zeta)$ if and only if

$$\sum_{k=2}^{\infty} k(1+\zeta\beta)\left[1+(k-1)\lambda\right]^n \delta(n,k) a_k \le \zeta\left(\beta+(1-\gamma\mu)\right) \tag{7}$$

where

$$\delta(n,k) = \binom{n+k-1}{n} = \frac{(n+1)_{k-1}}{(1)_{k-1}}.$$

Proof:

Supposed the inequality (7) holds true and $|z|=1$, then we have

$$\begin{aligned}
&\left|\left(D_\lambda^n f(z)\right)-1\right| - \zeta\left|\beta\left(D_\lambda^n f(z)\right)+(1-\gamma\mu)\right| \\
&= \left|\sum_{k=2}^{\infty} k\left[1+(k-1)\lambda\right]^n \delta(n,k) a_k z^{k-1}\right| \\
&\quad -\zeta\left|\beta-\beta\sum_{k=2}^{\infty} k\left[1+(k-1)\lambda\right]^n \delta(n,k) a_k z^{k-1}+(1-\gamma\mu)\right| \\
&\le \sum_{k=2}^{\infty} k(1+\zeta\beta)\left[1+(k-1)\lambda\right]^n \delta(n,k) a_k - \zeta\left(\beta+(1-\gamma\mu)\right) \le 0
\end{aligned}$$

But by maximun modullus principle, $f \in B_\lambda^n(\mu,\beta,\zeta,\gamma)$ establishing our desired result.

Conversely,

Let $f \in B_\lambda^n(\mu,\beta,\zeta,\gamma)$, then

$$\frac{\left|\left(D_\lambda^n f(z)\right)-1\right|}{\left|\beta\left(D_\lambda^n f(z)\right)+1-\lambda\mu\right|} < \varsigma,\ z \in \mathcal{U} \tag{8}$$

Then

$$\frac{\left|\sum_{k=2}^{\infty} k\left[1+(k-1)\right]^n \delta(n,k) a_k z^{k-1}\right|}{\left|\sum_{k=2}^{\infty} \beta\left(D^n f(z)\right)+\gamma\mu\right|} < \zeta$$

Recall that $\left|\Re f(z)\right| \le \left|f(z)\right|$, thus we have

$$\left|\Re\left\{\frac{\sum_{k=2}^{\infty} k\left[1+(k-1)\right]^n \delta(n,k) a_k z^{k-1}}{\sum_{k=2}^{\infty} \beta k\left[1+(k-1)\right]^n \delta(n,k) a_k + (1-\lambda\mu)}\right\}\right| \le \zeta \tag{9}$$

Choose z on the real axis and let $z \to 1^-$. Then we have

$$\frac{\sum_{k=2}^{\infty} k\left[1+(k-1)\right]^n \delta(n,k) a_k}{\sum_{k=2}^{\infty} \beta k\left[1+(k-1)\right]^n \delta(n,k) a_k + (1-\lambda\mu)} < \zeta \tag{10}$$

This yields;

$$\leq \sum_{k=2}^{\infty} k(1+\zeta\beta)\left[1+(k-1)\lambda\right]^n \delta(n,k) a_k \leq \zeta\left(\beta+(1-\gamma\mu)\right) \quad (11)$$

This establishes our proof.

Corollary 2.1.

If $f \in B_\lambda^n(\mu,\beta,\zeta)$ then

$$a_k \leq \frac{\zeta\left(\beta+(1-\gamma\mu)\right)}{k(1+\zeta\beta)\left[1+(k-1)\lambda\right]^n \delta(n,k)}, \quad k=2,3,\cdots \quad (12)$$

equality is attained for

$$f(z) = z + \frac{\zeta\left(\beta+(1-\gamma\mu)\right)}{k(1+\zeta\beta)\left[1+(k-1)\lambda\right]^n \delta(n,k)} z^k, \quad k=2,3,\cdots \quad (13)$$

We shall state the growth and distortion theorems for the class $B_\lambda^n(\vartheta,\varsigma,\gamma,\iota)$ The results of which follow easily on applying Theorem 2.1, therefore, we deem it necessary to omit the trivial proofs.

2.2. Growth and Distortion Theorems

Theorem 2.2.

Let the function $f(z) \in B_\lambda^n(\mu,\beta,\zeta,\gamma)$ then for $|z| = r$

$$r - \frac{\zeta\left(\beta+(1-\gamma\mu)\right)}{2(1+\zeta\beta)[1+\lambda]^n \delta(n,2)} r^2 \leq |f(z)| \leq r + \frac{\zeta\left(\beta+(1-\gamma\mu)\right)}{2(1+\zeta\beta)[1+\lambda]^n \delta(n,2)} r^2$$

Theorem 2.3.

Let the function $f(z) \in B_\lambda^n(\mu,\beta,\zeta)$ then for $|z| = r$

$$1 - \frac{\zeta\left(\beta+(1-\gamma\mu)\right)}{(1+\zeta\beta)[1+\lambda]^n \delta(n,2)} r \leq |f'(z)| \leq 1 + \frac{\zeta\left(\beta+(1-\gamma\mu)\right)}{(1+\zeta\beta)[1+\lambda]^n \delta(n,2)} r$$

When $f(z) = z + \dfrac{\zeta\left(\beta+(1-\gamma\mu)\right)}{2(1+\zeta\beta)[1+\lambda]^n \delta(n,2)} z^2$

we obtain a sharp result.

2.3. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 2.4.

Let the function $f(z) \in B_\lambda^n(\mu,\beta,\zeta,\gamma)$, then f is close-to-convex of order δ in $|z| < R_\tau 1$

where

$$R_\tau 1 = \inf_{k \geq 2} \left[\frac{k(1+\zeta\beta)(1-\delta)\left[1+(k-1)\lambda\right]^n \delta(n,k)}{\zeta\left(\beta+(1-\gamma\mu)\right)}\right]^{\frac{1}{k-1}}$$

The result obtained is sharp.

Proof.

It is sufficient to show that $|f'(z)-1| \leq 1-\delta$ for $|z| < R_\tau$ Thus we can write

$$\left|f'(z)-1\right|=\left|-\sum_{n=2}^{\infty}ka_kz^{k-1}\right|\le\sum_{n=2}^{\infty}ka_k|z|^{k-1}$$

Therefore $\left|f'(z)-1\right|\le 1-\delta$ if

$$\sum_{n=2}^{\infty}\left(\frac{k}{1-\delta}\right)a_k|z|^{k-1}\le 1 \tag{14}$$

But we have from theorem 2.1. that

$$\sum_{k=2}^{\infty}\frac{k(1+\zeta\beta)\left[1+(k-1)\lambda\right]^n\delta(n,k)a_k}{\zeta\left(\beta+(1-\gamma\mu)\right)}\le 1 \tag{15}$$

Relating (14) and (15) we have our desired result.

Theorem.2.5.

Let the function $f(z)\in B_\lambda^n(\mu,\beta,\zeta)$, then f is starlike of order δ, $0\le\delta<1$ in $|z|<R_\tau 2$

where

$$R_\tau 2=\inf_{k\ge 2}\left[\frac{k(1+\zeta\beta)(1-\delta)\left[1+(k-1)\lambda\right]^n\delta(n,k)}{(k-\delta)\zeta\left(\beta+(1-\gamma\mu)\right)}\right]^{\frac{1}{k-1}}$$

The result obtain here is sharp.

Proof.

We must show that $\left|\frac{zf'(z)}{f(z)}-1\right|\le 1-\delta$ for $|z|<R_\tau 2$. Equivalently, we have

$$\sum_{k=2}^{\infty}\frac{(k-\delta)a_k\left|z^{k-1}\right|}{1-\delta}\le 1 \tag{16}$$

But we have from theorem 2.1. that

$$\sum_{k=2}^{\infty}\frac{k(1+\zeta\beta)\left[1+(k-1)\lambda\right]^n\delta(n,k)a_k}{\zeta\left(\beta+(1-\gamma\mu)\right)}\le 1 \tag{17}$$

Relating (16) and (17) will have our desired result.

Theorem 2.6.

Let the function $f(z)\in B_\lambda^n(\vartheta,\varsigma,\gamma,\iota)$, then f is convex of order δ, $0\le\delta<1$ in $|z|<R_\tau 3$

where

$$R_\tau 3=\inf_{k\ge 2}\left[\frac{(1+\zeta\beta)(1-\delta)\left[1+(k-1)\lambda\right]^n\delta(n,k)}{(k-\delta)\zeta\left(\beta+(1-\gamma\mu)\right)}\right]^{\frac{1}{k-1}}$$

The result obtain here is sharp.

Proof.

By using the technique of theorem 2.5 we easily show that $\left|\frac{zf''(z)}{f'(z)}\right|\le 1-\delta$ this holds for $|z|<R_\tau 3$. The analogous details of theorem 2.5 are thus omitted,

hence the proof.

3. Integral Operator

Theorem 3.1.

Let the function $f(z)$ defined by (2) be in the class $T_n(\mu,\beta,\gamma\zeta)$ and let c be a real number such that $c>-1$. Then the function defined by

$$F(z)=\frac{c+1}{z^c}\int t^{c-1}f(t)\mathrm{d}t \tag{18}$$

also belong to the class $T_n(\mu,\beta,\gamma,\zeta)$

Proof.

From the representation and definition of $F(z)$ we have that

$$F(z)=z-\sum_{k=2}^{\infty}b_k z^k \tag{19}$$

where

$$b_k=\left(\frac{c+1}{c+k}\right)a_k \tag{20}$$

Thus we have

$$\sum_{k=2}^{\infty}k(1+\zeta\beta)\left[1+(k-1)\lambda\right]^n\delta(n,k)b_k \tag{21}$$

$$=\sum_{k=2}^{\infty}k(1+\zeta\beta)\left[1+(k-1)\lambda\right]^n\delta(n,k)\left(\frac{c+1}{c+k}\right)a_k$$

$$\le\sum_{k=2}^{\infty}k(1+\zeta\beta)\left[1+(k-1)\lambda\right]^n\delta(n,k)a_k\le\zeta\left(\beta+(1-\gamma\mu)\right) \tag{22}$$

since $f(z)\in T_n(\mu,\beta,\gamma,\zeta)$. By theorem 1.1 $F(z)\in T_n(\mu,\beta,\gamma,\zeta)$. This establishes our proof.

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