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Sharp Upper Bounds for Multiplicative Degree Distance of Graph Operations

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Abstract

In this paper, multiplicative version of degree distance of a graph is defined and tight upper bounds of the graph operations have been found.

Subject Areas

Discrete Mathematics

Keywords

Join, Disjunction, Composition, Symmetric Difference, Multiplicative Degree Distance, Zagreb Indices and Coindices

1. Introduction

A topological index of a graph is a numerical quantity which is structural invariant, *i.e.* it is fixed under graph automorphism. The simplest topological indices are the number of vertices and edges of a graph. In this paper, we define and study a new topological index called multiplicative degree distance. All graphs considered are simple and connected graphs.

We denote the vertex and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. $d_G(v)$ denotes the degree of a vertex v in G. The number of elements in the vertex set of a graph G is called the order of G and is denoted by $v(G)$. The number of elements in the edge set of a graph G is called the size of G and is denoted by $e(G)$. A graph with order n and size m edges is called a (n,m)-graph. For any $u,v \in V(G)$, the distance between u and v in G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G. The edge connective the vertices u and v will be denoted by uv. The complement $\overline{G}$ of the graph G is the graph with vertex set $V(G)$, in which two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in *G*.

The join of graphs G_1 and G_2 is denoted by G_1+G_2, and it is the graph with vertex set $V(G_1)\cup V(G_2)$ and the edge set $E(G_1+G_2)=E(G_1)\cup E(G_2)\cup\{u_1u_2 \mid u_1\in V(G_1), u_2\in V(G_2)\}$. The composition of graphs G_1 and G_2 is denoted by $G_1[G_2]$, and it is the graph with vertex set $V(G_1)\times V(G_2)$, and two vertices $u=(u_1,u_2)$ and $v=(v_1,v_2)$ are adjacent if (u_1 is adjacent to v_1) or ($u_1=v_1$ and u_2 and v_2 are adjacent). The disjunction of graphs G_1 and G_2 is denoted by $G_1\vee G_2$, and it is the graph with vertex set $V(G_1)\times V(G_2)$ and $E(G_1\vee G_2)=\{(u_1,u_2)(v_1,v_2) \mid u_1v_1\in E(G_1) \text{ or } u_2v_2\in E(G_2)\}$. The symmetric difference of graphs G_1 and G_2 is denoted by $G_1\oplus G_2$, and it is the graph with vertex set $V(G_1)\times V(G_2)$ and edge set

$E(G_1\oplus G_2)=\{(u_1,u_2)(v_1,v_2)) \mid u_1v_1\in E(G_1) \text{ or } u_2v_2\in E(G_2) \text{ but not both}\}$.

Let G be a connected graph. The Wiener index $W(G)$ of a graph G is defined as

$$W(G)=\sum_{\{u,v\}\subseteq V(G)} d_G(u,v)=\frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v).$$

Dobrynin and Kochetova [1] and Gutman [2] independently proposed a vertex-degree-Weighted version of Wiener index called degree distance or Schultz molecular topological index, which is defined for a connected graph G as

$$DD(G)=\sum_{\{u,v\}\subseteq V(G)} d_G(u,v)\left[d_G(u)+d_G(v)\right]=\frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)\left[d_G(u)+d_G(v)\right].$$

The Zagreb indices have been introduced more than thirth years ago by Gutman and Trianjestic [3]. The first Zagreb index $M_1(G)$ of a graph G is defined as

$$M_1(G)=\sum_{uv\in E(G)}\left[d_G(u)+d_G(v)\right]=\sum_{v\in V(G)} d_G^2(v).$$

The second Zagreb index $M_2(G)$ of a graph G is defined as

$$M_2(G)=\sum_{uv\in E(G)} d_G(u)d_G(v).$$

The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [4].

Note that contribution of nonadjacent vertex pair should be taken into account when computing the Weighted Wiener Polynomials of certain Composite graphs, see [5]. A.R. Ashrafi, T. Doslic, A. Hamzeha, [6] [7] defined the first Zagreb coindex of a graph G is

$$\bar{M}_1(G)=\sum_{uv\notin E(G)}\left[d_G(u)+d_G(v)\right]$$

The second Zagreb coindex of a graph G is

$$\bar{M}_2(G)=\sum_{uv\notin E(G)} d_G(u)d_G(v),$$

respectively.

In [8], Hamzeh, Iranmanesh Hossein-Zadeh and M.V. Diudea recently introduced the generalized degree distance of graphs. Asma Hamzeh, Ali Iranmanesh and Samaneh Hossein-Zadeh, Cartesian product, composition, join, disjunction and symmetric difference of graphs and introduce generalized and modified generalized degree distance Polynomials of graphs, such that their first derivatives at $x=1$, see [9].

In this paper, we defne a new graph invariant named multiplicative version of degree distance of a graph denoted by $DD^*(G)$ and defined by

$$\left[DD^*(G)\right]^2 = \prod_{u,v\in V(G),u\neq v} d_G(u,v)\left[d_G(u)+d_G(v)\right].$$

Therefore the study of this new topological index is important and we have obtained Sharp upper bounds for the graph operations such as join, disjunction, composition, symmetric difference of graphs.

2. The Multiplicative Degree Distance of Graph Operations

Lemma 2.1. [10] [11], Let G_1 and G_2 be two simple connected graphs. The number of vertices and edges of graph G_i is denoted by n_i and e_i respectively for $i=1,2$. Then we have

1. $$d_{G_1+G_2}(u,v)=\begin{cases}1, & uv\in E(G_1) \text{ or } uv\in E(G_2) \text{ or } \left(u\in V(G_1) \text{ and } v\in V(G_2)\right)\\ 2, & \text{otherwise}\end{cases}$$

For a vertex u of G_1, $d_{G_1+G_2}(u)=d_{G_1}(u)+n_2$, and for a vertex v of G_2, $d_{G_1+G_2}(v)=d_{G_2}(v)+n_1$.

2. $$d_{G_1[G_2]}\left((u_1,v_1),(u_2,v_2)\right)=\begin{cases}d_{G_1}(u_1,u_2), & u_1\neq u_2\\ 1, & u_1=u_2, v_1v_2\in E(G_2)\\ 2, & \text{otherwise}\end{cases}$$

$$d_{G_1[G_2]}(u,v)=n_2 d_{G_1}(u)+d_{G_2}(v).$$

3. $$d_{G_1\vee G_2}\left((u_1,v_1),(u_2,v_2)\right)=\begin{cases}1, & u_1u_2\in E(G_1) \text{ or } v_1v_2\in E(G_2)\\ 2, & \text{otherwise}\end{cases}$$

$$d_{G_1\vee G_2}\left((u,v)\right)=n_2 d_{G_1}(u)+n_1 d_{G_2}(v)-d_{G_1}(u)d_{G_2}(v).$$

4. $$d_{G_1\oplus G_2}\left((u_1,v_1),(u_2,v_2)\right)=\begin{cases}1, & u_1u_2\in E(G_1) \text{ or } v_1v_2\in E(G_2) \text{ but not both}\\ 2, & \text{otherwise}\end{cases}$$

$$d_{G_1\oplus G_2}\left((u,v)\right)=n_2 d_{G_1}(u)+n_1 d_{G_2}(v)-2d_{G_1}(u)d_{G_2}(v).$$

Lemma 2.2. (Arithmetic Geometric inequality)

Let $x_1,x_2,\cdots,x_n$ be non-negative numbers. Then

$$\frac{x_1+x_2+\cdots+x_n}{n}\geq\sqrt[n]{x_1x_2\cdots x_n}$$

Remark 2.3. For a graph G, let $A(G) = \{(x,y)\in V(G)\times V(G) \mid x$ and y are adjacent in $G\}$ and let $B(G) = \{(x,y)\in V(G)\times V(G) \mid x$ and y are not adjacent in $G\}$. For each $x\in V(G),(x,x)\in B(G)$. Clearly,

$A(G)\cup B(G)=V(G)\times V(G)$. Let $C(G)=\{(x,x)\mid x\in V(G)\}$ and $D(G)=B(G)-C(G)$. Clearly $B(G)=C(G)\cup D(G)$, $C(G)\cap D(G)=\varnothing$. The summation $\sum_{(x,y)\in A(G)}$ runs over the ordered pairs of $A(G)$. For simplicity, we write the summation $\sum_{(x,y)\in A(G)}$ as $\sum_{xy\in G}$. Similarly, we write the summation $\sum_{(x,y)\in B(G)}$ as $\sum_{xy\notin G}$. Also the summation $\sum_{xy\in E(G)}$ runs over the edges of *G*. We denote the summation $\sum_{x,y\in V(G)}$ by $\sum_{x,y\in G}$ and similarly $\sum_{x,y\in V(G)}$ by $\prod_{x,y\in G}$. The summation $\sum_{(x,y)\in D(G)}$ eqivalent to $\sum_{xy\notin G,x\neq y}$ and similarly the summation $\sum_{(x,y)\in C(G)}$ eqivalent to $\sum_{xy\notin G,x=y}$.

Lemma 2.4. Let G be a graph. Then

$$\sum_{xy\in G}1=2e(G)$$

Proof:

$$\sum_{xy\in G}1=2\sum_{xy\in E(G)}1=2e(G)$$

Lemma 2.5.

$$\sum_{xy\in G}d_G(x)=M_1(G)$$

Proof: Let $x\in V(G)$ and $t=d_G(x)$. Let $y_1,y_2,\cdots,y_t$ be the neighbours of *x*. Each orderd pair $(x,y_i),1\le i\le t$, contributes $d_G(x)$ to the sum. Thus these orderd pairs contribute $d_G^2(x)$ to the sum. Hence

$$\sum_{xy\in G}d_G(x)=\sum_{x\in V(G)}d_G^2(x)=M_1(G)$$

Lemma 2.6.

$$\sum_{xy\in G}d_G(x)d_G(y)=2M_2(G)$$

Proof: Clearly,

$$\sum_{xy\in G}d_G(x)d_G(y)=2\sum_{xy\in E(G)}d_G(x)d_G(y)=2M_2(G).$$

Lemma 2.7.

$$\sum_{xy\notin G}1=2e(\bar{G})+v(G)$$

Proof:

$$\sum_{xy\notin G}1=\sum_{(x,y)\in D(G)}1+\sum_{(x,x)\in C(G)}1=2e(\bar{G})+v(G)$$

Lemma 2.8.

$$\sum_{xy\notin G}d_G(x)=2e(\bar{G})(v(G)-1)+2e(G)-M_1(\bar{G})$$

Proof.

$$\begin{aligned}\sum_{xy\notin G} d_G(x) &= \sum_{(x,y)\in D(G)} d_G(x) + \sum_{(x,x)\in C(G)} d_G(x)\\ &= \sum_{(x,y)\in D(G)} \left\{v(G)-1-d_{\bar{G}}(x)\right\} + \sum_{(x,x)\in C(G)} d_G(x)\\ &= \left(v(G)-1\right)\sum_{(x,y)\in D(G)} 1 - \sum_{(x,y)\in D(G)} d_{\bar{G}}(x) + 2e(G)\\ &= \left(v(G)-1\right)2e\left(\bar{G}\right) - \sum_{(x,y)\in A(\bar{G})} d_{\bar{G}}^2(x) + 2e(G)\\ &= \left(v(G)-1\right)2e\left(\bar{G}\right) - \sum_{xy\in \bar{G}} d_{\bar{G}}^2(x) + 2e(G)\\ &= 2e\left(\bar{G}\right)\left(v(G)-1\right) + 2e(G) - M_1\left(\bar{G}\right) \text{ by Lemma 2.5}\end{aligned}$$

Lemma 2.9.

$$\sum_{xy\notin G} d_G(x)d_G(y) = 2\bar{M}_2(G) + M_1(G)$$

Proof:

$$\begin{aligned}\sum_{xy\notin G} d_G(x)d_G(y) &= \sum_{(x,y)\in D(G)} d_G(x)d_G(y) + \sum_{(x,x)\in C(G)} d_G(x)d_G(x)\\ &= 2\sum_{xy\notin E(G)} d_G(x)d_G(y) + \sum_{x\in V(G)} d_G^2(x)\\ &= 2\bar{M}_2(G) + M_1(G)\end{aligned}$$

Lemma 2.10.

$$\sum_{xy\notin G}\left[d_G(x)+d_G(y)\right] = 2\bar{M}_1(G) + 4e(G)$$

Proof:

$$\begin{aligned}\sum_{xy\notin G}\left[d_G(x)+d_G(y)\right] &= \sum_{(x,y)\in C(G)}\left[d_G(x)+d_G(y)\right] + \sum_{(x,y)\in D(G)}\left[d_G(x)+d_G(y)\right]\\ &= \sum_{x\in V(G)} 2d_G(x) + 2\sum_{xy\notin E(G)}\left[d_G(x)+d_G(y)\right]\\ &= 4e(G) + 2\bar{M}_1(G)\end{aligned}$$

3. The Multiplicative Degree Distance of Composition of Graph

Theorem 3.1. Let $G_i, i=1,2$, be a (n_i, m_i)-graph. Then

$$\begin{aligned}&\left[DD^*\left(G_1[G_2]\right)\right]^2\\ &\le \Bigg\{\frac{1}{n_1n_2(n_1n_2-1)}\Big[4M_1(G_2)W(G_1) + 4n_2m_2DD(G_1) + 4n_1\bar{M}_1(G_2)\\ &+ 8n_2^2m_1(n_2-1) + 2n_1M_1(G_2) + 8m_1n_2m_2 + 4W(G_1)\bar{M}_1(G_2)\\ &+ 2n_2DD(G_1)(2\bar{m}_2+n_2)\Big]\Bigg\}^{n_1n_2(n_1n_2-1)}\end{aligned}$$

Proof:

$$\begin{aligned}
&\left[DD^*\left(G_1\left[G_2\right]\right)\right]^2 \\
&=\prod_{x,y\in G_1}\prod_{u,v\in G_2,x\neq u(\text{or})y\neq v} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right] \\
&\leq\left\{\frac{1}{n_1n_2\left(n_1n_2-1\right)}\sum_{x,y\in G_1}\sum_{u,v\in G_2,x\neq u(\text{or})y\neq v} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right]\right\}^{n_1n_2(n_1n_2-1)} \\
&=\left\{\frac{1}{n_1n_2\left(n_1n_2-1\right)}\sum_{x,y\in G_1}\sum_{u,v\in G_2} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right]\right\}^{n_1n_2(n_1n_2-1)} \\
&=\left\{\frac{1}{n_1n_2\left(n_1n_2-1\right)}\sum_{x,y\in G_1}\left[\sum_{uv\in G_2} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left(d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right)\right.\right. \\
&\quad\left.\left.+\sum_{uv\notin G_2} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left(d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right)\right]\right\}^{n_1n_2(n_1n_2-1)} \\
&=\left\{\frac{1}{n_1n_2\left(n_1n_2-1\right)}\left[\sum_{x,y\in G_1,x=y}\sum_{uv\in G_2} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right]\right.\right. \\
&\quad+\sum_{x,y\in G_1,x\neq y}\sum_{uv\in G_2} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right] \\
&\quad+\sum_{x,y\in G_1,x=y}\sum_{uv\notin G_2} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right] \\
&\quad\left.\left.+\sum_{x,y\in G_1,x\neq y}\sum_{uv\notin G_2} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right]\right]\right\}^{n_1n_2(n_1n_2-1)}
\end{aligned}$$

$$\left[DD^*G_1\left[G_2\right]\right]^2\leq\left\{\frac{1}{n_1n_2\left(n_1n_2-1\right)}\left(S_3+S_1+S_2+S_4\right)\right\}^{n_1n_2(n_1n_2-1)}$$

where S_3,S_1,S_2,S_4 are terms of the above sums taken in order.

Next we calculate S_1,S_2,S_3 and S_4 separately.

$$\begin{aligned}
S_1&=\sum_{x,y\in G_1,x\neq y}\sum_{uv\in G_2} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right] \\
&=\sum_{x,y\in G_1,x\neq y}\sum_{uv\in G_2} d_{G_1}(x,y)\left[d_{G_2}(u)+n_2d_{G_1}(x)+d_{G_2}(v)+n_2d_{G_1}(y)\right] \text{ by Lemma 2.1} \\
&=n_2\sum_{x,y\in G_1,x\neq y} d_{G_1}(x,y)\left[d_{G_1}(x)+d_{G_1}(y)\right]\sum_{uv\in G_2}1+\sum_{x,y\in G_1,x\neq y} d_{G_1}(x,y)\sum_{uv\in G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right] \\
&=4n_2m_2DD\left(G_1\right)+4M_1\left(G_2\right)W\left(G_1\right)
\end{aligned}$$

$$\begin{aligned}
S_2&=\sum_{x,y\in G_1,x=y}\sum_{uv\notin G_2} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right] \\
&=\sum_{x,y\in G_1,x=y}\left\{\sum_{uv\notin G_2,u=v} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right]\right. \\
&\quad\left.+\sum_{uv\notin G_2,u\neq v} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right]\right\} \\
&=0+\sum_{x,y\in G_1,x=y}\sum_{uv\notin G_2,u\neq v} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right] \\
&=\sum_{x,y\in G_1,x=y}\sum_{uv\notin G_2,u\neq v} d_{G_1}(x,y)\left[d_{G_2}(u)+n_2d_{G_1}(x)+d_{G_2}(v)+n_2d_{G_1}(y)\right] \text{ by Lemma 2.1} \\
&=2\sum_{uv\notin G_2,u\neq v}\left[d_{G_2}(u)+d_{G_2}(v)\right]\sum_{x,y\in G_1,x=y}1+2n_2\left(\sum_{uv\notin G_2,u\neq v}1\right)\sum_{x,y\in G_1,x=y}\left[d_{G_1}(x)+d_{G_1}(y)\right] \\
&=4n_1\bar{M}_1\left(G_2\right)+8n_2m_1n_2\left(n_2-1\right)
\end{aligned}$$

$$S_3 = \sum_{x,y\in G_1,x=y} \sum_{uv\in G_2} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right]$$
$$=1\cdot \sum_{x,y\in G_1,x=y} \sum_{uv\in G_2} \left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right]$$
$$= \sum_{x,y\in G_1,x=y} \sum_{uv\in G_2} \left[d_{G_2}(u)+d_{G_1}(x)n_2+d_{G_2}(v)+n_2 d_{G_1}(y)\right] \text{ by Lemma 2.1}$$
$$=\left(\sum_{x,y\in G_1,x=y} 1\right)\sum_{uv\in G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]+n_2\sum_{x,y\in G_1,x=y}\left[d_{G_1}(x)+d_{G_1}(y)\right]\left(\sum_{uv\in G_2}1\right)$$
$$=2n_1M_1(G_2)+8n_2m_1m_2$$

$$S_4 = \sum_{x,y\in G_1,x\neq y} \sum_{uv\notin G_2} d_{G_1[G_2]}\left((x,u),(y,v)\right)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right]$$
$$= \sum_{x,y\in G_1,x\neq y} \sum_{uv\notin G_2} d_{G_1}(x,y)\left[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)\right]$$
$$= \sum_{x,y\in G_1,x\neq y} \sum_{uv\notin G_2} d_{G_1}(x,y)\left[d_{G_2}(u)+d_{G_1}(x)n_2+d_{G_2}(v)+n_2 d_{G_1}(y)\right] \text{ by Lemma 2.1}$$
$$=\left(\sum_{x,y\in G_1,x\neq y} d_{G_1}(x,y)\right)\sum_{uv\notin G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]$$
$$+n_2\sum_{x,y\in G_1,x\neq y} d_{G_1}(x,y)\left[d_{G_1}(x)+d_{G_1}(y)\right]\left(\sum_{uv\notin G_2}1\right)$$
$$=4W(G_1)\bar{M}_1(G_2)+2n_2DD(G_1)(2\bar{m}_2+n_2)$$

$$\left[DD^*(G_1[G_2])\right]^2 \le \left\{\frac{1}{n_1n_2(n_1n_2-1)}\left[4M_1(G_2)W(G_1)+4n_2m_2DD(G_1)+4n_1\bar{M}_1(G_2)\right.\right.$$
$$+8n_2^2m_1(n_2-1)+2n_1M_1(G_2)+8m_1n_2m_2+4W(G_1)\bar{M}_1(G_2)$$
$$\left.\left.+2n_2DD(G_1)(2\bar{m}_2+n_2)\right]\right\}^{n_1n_2(n_1n_2-1)}$$

Lemma 3.2.

$$DD^*K_n[K_r]=\left[2(nr-1)\right]^{\frac{nr(nr-1)}{2}}$$

Proof: Clearly the graph $K_n[K_r]$ is the complete graph K_{nr}.

$$\therefore DD^*(K_n[K_r])=DD^*(K_{nr})=\left[2(nr-1)\right]^{\frac{nr(nr-1)}{2}} \quad (1)$$

Remark 3.3. Let $G_1=K_n$ and $G_2=K_r$. We get,

$$DD(G_1)=2(n-1)\frac{n(n-1)}{2}=n(n-1)^2,\ m_1=\frac{n(n-1)}{2},\ W(G_1)=\frac{n(n-1)}{2}$$
$$M_1(G_2)=r(r-1)^2,\ \bar{M}_1(G_2)=0,\ \bar{m}_2=0,\ n_1=n,\ n_2=r,\ m_2=\frac{r(r-1)}{2}$$

$\therefore$ In Theorem 3.1, put $G_1=K_n$ and $G_2=K_r$, we get

$$DD^*K_n[K_r]\le\left[2(nr-1)\right]^{\frac{nr(nr-1)}{2}} \quad (2)$$

From (1) and (2) our bound is tight

4. The Multiplicative Degree Distance of Join of Graph

Theorem 4.1. Let $G_i, i=1,2$, be a (n_i,m_i)-graph and let $\bar{m}_i=e(\bar{G}_i)$. Then

$$\left[DD^*(G_1+G_2)\right]^2 \le \left\{\frac{1}{(n_1+n_2)(n_1+n_2-1)}\left[2M_1(G_1)+4n_2m_1+4\bar{M}_1(G_1)+8n_2\bar{m}_1 + 2m_1n_2+2m_2n_1+n_1n_2(n_1+n_2)+2M_1(G_2) + 4n_1m_2+4\bar{M}_1(G_2)+8n_1\bar{m}_2\right]\right\}^{(n_1+n_2)(n_1+n_2-1)}$$

Proof:

$$\begin{aligned}
&\left[DD^*(G_1+G_2)\right]^2\\
&= \prod_{x,y\in V(G_1+G_2),x\ne y} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\\
&\le \left[\frac{1}{(n_1+n_2)(n_1+n_2-1)}\sum_{x,y\in V(G_1+G_2),x\ne y} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\right]^{(n_1+n_2)(n_1+n_2-1)}\\
&= \left[\frac{1}{(n_1+n_2)(n_1+n_2-1)}\sum_{x,y\in V(G_1+G_2)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\right]^{(n_1+n_2)(n_1+n_2-1)}\\
&= \left[\frac{1}{(n_1+n_2)(n_1+n_2-1)}\sum_{x\in V(G_1+G_2)}\sum_{y\in V(G_1+G_2)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\right]^{(n_1+n_2)(n_1+n_2-1)}\\
&= \left[\frac{1}{(n_1+n_2)(n_1+n_2-1)}\sum_{x\in V(G_1+G_2)}\left\{\sum_{y\in V(G_1)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\right.\right.\\
&\quad \left.\left.+\sum_{y\in V(G_2)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\right\}\right]^{(n_1+n_2)(n_1+n_2-1)}\\
&= \left[\frac{1}{(n_1+n_2)(n_1+n_2-1)}\sum_{x\in V(G_1+G_2)}\sum_{y\in V(G_1)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\right.\\
&\quad \left.+\sum_{x\in V(G_1+G_2)}\sum_{y\in V(G_2)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\right]^{(n_1+n_2)(n_1+n_2-1)}\\
&= \left[\frac{1}{(n_1+n_2)(n_1+n_2-1)}\sum_{x\in V(G_1)}\sum_{y\in V(G_1)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\right.\\
&\quad +\sum_{x\in V(G_2)}\sum_{y\in V(G_1)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\\
&\quad +\sum_{x\in V(G_1)}\sum_{y\in V(G_2)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\\
&\quad \left.+\sum_{x\in V(G_2)}\sum_{y\in V(G_2)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\right]^{(n_1+n_2)(n_1+n_2-1)}\\
&= \left[\frac{1}{(n_1+n_2)(n_1+n_2-1)}\sum_{x\in V(G_1)}\sum_{y\in V(G_1)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\right.\\
&\quad +2\sum_{x\in V(G_1)}\sum_{y\in V(G_2)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\\
&\quad \left.+\sum_{x\in V(G_2)}\sum_{y\in V(G_2)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]\right]^{(n_1+n_2)(n_1+n_2-1)}
\end{aligned}$$

$$\left[DD^{*}\left(G_1+G_2\right)\right]^2 \le \left[\frac{1}{\left(n_1+n_2\right)\left(n_1+n_2-1\right)}\left(J_1+2J_2+J_3\right)\right]^{\left(n_1+n_2\right)\left(n_1+n_2-1\right)}$$

where J_1, J_2, J_3 are terms of the above sums taken in order.

Next we calculate J_1, J_2 and J_3 separately one by one. Now,

$$\begin{aligned}
J_1 &= \sum_{x\in V(G_1)}\sum_{y\in V(G_1)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right] \\
&= \sum_{x,y\in V(G_1)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right] \\
&= \sum_{xy\in G_1} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right] \\
&\quad + \sum_{xy\notin G_1, x\neq y} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right] \\
&\quad + \sum_{xy\notin G_1, x=y} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right] \\
&= 1\cdot \sum_{xy\in G_1}\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right] \\
&\quad + 2\cdot \sum_{xy\notin G_1, x\neq y}\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]+0 \\
&= \sum_{xy\in G_1}\left[d_{G_1}(x)+n_2+d_{G_2}(y)+n_2\right] \\
&\quad + 2\sum_{xy\notin G_1, x\neq y}\left[d_{G_1}(x)+n_2+d_{G_2}(y)+n_2\right] \text{ by Lemma 2.1} \\
&= \sum_{xy\in G_1}\left[d_{G_1}(x)+d_{G_2}(y)\right]+2n_2\sum_{xy\in G_1}1 \\
&\quad + 2\left\{\sum_{xy\notin G_1, x\neq y}\left[d_{G_1}(x)+d_{G_2}(y)\right]+2n_2\sum_{xy\notin G_1, x\neq y}1\right\} \\
&= 2M_1(G_1)+4n_2m_1+4\bar{M}_1(G_1)+8n_2\bar{m}_1
\end{aligned}$$

$$\begin{aligned}
J_2 &= \sum_{x\in V(G_1)}\sum_{y\in V(G_2)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right] \\
&= 1\sum_{x\in V(G_1)}\sum_{y\in V(G_2)}\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right] \\
&= \sum_{x\in V(G_1)}\sum_{y\in V(G_2)}\left[d_{G_1}(x)+n_2+d_{G_2}(y)+n_1\right] \text{ by Lemma 2.1} \\
&= \sum_{x\in V(G_1)} d_{G_1}(x)\sum_{y\in V(G_2)}1+\sum_{x\in V(G_1)}1\sum_{y\in V(G_2)} d_{G_2}(y)+\left(n_1+n_2\right)\sum_{x\in V(G_1)}1\sum_{y\in V(G_2)}1 \\
&= 2m_1n_2+2m_2n_1+\left(n_1+n_2\right)n_1n_2
\end{aligned}$$

$$\begin{aligned}
J_3 &= \sum_{x\in V(G_2)}\sum_{y\in V(G_2)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right] \\
&= \sum_{x,y\in V(G_2)} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right] \\
&= \sum_{xy\in G_2} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right] \\
&\quad + \sum_{xy\notin G_2, x\neq y} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right] \\
&\quad + \sum_{xy\notin G_2, x=y} d_{(G_1+G_2)}(x,y)\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]
\end{aligned}$$

$$= 1\sum_{xy\in G_2}\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]$$

$$+2\sum_{xy\notin G_2, x\neq y}\left[d_{(G_1+G_2)}(x)+d_{(G_1+G_2)}(y)\right]+0$$

$$= \sum_{xy\in G_2}\left[d_{G_2}(x)+n_1+d_{G_2}(y)+n_1\right]$$

$$+2\sum_{xy\notin G_2, x\neq y}\left[d_{G_2}(x)+n_1+d_{G_2}(y)+n_1\right] \text{ by Lemma 2.1}$$

$$= \sum_{xy\in G_2}\left[d_{G_2}(x)+d_{G_2}(y)\right]+2n_1\sum_{xy\in G_2}1$$

$$+2\left\{\sum_{xy\notin G_2, x\neq y}\left[d_{G_2}(x)+d_{G_2}(y)\right]+2n_1\sum_{xy\notin G_2, x\neq y}1\right\}$$

$$= 2M_1(G_2)+4n_1m_2+4\bar{M}_1(G_2)+8n_1\bar{m}_2$$

$$\left[DD^*(G_1+G_2)\right]^2 \le \left\{\frac{1}{(n_1+n_2)(n_1+n_2-1)}\left[2M_1(G_1)+4n_2m_1+4\bar{M}_1(G_1)+8n_2\bar{m}_1 + 2m_1n_2+2m_2n_1+n_1n_2(n_1+n_2)+2M_1(G_2) + 4n_1m_2+4\bar{M}_1(G_2)+8n_1\bar{m}_2\right]\right\}^{(n_1+n_2)(n_1+n_2-1)}$$

Lemma 4.2.

$$DD^*[K_n+K_r] = \left[2(n+r-1)\right]^{\frac{(n+r)(n+r-1)}{2}}$$

Proof: Clearly the graph K_n+K_r is the complete graph K_{n+r}

$$DD^*[K_n+K_r] = DD^*[K_{n+r}] = \left[2(n+r-1)\right]^{\frac{(n+r)(n+r-1)}{2}} \tag{3}$$

Remark 4.3. Let $G_1=K_n$ and $G_2=K_r$. We get,

$$M_1(G_1)=n(n-1)^2,$$

$$m_1=\frac{n(n-1)}{2},$$

$$M_1(G_2)=r(r-1)^2,$$

$$\bar{M}_1(G_2)=0,$$

$$m_2=\frac{r(r-1)}{2}, \bar{M}_1(G_1)=0, n_1=n, n_2=r, \bar{m}_1=0, \bar{m}_2=0.$$

$\therefore$ In Theorem 4.1, put $G_1=K_n, G_2=K_r$, we get

$$DD^*[K_n+K_r]\le\left[2(n+r-1)\right]^{\frac{(n+r)(n+r-1)}{2}} \tag{4}$$

From (3) and (4) our bound is tight.

5. The Multiplicative Degree Distance of Disjunction of Graph

Theorem 5.1. Let $G_i, i=1,2$, be a (n_i,m_i)-graph and let $\bar{m}_i=e(\bar{G}_i)$. Then

$$\left[DD^*(G_1 \vee G_2)\right]^2$$

$$\leq \left[\frac{1}{n_1 n_2 (n_1 n_2 - 1)}\left\{2m_2 n_2 \left(2\bar{M}_1(G_1) + 4m_1\right) + 2n_1\left(2\bar{m}_1 + n_1\right)M_1(G_2)\right.\right.$$

$$-2M_1(G_2)\left(2\bar{m}_1(n_1-1) + 2m_1 - M_1(\bar{G}_1)\right) + 2n_2 M_1(G_1)\left(2\bar{m}_2 + n_2\right)$$

$$+2m_1 n_1\left(2\bar{M}_1(G_2) + 4m_2\right) - 2M_1(G_1)\left(2(n_2-1)\bar{m}_2 + 2m_2 - M_1(\bar{G}_2)\right)$$

$$+4n_2 M_1(G_1)m_2 + 4n_1 m_1 M_1(G_2) - 2M_1(G_1)M_1(G_2)$$

$$+2n_2\left(2\bar{M}_1(G_1) + 4m_1\right)\left(2\bar{m}_2 + 2n_2\right) + 2n_1\left(2\bar{M}_1(G_2) + 4m_2\right)\left(2\bar{m}_1 + n_1\right)$$

$$-4\left(2\bar{m}_1(n_1-1) + 2m_1 - M_1(\bar{G}_1)\right)\left(2\bar{m}_2(n_2-1) + 2m_2 - M_1(\bar{G}_2)\right)$$

$$\left.\left.-8m_1 n_2^2 - 4n_1^2 m_2 + 16m_1 m_2\right\}\right]^{n_1 n_2 (n_1 n_2 - 1)}$$

Proof:

$$\left[DD^*(G_1 \vee G_2)\right]^2$$

$$= \prod_{x,y \in G_1} \prod_{u,v \in G_2, x \neq y (\text{or}) u \neq v} d_{(G_1 \vee G_2)}((x,u),(y,v))\left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]$$

$$\leq \left\{\frac{1}{n_1 n_2 (n_1 n_2 - 1)}\left[\sum_{x,y \in G_1} \sum_{u,v \in G_2, x \neq y (\text{or}) u \neq v} d_{(G_1 \vee G_2)}((x,u),(y,v))\left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]\right]\right\}^{n_1 n_2 (n_1 n_2 - 1)}$$

$$= \left\{\frac{1}{n_1 n_2 (n_1 n_2 - 1)}\left[\sum_{x,y \in G_1} \sum_{u,v \in G_2} d_{(G_1 \vee G_2)}((x,u),(y,v))\left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]\right]\right\}^{n_1 n_2 (n_1 n_2 - 1)}$$

$$= \left\{\frac{1}{n_1 n_2 (n_1 n_2 - 1)}\left[\sum_{x,y \in G_1}\left(\sum_{uv \in G_2} d_{(G_1 \vee G_2)}((x,u),(y,v))\left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]\right.\right.\right.$$

$$\left.\left.\left.+\sum_{uv \notin G_2} d_{(G_1 \vee G_2)}((x,u),(y,v))\left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]\right)\right]\right\}^{n_1 n_2 (n_1 n_2 - 1)}$$

$$= \left\{\frac{1}{n_1 n_2 (n_1 n_2 - 1)}\left[\sum_{xy \in G_1} \sum_{uv \in G_2} d_{(G_1 \vee G_2)}((x,u),(y,v))\left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]\right.\right.$$

$$+\sum_{xy \notin G_1} \sum_{uv \in G_2} d_{(G_1 \vee G_2)}((x,u),(y,v))\left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]$$

$$+\sum_{xy \in G_1} \sum_{uv \notin G_2} d_{(G_1 \vee G_2)}((x,u),(y,v))\left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]$$

$$\left.\left.+\sum_{xy \notin G_1} \sum_{uv \notin G_2} d_{(G_1 \vee G_2)}((x,u),(y,v))\left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]\right]\right\}^{n_1 n_2 (n_1 n_2 - 1)}$$

$$\left[DD^*(G_1 \vee G_2)\right]^2 \leq \left[\frac{1}{n_1 n_2 (n_1 n_2 - 1)}\left(A_3 + A_1 + A_2 + A_4\right)\right]^{n_1 n_2 (n_1 n_2 - 1)}$$

where A_3, A_1, A_2, A_4 are terms of the above sums taken in order.

Next we calculate A_1, A_2, A_3 and A_4 separately one by one. Now,

$$
\begin{aligned}
A_1 &= \sum_{xy\notin G_1}\sum_{uv\in G_2} d_{(G_1\vee G_2)}\left((x,u),(y,v)\right)\left[d_{(G_1\vee G_2)}(x,u)+d_{(G_1\vee G_2)}(y,v)\right]\\
&= \sum_{xy\notin G_1}\sum_{uv\in G_2} 1\cdot\left[n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)+n_2 d_{G_1}(y)\right.\\
&\quad \left.+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right] \text{ by Lemma 2.1}\\
&= n_2\sum_{xy\notin G_1}\sum_{uv\in G_2}\left[d_{G_1}(x)+d_{G_1}(y)\right]+n_1\sum_{xy\notin G_1}\sum_{uv\in G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]\\
&\quad -\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_2}(u)-\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1}(y)d_{G_2}(v)\\
&= n_2\left(\sum_{xy\notin G_1}\left[d_{G_1}(x)+d_{G_1}(y)\right]\right)\left(\sum_{uv\in G_2}1\right)+n_1\left(\sum_{xy\notin G_1}1\right)\left(\sum_{uv\in G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]\right)\\
&\quad -\left(\sum_{xy\notin G_1} d_{G_1}(x)\right)\left(\sum_{uv\in G_2} d_{G_2}(u)\right)-\left(\sum_{xy\notin G_1} d_{G_1}(y)\right)\left(\sum_{uv\in G_2} d_{G_2}(v)\right)\\
&= 2m_2 n_2\left(2\bar{M}_1(G_1)+4m_1\right)+2n_1\left(2\bar{m}_1+n_1\right)M_1(G_2)\\
&\quad -2M_1(G_2)\left(2\bar{m}_1(n_1-1)+2m_1-M_1(\bar{G}_1)\right)
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= \sum_{xy\in G_1}\sum_{uv\notin G_2} d_{(G_1\vee G_2)}\left((x,u),(y,v)\right)\left[d_{(G_1\vee G_2)}(x,u)+d_{(G_1\vee G_2)}(y,v)\right]\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2}\left[n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)+n_2 d_{G_1}(y)\right.\\
&\quad \left.+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right] \text{ by Lemma 2.1}\\
&= n_2\sum_{xy\in G_1}\sum_{uv\in G_2}\left[d_{G_1}(x)+d_{G_2}(y)\right]+n_1\sum_{xy\in G_1}\sum_{uv\notin G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]\\
&\quad -\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1}(x)d_{G_2}(u)-\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1}(y)d_{G_2}(v)\\
&= n_2\left(\sum_{xy\in G_1}\left[d_{G_1}(x)+d_{G_2}(y)\right]\right)\left(\sum_{uv\notin G_2}1\right)+n_1\left(\sum_{xy\in G_1}1\right)\left(\sum_{uv\notin G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]\right)\\
&\quad -\left(\sum_{xy\in G_1} d_{G_1}(x)\right)\left(\sum_{uv\notin G_2} d_{G_2}(u)\right)-\left(\sum_{xy\in G_1} d_{G_1}(y)\right)\left(\sum_{uv\notin G_2} d_{G_2}(v)\right)\\
&= 2n_2 M_1(G_1)\left(2\bar{m}_2+n_2\right)+2n_1 m_1\left(2\bar{M}_1(G_2)+4m_2\right)\\
&\quad -2M_1(G_1)\left[2(n_2-1)\bar{m}_2+2m_2-M_1(\bar{G}_2)\right]
\end{aligned}
$$

$$
\begin{aligned}
A_3 &= \sum_{xy\in G_1}\sum_{uv\in G_2} d_{(G_1\vee G_2)}\left((x,u),(y,v)\right)\left[d_{(G_1\vee G_2)}(x,u)+d_{(G_1\vee G_2)}(y,v)\right]\\
&= \sum_{xy\in G_1}\sum_{uv\in G_2}\left[n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)+n_2 d_{G_1}(y)\right.\\
&\quad \left.+n_1 d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right] \text{ by Lemma 2.1}\\
&= n_2\sum_{xy\in G_1}\sum_{uv\in G_2}\left[d_{G_1}(x)+d_{G_2}(y)\right]+n_1\sum_{xy\in G_1}\sum_{uv\in G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]\\
&\quad -\sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_2}(u)-\sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_1}(y)d_{G_2}(v)\\
&= n_2\left(\sum_{xy\in G_1}\left[d_{G_1}(x)+d_{G_2}(y)\right]\right)\left(\sum_{uv\in G_2}1\right)+n_1\left(\sum_{xy\in G_1}1\right)\left(\sum_{uv\in G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]\right)\\
&\quad -\left(\sum_{xy\in G_1} d_{G_1}(x)\right)\left(\sum_{uv\in G_2} d_{G_2}(u)\right)-\left(\sum_{xy\in G_1} d_{G_1}(y)\right)\left(\sum_{uv\in G_2} d_{G_2}(v)\right)\\
&= 4n_2 m_2 M_1(G_1)+4n_1 m_1 M_1(G_2)-2M_1(G_1)M_1(G_2)
\end{aligned}
$$

$$A_4 = \sum_{xy \notin G_1} \sum_{uv \notin G_2} d_{(G_1 \vee G_2)}\left((x,u),(y,v)\right)\left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]$$
$$= 2\sum_{xy \notin G_1} \sum_{uv \notin G_2} \left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]$$
$$-2\sum_{xy \notin G_1, x=y} \sum_{uv \notin G_2, u=v} \left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]$$

$A_4 = 2A_5 - 2A_6$, where A_5 and A_6 are terms of the above sums taken in order.

Now,

$$A_5 = \sum_{xy \notin G_1} \sum_{uv \notin G_2} \left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]$$
$$= \sum_{xy \notin G_1} \sum_{uv \notin G_2} \left[n_2 d_{G_1}(x) + n_1 d_{G_2}(u) - d_{G_1}(x) d_{G_2}(u) + n_2 d_{G_1}(y)\right.$$
$$\left. + n_1 d_{G_2}(v) - d_{G_1}(y) d_{G_2}(v)\right] \text{ by Lemma 2.1}$$
$$= n_2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} \left[d_{G_1}(x) + d_{G_1}(y)\right] + n_1 \sum_{xy \notin G_1} \sum_{uv \notin G_2} \left[d_{G_2}(u) + d_{G_2}(v)\right]$$
$$- \sum_{xy \notin G_1} \sum_{uv \notin G_2} d_{G_1}(x) d_{G_2}(u) - \sum_{xy \notin G_1} \sum_{uv \notin G_2} d_{G_1}(y) d_{G_2}(v)$$
$$= n_2 \left(\sum_{xy \notin G_1} \left[d_{G_1}(x) + d_{G_2}(y)\right]\right)\left(\sum_{uv \notin G_2} 1\right) + n_1 \left(\sum_{xy \notin G_1} 1\right)\left(\sum_{uv \notin G_2} \left[d_{G_2}(u) + d_{G_2}(v)\right]\right)$$
$$-\left(\sum_{xy \notin G_1} d_{G_1}(x)\right)\left(\sum_{uv \notin G_2} d_{G_2}(u)\right) - \left(\sum_{xy \notin G_1} d_{G_1}(y)\right)\left(\sum_{uv \notin G_2} d_{G_2}(v)\right)$$
$$= n_2\left(2\bar{M}_1(G_1) + 4m_1\right)\left(2\bar{m}_2 + n_2\right) + n_1\left(2\bar{M}_1(G_2) + 4m_2\right)\left(2\bar{m}_1 + n_1\right)$$
$$-2\left(2\bar{m}_1(n_1 - 1) + 2m_1 - M_1(\bar{G}_1)\right)\left(2\bar{m}_2(n_2 - 1) + 2m_2 - M_1(\bar{G}_2)\right)$$

$$A_6 = \sum_{xy \notin G_1, x=y} \sum_{uv \notin G_2, u=v} \left[d_{(G_1 \vee G_2)}(x,u) + d_{(G_1 \vee G_2)}(y,v)\right]$$
$$= \sum_{xy \notin G_1, u=v} \sum_{uv \notin G_2, u=v} \left[n_2 d_{G_1}(x) + n_1 d_{G_2}(u) - d_{G_1}(x) d_{G_2}(u) + n_2 d_{G_1}(y)\right.$$
$$\left. + n_1 d_{G_2}(v) - d_{G_1}(y) d_{G_2}(v)\right] \text{ by Lemma 2.1}$$
$$= n_2 \sum_{xy \notin G_1, x=y} \sum_{uv \notin G_2, u=v} \left[d_{G_1}(x) + d_{G_1}(y)\right] + n_1 \sum_{xy \notin G_1, x=y} \sum_{uv \notin G_2, u=v} \left[d_{G_2}(u) + d_{G_2}(v)\right]$$
$$- \sum_{xy \notin G_1, x=y} \sum_{uv \notin G_2, u=v} d_{G_1}(x) d_{G_2}(u) - \sum_{xy \notin G_1, x=y} \sum_{uv \notin G_2, u=v} d_{G_1}(y) d_{G_2}(v)$$
$$= n_2 \left(\sum_{xy \notin G_1, x=y} \left[d_{G_1}(x) + d_{G_2}(y)\right]\right)\left(\sum_{uv \notin G_2, u=v} 1\right) + n_1 \left(\sum_{xy \notin G_1, x=y} 1\right)\left(\sum_{uv \notin G_2, u=v} \left[d_{G_2}(u) + d_{G_2}(v)\right]\right)$$
$$-\left(\sum_{xy \notin G_1, x=y} d_{G_1}(x)\right)\left(\sum_{uv \notin G_2, u=v} d_{G_2}(u)\right) - \left(\sum_{xy \notin G_1, x=y} d_{G_1}(y)\right)\left(\sum_{uv \notin G_2, u=v} d_{G_2}(v)\right)$$
$$= 4m_1 n_2^2 + 4m_2 n_1^2 - 8m_1 m_2$$

$$\left[DD^*(G_1 \vee G_2)\right]^2$$
$$\le \Bigg[\frac{1}{n_1n_2(n_1n_2-1)}\Big\{2m_2n_2\left(2\bar{M}_1(G_1)+4m_1\right)+2n_1\left(2\bar{m}_1+n_1\right)M_1(G_2)$$
$$-2M_1(G_2)\left(2\bar{m}_1(n_1-1)+2m_1-M_1(\bar{G}_1)\right)+2n_2M_1(G_1)\left(2\bar{m}_2+n_2\right)$$
$$+2m_1n_1\left(2\bar{M}_1(G_2)+4m_2\right)-2M_1(G_1)\left(2(n_2-1)\bar{m}_2+2m_2-M_1(\bar{G}_2)\right)$$
$$+4n_2M_1(G_1)m_2+4n_1m_1M_1(G_2)-2M_1(G_1)M_1(G_2)$$
$$+2n_2\left(2\bar{M}_1(G_1)+4m_1\right)\left(2\bar{m}_2+2n_2\right)+2n_1\left(2\bar{M}_1(G_2)+4m_2\right)\left(2\bar{m}_1+n_1\right)$$
$$-4\left(2\bar{m}_1(n_1-1)+2m_1-M_1(\bar{G}_1)\right)\left(2\bar{m}_2(n_2-1)+2m_2-M_1(\bar{G}_2)\right)$$
$$-8m_1n_2^2-4n_1^2m_2+16m_1m_2\Big\}\Bigg]^{n_1n_2(n_1n_2-1)}$$

Lemma 5.2.

$$DD^*[K_m \vee K_n]=(2mn-2)^{\frac{mn(mn-1)}{2}}$$

Proof: Clearly the graph $K_m \vee K_n$ is the complete graph K_{mn}.

$$DD^*(K_m \vee K_n)=DD^*(K_{mn})=(2mn-2)^{\frac{mn(mn-1)}{2}} \quad (5)$$

Remark 5.3. Let $G_1=K_m$ and $G_2=K_n$. We get

$$n_1=m, n_2=n, m_1=\frac{m(m-1)}{2}, m_2=\frac{n(n-1)}{2}, \bar{m}_1=0, \bar{m}_2=0$$

$$M_1(G_1)=M_1(K_m)=m(m-1)^2, M_1(G_2)=M_1(K_n)=n(n-1)^2$$

$$M_1(\bar{G}_1)=M_1(\bar{K}_m)=0, M_1(\bar{G}_2)=M_1(\bar{K}_n)=0, \bar{M}_1(G_1)=\bar{M}_1(K_m)=0$$

$\therefore$ In Theorem 5.1, put $G_1=K_m$ and $G_2=K_n$, we get

$$DD^*[K_m \vee K_n]\le(2mn-2)^{\frac{mn(mn-1)}{2}} \quad (6)$$

From (5) and (6) our bound is tight.

6. The Multiplicative Degree Distance of Symmetric difference of Graph

Theorem 6.1.

$$\left[DD^*(G_1 \oplus G_2)\right]^2$$
$$\le \Bigg\{\frac{1}{n_1n_2(n_1n_2-1)}\Big[2n_2m_2\left(2\bar{M}_1(G_1)+4m_1\right)+2n_1M_1(G_2)\left(2\bar{m}_1+n_1\right)$$
$$-4M_1(G_2)\left(2\bar{m}_1(n_1-1)+2m_1-M_1(\bar{G}_1)\right)+2n_2M_1(G_1)\left(2\bar{m}_2+n_2\right)$$
$$+2n_1m_1\left(2\bar{M}_1(G_2)+4m_2\right)-4M_1(G_1)\left(2(n_2-1)\bar{m}_2+2m_2-M_1(\bar{G}_2)\right)$$
$$+4n_2M_1(G_1)m_2+4n_1m_1M_1(G_2)-4M_1(G_1)M_1(G_2)$$
$$+2\Big[n_2\left(2\bar{M}_1(G_1)+4m_1\right)\left(2m_2+n_2\right)+n_1\left(2\bar{M}_1(G_2)+4m_2\right)\left(2m_1+n_1\right)$$
$$-4\left(2\bar{m}_1(n_1-1)+2m_1-M_1(\bar{G}_1)\right)\left(2\bar{m}_2(n_2-1)+2m_2-M_1(\bar{G}_2)\right)\Big]$$
$$-2\left(4n_2^2m_1+4n_1^2m_2-16m_1m_2\right)\Big]\Bigg\}^{n_1n_2(n_1n_2-1)}$$

Proof:

$$\left[DD^*(G_1 \oplus G_2)\right]^2$$

$$= \prod_{x,y\in G_1} \prod_{u,v\in G_2, x\neq y(\text{or})u\neq v} d_{(G_1\oplus G_2)}((x,u),(y,v))\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]$$

$$\leq \left\{\frac{1}{n_1n_2(n_1n_2-1)}\left[\sum_{x,y\in G_1}\sum_{u,v\in G_2, x\neq y(\text{or})u\neq v} d_{(G_1\oplus G_2)}((x,u),(y,v))\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]\right]\right\}^{n_1n_2(n_1n_2-1)}$$

$$= \left\{\frac{1}{n_1n_2(n_1n_2-1)}\left[\sum_{x,y\in G_1}\sum_{u,v\in G_2} d_{(G_1\oplus G_2)}((x,u),(y,v))\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]\right]\right\}^{n_1n_2(n_1n_2-1)}$$

$$= \left\{\frac{1}{n_1n_2(n_1n_2-1)}\left[\sum_{x,y\in G_1}\left(\sum_{uv\in G_2} d_{(G_1\oplus G_2)}((x,u),(y,v))\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]\right.\right.\right.$$

$$\left.\left.\left.+\sum_{uv\notin G_2} d_{(G_1\oplus G_2)}((x,u),(y,v))\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]\right)\right]\right\}^{n_1n_2(n_1n_2-1)}$$

$$= \left\{\frac{1}{n_1n_2(n_1n_2-1)}\left[\sum_{xy\in G_1}\sum_{uv\in G_2} d_{(G_1\oplus G_2)}((x,u),(y,v))\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]\right.\right.$$

$$+\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{(G_1\oplus G_2)}((x,u),(y,v))\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]$$

$$+\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{(G_1\oplus G_2)}((x,u),(y,v))\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]$$

$$\left.\left.+\sum_{xy\notin G_1}\sum_{uv\notin G_2} d_{(G_1\oplus G_2)}((x,u),(y,v))\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]\right]\right\}^{n_1n_2(n_1n_2-1)}$$

$$\left[DD^*(G_1 \oplus G_2)\right]^2 \leq \left[\frac{1}{n_1n_2(n_1n_2-1)}(C_3+C_1+C_2+C_4)\right]^{n_1n_2(n_1n_2-1)}$$

where C_3, C_1, C_2, C_4 are terms of the above sums taken in order.

Next we calculate C_1, C_2, C_3 and C_4 separately.

$$C_1 = \sum_{xy\notin G_1}\sum_{uv\in G_2} d_{(G_1\oplus G_2)}((x,u),(y,v))\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]$$

$$= \sum_{xy\notin G_1}\sum_{uv\in G_2} 1\cdot\left[n_2d_{G_1}(x)+n_1d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)+n_2d_{G_1}(y)\right.$$

$$\left.+n_1d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\right] \text{ by Lemma 2.1}$$

$$= n_2\sum_{xy\notin G_1}\sum_{uv\in G_2}\left[d_{G_1}(x)+d_{G_1}(y)\right]+n_1\sum_{xy\notin G_1}\sum_{uv\in G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]$$

$$-2\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_2}(u)-2\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1}(y)d_{G_2}(v)$$

$$= n_2\left(\sum_{xy\notin G_1}\left[d_{G_1}(x)+d_{G_1}(y)\right]\right)\left(\sum_{uv\in G_2}1\right)+n_1\left(\sum_{xy\notin G_1}1\right)\left(\sum_{uv\in G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]\right)$$

$$-2\left(\sum_{xy\notin G_1} d_{G_1}(x)\right)\left(\sum_{uv\in G_2} d_{G_2}(u)\right)-2\left(\sum_{xy\notin G_1} d_{G_1}(y)\right)\left(\sum_{uv\in G_2} d_{G_2}(v)\right)$$

$$= 2n_2m_2\left(2\bar{M}_1(G_1)+4m_1\right)+2n_1M_1(G_2)\left(2\bar{m}_1+n_1\right)$$

$$-4M_1(G_2)\left(2\bar{m}_1(n_1-1)+2m_1-M_1(\bar{G}_1)\right)$$

$$C_2 = \sum_{xy\in G_1}\sum_{uv\notin G_2} d_{(G_1\oplus G_2)}\left((x,u),(y,v)\right)\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]$$

$$= \sum_{xy\in G_1}\sum_{uv\notin G_2}\left[n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)\right.$$

$$\left.+n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\right] \text{ by Lemma 2.1}$$

$$= n_2\sum_{xy\in G_1}\sum_{uv\notin G_2}\left[d_{G_1}(x)+d_{G_2}(y)\right]+n_1\sum_{xy\in G_1}\sum_{uv\notin G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]$$

$$-2\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1}(x)d_{G_2}(u)-2\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1}(y)d_{G_2}(v)$$

$$= n_2\left(\sum_{xy\in G_1}\left[d_{G_1}(x)+d_{G_2}(y)\right]\right)\left(\sum_{uv\notin G_2}1\right)+n_1\left(\sum_{xy\in G_1}1\right)\left(\sum_{uv\notin G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]\right)$$

$$-2\left(\sum_{xy\in G_1} d_{G_1}(x)\right)\left(\sum_{uv\notin G_2} d_{G_2}(u)\right)-2\left(\sum_{xy\in G_1} d_{G_1}(y)\right)\left(\sum_{uv\notin G_2} d_{G_2}(v)\right)$$

$$= 2n_2 M_1(G_1)(\bar{m}_2+n_2)+2n_1 m_1\left(2\bar{M}_1(G_2)+4m_2\right)$$

$$-4M_1(G_1)\left(2(n_2-1)\bar{m}_2+2m_2-M_1(\bar{G}_2)\right)$$

$$C_3 = \sum_{xy\in G_1}\sum_{uv\in G_2} d_{(G_1\oplus G_2)}\left((x,u),(y,v)\right)\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]$$

$$= \sum_{xy\in G_1}\sum_{uv\in G_2}\left[n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)\right.$$

$$\left.+n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\right] \text{ by Lemma 2.1}$$

$$= n_2\sum_{xy\in G_1}\sum_{uv\in G_2}\left[d_{G_1}(x)+d_{G_2}(y)\right]+n_1\sum_{xy\in G_1}\sum_{uv\in G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]$$

$$-2\sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_2}(u)-2\sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_1}(y)d_{G_2}(v)$$

$$= n_2\left(\sum_{xy\in G_1}\left[d_{G_1}(x)+d_{G_2}(y)\right]\right)\left(\sum_{uv\in G_2}1\right)+n_1\left(\sum_{xy\in G_1}1\right)\left(\sum_{uv\in G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]\right)$$

$$-2\left(\sum_{xy\in G_1} d_{G_1}(x)\right)\left(\sum_{uv\in G_2} d_{G_2}(u)\right)-2\left(\sum_{xy\in G_1} d_{G_1}(y)\right)\left(\sum_{uv\in G_2} d_{G_2}(v)\right)$$

$$= 4n_2 M_1(G_1)m_2+4n_1 m_1 M_1(G_2)-4M_1(G_1)M_1(G_2)$$

$$C_4 = \sum_{xy\notin G_1}\sum_{uv\notin G_2} d_{(G_1\oplus G_2)}\left((x,u),(y,v)\right)\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]$$

$$= 2\sum_{xy\notin G_1}\sum_{uv\notin G_2}\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]$$

$$-2\sum_{xy\notin G_1,x=y}\ \sum_{uv\notin G_2,u=v}\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]$$

$C_4 = 2C_5 - 2C_6$, where C_5 and C_6 denote the sums of the above terms in order.

Now,

$$C_5 = \sum_{xy\notin G_1}\sum_{uv\notin G_2}\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]$$

$$= \sum_{xy\notin G_1}\sum_{uv\notin G_2}\left[n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)\right.$$

$$\left.+n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\right] \text{ by Lemma 2.1}$$

$$= n_2\sum_{xy\notin G_1}\sum_{uv\notin G_2}\left[d_{G_1}(x)+d_{G_2}(y)\right]+n_1\sum_{xy\notin G_1}\sum_{uv\notin G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]$$

$$-2\sum_{xy\notin G_1}\sum_{uv\notin G_2}d_{G_1}(x)d_{G_2}(u)-2\sum_{xy\notin G_1}\sum_{uv\notin G_2}d_{G_1}(y)d_{G_2}(v)$$

$$= n_2\left(\sum_{xy\notin G_1}\left[d_{G_1}(x)+d_{G_2}(y)\right]\right)\left(\sum_{uv\notin G_2}1\right)+n_1\left(\sum_{xy\notin G_1}1\right)\left(\sum_{uv\notin G_2}\left[d_{G_2}(u)+d_{G_2}(v)\right]\right)$$

$$-2\left(\sum_{xy\notin G_1}d_{G_1}(x)\right)\left(\sum_{uv\notin G_2}d_{G_2}(u)\right)-2\left(\sum_{xy\notin G_1}d_{G_1}(y)\right)\left(\sum_{uv\notin G_2}d_{G_2}(v)\right)$$

$$= n_2\left(2\bar{M}_1(G_1)+4m_1\right)\left(2m_2+n_2\right)+n_1\left(2\bar{M}_1(G_2)+4m_2\right)\left(2m_1+n_1\right)$$

$$-4(\left(2\bar{m}_1(n_1-1)+2m_1-M_1(\bar{G}_1)\right)\left(2\bar{m}_2(n_2-1)+2m_2-M_1(\bar{G}_2)\right)$$

$$C_6 = \sum_{xy\notin G_1,x=y}\sum_{uv\notin G_2,u=v}\left[d_{(G_1\oplus G_2)}(x,u)+d_{(G_1\oplus G_2)}(y,v)\right]$$

$$= \sum_{xy\notin G_1,x=y}\sum_{uv\notin G_2,u=v}[n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)$$

$$\left.+n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\right] \text{ by Lemma 2.1}$$

$$= n_2\sum_{xy\notin G_1,x=y}\sum_{uv\notin G_2,u=v}\left[d_{G_1}(x)+d_{G_2}(y)\right]+n_1\sum_{xy\notin G_1,x=y}\sum_{uv\notin G_2,u=v}\left[d_{G_2}(u)+d_{G_2}(v)\right]$$

$$-2\sum_{xy\notin G_1,x=y}\sum_{uv\notin G_2}d_{G_1,u=v}(x)d_{G_2}(u)-2\sum_{xy\notin G_1,x=y}\sum_{uv\notin G_2,u=v}d_{G_1}(y)d_{G_2}(v)$$

$$= n_2\left(\sum_{xy\notin G_1,x=y}\left[d_{G_1}(x)+d_{G_2}(y)\right]\right)\left(\sum_{uv\notin G_2,u=v}1\right)+n_1\left(\sum_{xy\notin G_1,x=y}1\right)\left(\sum_{uv\notin G_2,u=v}\left[d_{G_2}(u)+d_{G_2}(v)\right]\right)$$

$$-2\left(\sum_{xy\notin G_1,x=y}d_{G_1}(x)\right)\left(\sum_{uv\notin G_2,u=v}d_{G_2}(u)\right)-2\left(\sum_{xy\notin G_1,x=y}d_{G_1}(y)\right)\left(\sum_{uv\notin G_2,u=v}d_{G_2}(v)\right)$$

$$= 4n_2^2 m_1+4n_1^2 m_2-16m_1m_2$$

$$\left[DD^*(G_1\oplus G_2)\right]^2$$

$$\leq\left\{\frac{1}{n_1n_2(n_1n_2-1)}\left[2n_2m_2\left(2\bar{M}_1(G_1)+4m_1\right)+2n_1M_1(G_2)\left(2\bar{m}_1+n_1\right)\right.\right.$$

$$-4M_1(G_2)\left(2\bar{m}_1(n_1-1)+2m_1-M_1(\bar{G}_1)\right)+2n_2M_1(G_1)\left(2\bar{m}_2+n_2\right)$$

$$+2n_1m_1\left(2\bar{M}_1(G_2)+4m_2\right)-4M_1(G_1)\left(2(n_2-1)\bar{m}_2+2m_2-M_1(\bar{G}_2)\right)$$

$$+4n_2M_1(G_1)m_2+4n_1m_1M_1(G_2)-4M_1(G_1)M_1(G_2)$$

$$+2\left[n_2\left(2\bar{M}_1(G_1)+4m_1\right)\left(2m_2+n_2\right)+n_1\left(2\bar{M}_1(G_2)+4m_2\right)\left(2m_1+n_1\right)\right.$$

$$\left.-4\left(2\bar{m}_1(n_1-1)+2m_1-M_1(\bar{G}_1)\right)\left(2\bar{m}_2(n_2-1)+2m_2-M_1(\bar{G}_2)\right)\right]$$

$$\left.\left.-2\left(4n_2^2m_1+4n_1^2m_2-16m_1m_2\right)\right]\right\}^{n_1n_2(n_1n_2-1)}$$

Lemma 6.2.

$$DD^*[K_m\oplus K_1]=(2m-2)^{\frac{m(m-1)}{2}}$$

Proof: Clearly the graph $K_m \oplus K_1$ is the complete graph K_m

$$DD^*[K_m \oplus K_1] = DD^* K_m = (2m-2)^{\frac{m(m-1)}{2}} \tag{7}$$

Remark 6.3. Let $G_1 = K_m$ and $G_2 = K_1$. We get

$$n_1 = m,\ n_2 = 1, m_1 = \frac{m(m-1)}{2},\ m_2 = 0,\ \bar{m}_1 = 0,\ \bar{m}_2 = 0$$

$$M_1(G_1) = M_1(K_m) = m(m-1)^2, M_1(G_2) = M_1(K_1) = 0$$

$$M_1(\bar{G}_1) = M_1(\bar{K}_m) = 0, M_1(\bar{G}_2) = M_1(\bar{K}_1) = 0$$

$$\bar{M}_1(G_1) = \bar{M}_1(K_m) = 0, \bar{M}_1(G_2) = \bar{M}_1(K_1) = 0$$

$\therefore$ In Theorem 6.1, put $G_1 = K_m$ and $G_2 = K_1$, we get

$$DD^*[K_m \oplus K_1] \le (2m-2)^{\frac{m(m-1)}{2}} \tag{8}$$

From (7) and (8) our bound is tight.

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